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Splitting of some more spaces

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In (4) we showed how to stably split certain spaces $C\mathbf{X}$ built up from a 'coefficient system' $\mathscr{C}$ and a 'Π-space' $\mathbf{X}$. Via an approximation theorem relating particular examples to loop spaces, there resulted stable splittings of $\Omega^n\Sigma^n X$ for all $n \geqslant 1$ and all (path) connected based spaces X.

In this short sequel to (4) we shall introduce and split a new construction $\bar{C}\mathbf{X}$. Via an approximation theorem, there will result a stable splitting of the basepoint component $\Omega_0^n\,\Sigma^n(X^+)$ of $\Omega^n\Sigma^n(X^+)$ for all $n \geqslant 2$ and all connected X, where X^+ denotes the union of X and a disjoint basepoint. In the case $n = \infty$ and $X^+ = S^0$, such a splitting was announced by Kahn and Priddy (5), but details have not appeared (compare (6)). The first author promised more such splittings in (2). In the case $n = 2$ and $X^+ = S^0$, such a splitting is already obvious from the homotopy equivalence $\Omega^2S^3 \simeq \Omega_0^2 S^2$ (and a comparison between the old splitting of Ω^2S^3 and the new splitting of $\Omega_0^2 S^2$ will be given in Proposition 3.3).

The methods here follow those of (4) in philosophy and in technical detail. For coefficient systems $\mathscr{C}$ equipped with suitable maps $\mathscr{C}_r \to \mathscr{C}_{r+1}$ and for Π-spaces $\mathbf{X}$, we shall construct spaces

$$\bar{C}\mathbf{X} = \operatorname{Colim} \mathscr{C}_r \times_{\Sigma_r} X_r.$$

For certain spaces $\bar{D}_q(\mathscr{C}, \mathbf{X})$ equivalent to the cofibres of the maps

$$\mathscr{C}_{q-1} \times_{\Sigma_{q-1}} X_{q-1} \to \mathscr{C}_q \times_{\Sigma_q} X_q$$

of the colimit system, we shall prove that $\bar{C}\mathbf{X}$ splits homologically as the wedge of the $\bar{D}_q(\mathscr{C}, \mathbf{X})$ and that, if $\mathscr{C}$ is Σ-free, this splitting is realized by a stable splitting of spaces. The constructions are somewhat more delicate than in (4) since the requirements for a well-behaved colimit system and for compatible James maps (as in Lemma 2·3 below) tend to be in conflict with each other. The fussy details of Section 1 are designed to arrange this precise compatibility.

We note the following example of our stable splitting theorem.

COROLLARY. *Let G be any topological monoid. Then $B(\Sigma_\infty \int G)$ splits stably as the wedge of the cofibres of the natural maps $B(\Sigma_{q-1} \int G) \to B(\Sigma_q \int G)$.*

When G is the trivial group, the resulting stable splitting of $B\Sigma_\infty$, and hence of Q_0S^0, is the cited result of Kahn and Priddy. Analogous stable splittings of BO, BU, and BSp have been given by Snaith (7).

1. *Directed coefficient systems.* The basic constructions of this paper depend on coefficient systems with certain additional structure. We describe this structure and give a number of examples and counterexamples here.

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Definition 1·1. A coefficient system $\mathscr{C}$ (see (4), 1·3) is said to be directed if there are subspaces $\mathscr{A}_{r+1}$ of $\mathscr{C}_{r+1}$ and maps $\lambda_r: \mathscr{C}_r \to \mathscr{A}_{r+1}$ for $r \geq 0$ which satisfy the following properties.

(i) $\mathscr{A}_{r+1}$ is a sub Σ_r-space of $\mathscr{C}_{r+1}$ and the inclusion of $\mathscr{A}_{r+1}\tau^{i_1} \cap \ldots \cap \mathscr{A}_{r+1}\tau^{i_q}$ in $\mathscr{C}_{r+1}$ is a $(\tau^{i_1}\Sigma_r\tau^{-i_1} \cap \ldots \cap \tau^{i_q}\Sigma_r\tau^{-i_q})$-equivariant cofibration for all sequences

$$0 \leq i_1 < \ldots < i_q \leq r,$$

where $\Sigma_r \subset \Sigma_{r+1}$ is the subgroup fixing the last letter and $\tau \in \Sigma_{r+1}$ is the cyclic permutation $(1, 2, \ldots, r+1)$.

(ii) $\lambda_r: \mathscr{C}_r \to \mathscr{A}_{r+1}$ is a Σ_r-equivariant homotopy equivalence and $\phi_r \lambda_r = 1$ on $\mathscr{C}_r$, where $\phi_r: \mathscr{C}_{r+1} \to \mathscr{C}_r$ is induced by the injection $\phi_r: \mathbf{r} \to \mathbf{r+1}$ specified by $\phi_r(i) = i$ for $0 \leq i \leq r$.

(iii) If $\omega: \mathbf{q} \to \mathbf{r+1}$ is an ordered injection such that $\omega(q) = r+1$, then the composite $\mathscr{C}_r \xrightarrow{\lambda_r} \mathscr{C}_{r+1} \xrightarrow{\omega} \mathscr{C}_q$ takes values in $\mathscr{A}_q$.

A map $g: \mathscr{C} \to \mathscr{C}'$ of directed coefficient systems is a map of coefficient systems such that $g(\mathscr{A}_{r+1}) \subset \mathscr{A}'_{r+1}$ and $g\lambda_r = \lambda'_r g$ for $r \geq 0$.

Remarks 1·2. By a result of Boardman and Vogt (1), 2·7 (p. 234), the cofibration condition of (i) implies that the inclusion of $\mathscr{A}_{r+1}\Sigma_{r+1}$ in $\mathscr{C}_{r+1}$ is a Σ_{r+1}-equivariant cofibration, where

$$\mathscr{A}_{r+1}\Sigma_{r+1} = \bigcup_{0 \leq i \leq r} \mathscr{A}_{r+1}\tau^i$$

is the saturation of $\mathscr{A}_{r+1}$ under the action of Σ_{r+1}. In turn, (i) is implied by the following two conditions.

(i*a*) For $c \in \mathscr{A}_{r+1}$ and $\sigma \in \Sigma_{r+1}$, $c\sigma \in \mathscr{A}_{r+1}$ if and only if $\sigma \in \Sigma_r$.

(i*b*) The inclusion of $\mathscr{A}_{r+1}$ in $\mathscr{C}_{r+1}$ is a Σ_r-equivariant cofibration.

The need for precisely these conditions will gradually become apparent. The $\mathscr{A}_r$ are not really needed for our first example.

Example 1·3. Suppose $\mathscr{C}: \Lambda \to \mathscr{U}$ extends to a contravariant functor $\Pi \to \mathscr{U}$ (see (4), 1·1–1·3). Let $\lambda_r: \mathscr{C}_r \to \mathscr{C}_{r+1}$ be induced by the projection $\lambda_r: \mathbf{r+1} \to \mathbf{r}$ specified by $\lambda_r(i) = i$ for $0 \leq i \leq r$ and $\lambda_r(r+1) = 0$ and let $\mathscr{A}_{r+1}$ be the image of λ_r. Then (ii) and (iii) hold trivially and only the cofibration condition of (i) need be assumed. Since Π is isomorphic to its own opposite, any Π-space (see (4), 1·8) thus gives a directed coefficient system. The operads $\mathscr{M}$ and $\mathscr{N}$ are other such examples.

The $\mathscr{A}_r$ are less obvious in the following example.

Example 1·4. Let Y be a space which contains a copy of $R \times Z$, where R is the real numbers and Z is a non-degenerately based space. Assume the following.

(*a*) There is a map $p: Y \to R$ which restricts to the projection on $R \times Z$; this holds, for example, if the inclusion $R \times Z \to Y$ is a cofibration.

(*b*) $(Z, *)$ admits a representation as an NDR-pair by maps $k: I \times Z \to Z$ and $w: Z \to I$ such that $R \times w^{-1}[0, 1)$ is an open subset of Y.

Then the configuration space coefficient system $\mathscr{C}(Y)$ of (5), 1·6, is directed. In particular, $\mathscr{C}(R \times Z)$ is directed, naturally with respect to based injections of Z, and $\mathscr{C}(M)$ is directed if M is an open (paracompact but not compact) *PL* manifold or an open topological manifold of dimension other than four.

Proof. Define
$$\lambda_r\langle y_1,\ldots,y_r\rangle = \langle y_1,\ldots,y_r,(m+1,*)\rangle,$$
where, here and below, $m = \max_{1\leqslant i\leqslant r} p(y_i)$; define
$$\mathscr{A}_{r+1} = \{\langle y_1,\ldots,y_r,(t,*)\rangle \mid t \geqslant m+1\}.$$
Clearly $\phi_r\lambda_r = 1$ on $F(Y,r)$, and a Σ_r-equivariant deformation d: $1 \simeq \lambda_r\phi_r$ on $\mathscr{A}_{r+1}$ is specified by
$$d(s,\langle y_1,\ldots,y_r,(t,*)\rangle) = \langle y_1,\ldots,y_r,(t-st+s+sm,*)\rangle.$$
This verifies (ii), and (iii) holds since if ω: $\mathbf{q}\to\mathbf{r}+\mathbf{1}$ satisfies $\omega(q) = r+1$, then
$$(\lambda_r\langle y_1,\ldots,y_r\rangle)\,\omega = \langle y_{\omega(1)},\ldots,y_{\omega(q-1)},(1+m,*)\rangle,$$
which is in $\mathscr{A}_q$ since
$$m = \max_{1\leqslant i\leqslant r} p(y_i) \geqslant \max_{1\leqslant j<q} p(y_{\omega(j)}).$$
Since (ia) obviously holds, it remains to show (ib). We do this by displaying
$$(F(Y,r+1),\mathscr{A}_{r+1})$$
as a Σ_r-NDR-pair. Define a map δ: $Y\to I$ by
$$\delta(y) = \begin{cases} 1 & \text{if } y\notin R\times Z \\ 2w(z)-1 & \text{if } y=(t,z) \text{ with } w(z)\geqslant \tfrac12 \\ 0 & \text{if } y=(t,z) \text{ with } w(z)\leqslant\tfrac12. \end{cases}$$
The openness condition (b) ensures that δ is continuous. Define
$$h\colon I\times F(Y,r+1)\to F(Y,r+1) \quad\text{and}\quad u\colon F(Y,r+1)\to I$$
on points
$$y = \langle y_1,\ldots,y_r,y_{r+1}\rangle \quad\text{with}\quad m = \max_{1\leqslant i\leqslant r} p(y_i)$$
by
$$h(s,y) = \langle y_1,\ldots,y_r,j(s,m,y_{r+1})\rangle \quad\text{and}\quad u(y) = v(m,y_{r+1}),$$
where
$$j(s,m,y_{r+1}) = y_{r+1} \quad\text{and}\quad v(m,y_{r+1}) = 1 \quad\text{if}\quad y_{r+1}\notin R\times Z,$$
and where, if $y_{r+1} = (t,z)$ with $\delta(y_{r+1}) = d$,
$$j(s,m,y_{r+1}) = \begin{cases} (t,z) & \text{if } t\leqslant m \\ (td+(1-d)(t+st-sm), \\ \quad k((1-d)(2st-2sm),z)) & \text{if } m\leqslant t\leqslant m+\tfrac12 \\ (td+(1-d)(t+sm+s-st), \\ \quad k((1-d)s,z)) & \text{if } m+\tfrac12\leqslant t\leqslant m+1 \\ (t,k((1-d)s,z)) & \text{if } m+1\leqslant t \end{cases}$$
and
$$v(m,y_{r+1}) = \begin{cases} 1 & \text{if } t\leqslant m+\tfrac12 \text{ or } w(z)\geqslant\tfrac12 \\ 1-(2t-2m-1)(1-2w(z)) & \text{if } m+\tfrac12\leqslant t\leqslant m+1 \text{ and } w(z)\leqslant\tfrac12 \\ 2w(z) & \text{if } m+1\leqslant t \text{ and } w(z)\leqslant\tfrac12. \end{cases}$$
Obviously h and u are Σ_r-equivariant, and it is easy to check that $h(0,y) = y$, $h(s,y) = y$ if $y\in\mathscr{A}_{r+1}$, $h(1,y)\in\mathscr{A}_{r+1}$ if $u(y) < 1$ (this holding if and only if $m+\frac12 < t$ and $w(z) < \frac12$), and $u^{-1}(0) = \mathscr{A}_{r+1}$.

The hypotheses on Y cannot be greatly weakened.

Counter-example 1·5. Let Y be a compact ANR. Then $\mathscr{C}(Y)$ cannot be directed.

Proof. Suppose we have $\mathscr{A}_1$, $\mathscr{A}_2$, and $\lambda_1: Y \to \mathscr{A}_2$ satisfying (ii) and (iii). The equivalence $\lambda_0: \mathscr{C}_0 = \{*\} \to \mathscr{A}_1$ shows that $\mathscr{A}_1$ is contractible. We have

$$\mathscr{A}_2 \subset F(Y,2) \subset Y \times Y,$$

and we let $\pi_i: Y \times Y \to Y$ be the projections. Clearly $\pi_1 = \phi_2$, hence $\pi_1 \lambda_1$ is the identity on Y. Clearly $\pi_2 = \omega$, where $\omega: F(Y,2) \to Y$ is induced by the injection $\omega: \mathbf{1} \to \mathbf{2}$ with $\omega(1) = 2$, hence $\pi_2 \lambda_1$ factors through $\mathscr{A}_1$ and is thus null homotopic. Moreover, $\pi_2 \lambda_1: Y \to Y$ cannot have a fixed point since λ_1 does take values in $F(Y,2)$. This contradicts the Lefschetz fixed point theorem.

Other important examples also fail to be directed.

Counter-example 1·6. The little cubes operads $\mathscr{C}_n$ cannot be directed. Indeed, $1: I^n \to I^n$ is a point of $\mathscr{C}_{n,1}$, and the condition $\phi_1 \lambda_1 = 1$ on $\mathscr{C}_{n,1}$ would force $\lambda_1(1)$ to be a pair of little cubes $\langle 1, c \rangle$ with disjoint interiors, an obvious impossibility.

However, we do have the following closure property.

Example 1·7. The product of directed coefficient systems is directed. Since $\mathscr{C}(R^\infty)$ is directed, by Example 1·4, we can use products with $\mathscr{C}(R^\infty)$ just as in (4) to prove our splitting theorems for arbitrary Σ-free directed coefficient systems once they are known for separated directed systems (see (4), 5·2–5·4).

2. *The general splitting theorems*. We construct analogs $\bar{C}\mathbf{X}$ and $\bar{D}_q(\mathscr{C}, \mathbf{X})$ of the spaces $C\mathbf{X}$ and $D_q(\mathscr{C}, \mathbf{X})$ introduced in (4), 2·1–2·3, and define the relevant James splitting maps. The splitting theorems will then follow by the same pattern of argument as in (4). We assume given a directed coefficient system $\mathscr{C}$ and a Π-space $\mathbf{X}$.

Definition 2·1. (i) With ϕ_r as in Definition 1·1 (ii), define

$$\zeta_r = \lambda_r \times \phi_r: \mathscr{C}_r \times_{\Sigma_r} X_r \to \mathscr{C}_{r+1} \times_{\Sigma_{r+1}} X_{r+1} \quad (r \geqslant 0).$$

Then define $\bar{C}\mathbf{X}$ to be the colimit over r of the inclusions ζ_r.

(ii) Define $\bar{D}_q(\mathscr{C}, \mathbf{X})$, or $\bar{D}_q\mathbf{X}$ for short, to be the quotient space

$$\mathscr{C}_q \times_{\Sigma_q} X_q / \mathscr{A}_q \times_{\Sigma_{q-1}} \phi_{q-1} X_{q-1} = (\mathscr{C}_q \times X_q / [\mathscr{A}_q \times \phi_{q-1} X_{q-1}]\Sigma_q)/\Sigma_q.$$

By Definition 1·1 (i) the inclusion of the saturation $[\mathscr{A}_q \times \phi_{q-1} X_{q-1}]\Sigma_q$ in $\mathscr{C}_q \times X_q$ is a Σ_q-equivariant cofibration, hence $\bar{D}_q\mathbf{X}$ is equivalent to the cofibre (or mapping cone) of the inclusion of $\mathscr{A}_q \times_{\Sigma_{q-1}} \phi_{q-1} X_{q-1}$ in $\mathscr{C}_q \times_{\Sigma_q} X_q$. Since ζ_{q-1} is the composite of the latter inclusion and the equivalence

$$\lambda_{q-1} \times \phi_{q-1}: \mathscr{C}_{q-1} \times_{\Sigma_{q-1}} X_{q-1} \to \mathscr{A}_q \times_{\Sigma_{q-1}} \phi_{q-1} X_{q-1},$$

it follows that $\bar{D}_q\mathbf{X}$ is equivalent to the cofibre of ζ_{q-1}.

Clearly $\bar{C}\mathbf{X}$ and the $\bar{D}_q\mathbf{X}$ are functors of $\mathscr{C}$ and $\mathbf{X}$. We write them as $\bar{C}X$ and $\bar{D}_qX$ when $\mathbf{X}$ arises as in (4), 1·9, from a space X. We now give a variant of the generalized James maps of (4), 4·2.

Definition 2·2. Assume given a James system $\xi_{qr}\colon \mathscr{C}_r \to \mathscr{C}'_m$, $m = (r-q, q)$, as in (4), 4·1 (where $\mathscr{C}'$ need not be directed). Define maps

$$\bar{\jmath}_{qr}\colon \mathscr{C}_r \times_{\Sigma_r} X_r \to C'\bar{D}_q(\mathscr{C}, \mathbf{X})$$

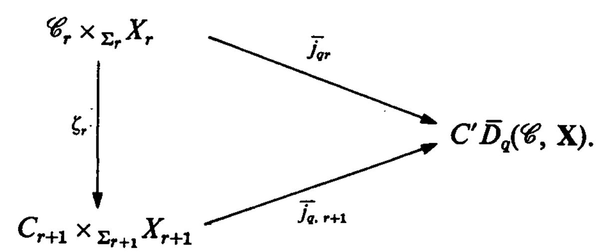
as follows. For $r < q$, $\bar{\jmath}_{qr}$ is the trivial map. For $r \geqslant q$,

$$\bar{\jmath}_{qr}(c, x) = (\xi_{qr}(c), [c\psi_1; \psi_1^{-1} x], \ldots, [c\psi_m; \psi_m^{-1} x]),$$

where $\{\psi_i\}$ is the set of ordered injections $\mathbf{q} \to \mathbf{r}$ in reverse lexicographic order and where $[c\psi_i; \psi_i^{-1} x]$ denotes the image in $\bar{D}_q(\mathscr{C}, \mathbf{X})$ of $(c\psi_i, \psi_i^{-1} x)$ in $\mathscr{C}_q \times X_q$ (see (4), 3·1). That $\bar{\jmath}_{qr}$ factors over Σ_r is immediate from the argument of (4), 4·2, applied to a permutation $\phi\colon \mathbf{r} \to \mathbf{r}$.

The following lemma is crucial.

Lemma 2·3. *The following diagram commutes for* $r \geqslant 0$.



Thus, by passage to colimits, the $\bar{\jmath}_{q,r}$ *induce a map*

$$\bar{\jmath}_q\colon \bar{C}\mathbf{X} \to C'\bar{D}_q(\mathscr{C}, \mathbf{X}).$$

Proof. The idea is to imitate the case $\phi = \phi_r\colon \mathbf{r} \to \mathbf{r}+\mathbf{1}$ of the argument in (4), 4·2. Let $\{\psi_i\}$ and $\{\omega_j\}$ be the ordered sets of ordered injections $\mathbf{q} \to \mathbf{r}$ and $\mathbf{q} \to \mathbf{r}+\mathbf{1}$. Then

$$\bar{\jmath}_{qr}(c, x) = \left(\xi_{qr}(c), \ \bigtimes_{i=1}^{m} [c\psi_i; \psi_i^{-1} x]\right), \quad m = (r-q, q),$$

and

$$\bar{\jmath}_{q,r+1} \zeta_r(c, x) = \left(\xi_{q,r+1} \lambda_r(c), \ \bigtimes_{j=1}^{n} [\lambda_r(c)\,\omega_j, \omega_j^{-1} \phi_r x]\right), \quad n = (r+1-q, q).$$

If $\omega_j(q) = r+1$, then $j > m$, $\omega_j^{-1} \phi_r x = \phi_{q-1} x'$ for some $x' \in X_{q-1}$, and $\lambda_r(c)\,\omega_j \in \mathscr{A}_q$ by Definition 1·1 (iii). Thus, for such j,

$$[\lambda_r(c)\,\omega_j, \omega_j^{-1} \phi_r x] = * \in \bar{D}_q \mathbf{X}.$$

If $\omega_j(q) \leqslant r$, then $j \leqslant m$ and $\omega_j = \phi_r \psi_j$. Here $\omega_j^{-1} \phi_r x = \psi_j^{-1} x$ as in (4), 3·2, while Definition 1·1 (ii) and the definition of a James system give

$$\lambda_r(c)\,\omega_j = \lambda_r(c)\,\phi_r \psi_j = c\psi_j$$

and

$$(\xi_{q,r+1} \lambda_r(c))\,\bar{\phi}_r = \xi_{qr}(\lambda_r(c)\,\phi_r) = \xi_{qr}(c),$$

where $\bar{\phi}_r\colon \mathbf{m} \to \mathbf{n}$ is specified by $\bar{\phi}_r(k) = k$ for $0 \leqslant k \leqslant m$. The desired equality follows from the construction of $C'\bar{D}_q\mathbf{X}$.

Remarks 2·4. Recall that $C\mathbf{X}$ is defined in terms of maps $\mathscr{C}_r \times_{\Sigma_r} X_r \to C\mathbf{X}$. It is immediate from Definition 1·1 (ii) that these maps pass to colimits to yield a filtration-preserving quotient map $\bar{C}\mathbf{X} \to C\mathbf{X}$. There are also evident quotient maps $\bar{D}_q\mathbf{X} \to D_q\mathbf{X}$, and it is obvious from the definitions that the following natural diagram commutes

$$\begin{array}{ccc} \bar{C}\mathbf{X} & \xrightarrow{\bar{j}_q} & C'\bar{D}_q\mathbf{X} \\ \downarrow & & \downarrow \\ C\mathbf{X} & \xrightarrow{j_q} & C'D_q\mathbf{X} \end{array}$$

It is now a simple matter to mimic the arguments of (4) to prove splitting theorems for the spaces $\bar{C}\mathbf{X}$. As in propositions 3·5, 4·6, and 6·4 of (4), one first constructs commutative diagrams of the following general form.

$$\begin{array}{ccccc} \mathscr{C}_{r-1} \times_{\Sigma_{r-1}} X_{r-1} & \xrightarrow{\zeta_{r-1}} & \mathscr{C}_r \times_{\Sigma_r} X_r & \xrightarrow{\pi} & \bar{D}_r\mathbf{X} \\ \Big\downarrow \bar{k}_{r-1} & & \Big\downarrow \bar{k}_r & & \Big\downarrow \bar{g}_r \\ C'\left(\bigvee_{q=1}^{r-1} \bar{D}_q\mathbf{X}\right) & \xrightarrow{C'\iota} & C'\left(\bigvee_{q=1}^{r} \bar{D}_q\mathbf{X}\right) & \xrightarrow{C'\pi} & C'\bar{D}_r\mathbf{X} \end{array}$$

Here ι is the inclusion, the π are quotient maps, the $\bar{k}_r$ are obtained by adding up the James maps $\bar{j}_q$ by use of H-space structures on $C'X$ for spaces X, and $\bar{g}_r$ is a map homotopic to the standard inclusion η. Precisely as in (4), one then uses adjunctions based on special properties of C' to pass to diagrams featuring the desired splitting maps $\tilde{k}_r$. In this way, one obtains the following four theorems, which are respective analogs of Theorems 3·7, 4·10, 7·1, and 8·2 of (4). Moreover, the previous remarks yield compatibility diagrams which show that the old splittings of $C\mathbf{X}$ are quotients of the new splittings of $\bar{C}\mathbf{X}$.

For our first theorem, we take $\mathscr{C} = \mathscr{C}' = \mathscr{M}$ (directed as in Example 1·2).

THEOREM 2·4. *For all* Π*-spaces* $\mathbf{X}$, *there are equivalences*

$$\tilde{k}_r\colon \Sigma X_r \to \bigvee_{q=1}^{r} \Sigma D_q(\mathscr{M}, \mathbf{X}) \quad \text{and} \quad \tilde{k}_\infty\colon \Sigma\bar{M}\mathbf{X} \to \bigvee_{q\geq 1} \Sigma D_q(\mathscr{M}, \mathbf{X}),$$

where $D_q(\mathscr{M}, \mathbf{X}) = X_q/\phi_{q-1}X_{q-1}$. *Moreover,* $\tilde{k}_r$ *is the sum over* q *of restrictions of James–Hopf maps*

$$h_q\colon \Sigma\bar{M}\mathbf{X} \to \Sigma D_q(\mathscr{M}, \mathbf{X}).$$

For spaces X, such an equivalence between

$$\Sigma X^r \quad \text{and} \quad \bigvee_{q=1}^{r} \Sigma(X^q/X^{q-1})$$

can also be obtained by direct cofibration sequence arguments. Note that $\bar{M}X$ is just the weak infinite product (all but finitely many coordinates of each point at the basepoint) of countably many copies of X.

For our second theorem, we take $\mathscr{C}' = \mathscr{N}$.

THEOREM 2·5. *For all directed coefficient systems* $\mathscr{C}$, Π*-spaces* $\mathbf{X}$, *and Abelian groups* G, *there are isomorphisms*

$$\tilde{H}_*(\mathscr{C}_r \times_{\Sigma_r} X_r; G) \cong \sum_{q=1}^{r} \tilde{H}_*(D_q\mathbf{X}; G) \quad \text{and} \quad \tilde{H}_*(\bar{C}\mathbf{X}; G) \cong \sum_{q\geq 1} \tilde{H}_*(D_q\mathbf{X}; G).$$

These isomorphisms are natural in $\mathscr{C}$, $\mathbf{X}$, *and* G *and commute with Bockstein homomorphisms.*

Consider $\mathscr{C} = \mathscr{N}$ (directed as in Example 1·2). For spaces X, $D_q(\mathscr{N}, X)$ is the quotient $(X^q/\Sigma_q)/(X^{q-1}/\Sigma_{q-1})$ of unreduced symmetric products. Thus the theorem implies Steenrod's isomorphisms (8)

$$\tilde{H}_*(X^r/\Sigma_r; G) \cong \sum_{q=1}^{r} \tilde{H}_*(X^q/\Sigma_q, X^{q-1}/\Sigma_{q-1}; G).$$

The analog for the reduced symmetric products $F_r NX$ was a consequence of (4), 4·10.

For our third theorem, we take $\mathscr{C}' = \mathscr{C}(R^t)$.

THEOREM 2·6. *Assume that* $\mathscr{C}$ *is separated and directed and that* $\mathscr{B}_q = \mathscr{C}_q/\Sigma_q$ *embeds in* R^t *for all* $q \leq r$. *For all* Π*-spaces* $\mathbf{X}$, *there is an equivalence*

$$\tilde{k}_r\colon \Sigma^t(\mathscr{C}_r \times_{\Sigma_r} X_r) \to \bigvee_{q=1}^{r} \Sigma^t \bar{D}_q\mathbf{X}.$$

Moreover, $\tilde{k}_r$ *is the sum over* q *of restrictions of James–Hopf maps*

$$h_q\colon \Sigma^t\bar{C}\mathbf{X} \to \Sigma^t\bar{D}_q\mathbf{X}.$$

For example, this applies to $\mathscr{C}(R^n)$ with t taken to be the embedding dimension of the braid space $B(R^n, r)$; compare (4), 5·6–5·11.

For our last and main theorem, we use the methods specified at the beginning of (4), § 8, with $Q_\infty X$ being the suspension spectrum associated to a based space X.

THEOREM 2·7. *For all* Σ*-free directed coefficient systems* $\mathscr{C}$ *and all* Π*-spaces* $\mathbf{X}$, *there are isomorphisms in the stable category*

$$\tilde{k}_r\colon Q_\infty\mathscr{C}_r \times_{\Sigma_r} X_r \to \bigvee_{q=1}^{r} Q_\infty\bar{D}_q\mathbf{X} \quad \text{and} \quad \tilde{k}_\infty\colon Q_\infty\bar{C}\mathbf{X} \to \bigvee_{q\geq 1} Q_\infty\bar{D}_q\mathbf{X}.$$

Moreover, $\tilde{k}_r$ *is the sum over* q *of restrictions of stable James–Hopf maps*

$$h_q^s\colon Q_\infty\bar{C}\mathbf{X} \to Q_\infty\bar{D}_q\mathbf{X}.$$

The h_q^s *and* $\tilde{k}_r$ *are natural with respect to maps of directed coefficient systems and maps of* Π*-spaces.*

Remarks 2·8. The uniqueness results for the h_q^s discussed in (4), 8·3(i), also apply here. As in (4), 8·3(ii), if $\mathscr{C}$ is separated and $\mathscr{B}_q$ embeds in R^t, then h_q^s is the stabilization of the unstable James–Hopf map $h_q\colon \Sigma^t\bar{C}\mathbf{X} \to \Sigma^t\bar{D}_q\mathbf{X}$.

3. *Special cases of the splitting theorem.* We must still show how the results promised in the introduction drop out of the general theory. For this, we write $\bar{C}\mathbf{X} = \bar{C}(Y, \mathbf{X})$ and $\bar{D}_q\mathbf{X} = \bar{D}_q(Y, \mathbf{X})$ when $\mathscr{C} = \mathscr{C}(Y)$ for a suitable space Y (directed as in Example 1·4), and we consider the case $Y = R^n$. Thus, for a based space X, $\bar{C}(R^n, X)$ is the colimit of the spaces $F(R^n, r) \times_{\Sigma_r} X^r$ and $\bar{D}_q(R^n, X)$ is equivalent to the cofibre of the map

$$\zeta_{q-1}: F(R^n, q-1) \times_{\Sigma_{q-1}} X^{q-1} \to F(R^n, q) \times_{\Sigma_q} X^q.$$

If G is a topological monoid, then $\Sigma_\infty \int G$ is the colimit of the monoids $\Sigma_q \int G$ (as specified in (3), p. 51). Since $F(R^\infty, q)$ is a contractible space with a free Σ_q-action,

$$F(R^\infty, q) \times_{\Sigma_q} (BG)^q \simeq B(\Sigma_q \textstyle\int G) \quad \text{and} \quad \bar{C}(R^\infty, BG) \simeq B(\Sigma_\infty \textstyle\int G).$$

Thus the corollary of the introduction is an immediate consequence of Theorem 2·7.

Finally, for the promised application to loop spaces, we exploit a space $\bar{C}_n(X^+)$ analogous to (but not of precisely the same form as) our present spaces $\bar{C}X$, where X^+ is the union of X and a disjoint basepoint.

THEOREM 3·1. *For $n \geqslant 2$ or $n = \infty$ and for all connected based spaces X, there is a space $\bar{C}_n(X^+)$ and there are maps*

$$\bar{C}(R^n, X) \xleftarrow{\bar{g}} \bar{C}_n(X^+) \xrightarrow{\bar{\alpha}_n} \Omega_0^n \Sigma^n(X^+)$$

such that $\bar{g}$ is a weak equivalence and $\bar{\alpha}_n$ induces an isomorphism on integral homology. Therefore $\bar{g}$ and $\bar{\alpha}_n$ are stable equivalences.

Proof. $\bar{C}_n(X^+)$ is constructed in (3), p. 56, as the telescope of the spaces $\mathscr{C}_{n,r} \times_{\Sigma_r} X^r$ under the 'right translations'

$$\rho(1): \mathscr{C}_{n,q-1} \times_{\Sigma_{q-1}} X^{q-1} \to \mathscr{C}_{n,q} \times_{\Sigma_q} X^q$$

specified by $\rho(1)\ (c, x) = (\gamma(c_2; c, 1),\ (x, 1))$ for some fixed $c_2 \in \mathscr{C}_{n,2}$. The homology isomorphism $\bar{\alpha}_n$ is given by (3), I·5·10, when $n = \infty$ and by (3), I·5·11, when $2 \leqslant n < \infty$. For the latter, the case $n = 1$ would be awkward due to noncommutativity and the present restriction to spaces of the form X^+ for connected X is essential since if more components were present the Browder operations of (3), III·1·2, would mix components non-trivially. Since $\mathscr{C}_n$ may be viewed as acting on $\mathscr{C}(R^n)$, by (4), 6·2, we may define translations $\rho(1)$ as above with $\mathscr{C}_{n,q}$ replaced by $F(R^n, q)$. The map $g: \mathscr{C}_n \to \mathscr{C}(R^n)$ of (4), 1·7, clearly induces an equivalence

$$\operatorname*{Tel}_{\rho(1)} \mathscr{C}_{n,r} \times_{\Sigma_r} X^r \to \operatorname*{Tel}_{\rho(1)} F(R^n, r) \times_{\Sigma_r} X^r,$$

and the natural map

$$\operatorname*{Tel}_{\zeta_r} F(R^n, r) \times_{\Sigma_r} X^r \to \operatorname*{Colim}_{\zeta_r} F(R^n, r) \times_{\Sigma_r} X^r$$

is a weak equivalence by a standard compactness of spheres argument. In view of Definitions 2·1, we need only show that the maps λ_r and γ_r from $F(R^n, r)$ to $F(R^n, r+1)$ are Σ_r-equivariantly homotopic, where $\gamma_r(c)$ is given by the action of c_2 on $(c, 0)$. Here if $c_2 = \langle b, b' \rangle$, then b acts on the points of $c \in F(R^n, r)$ and b' acts on $0 \in R^n = F(R^n, 1)$.

By use of a homeomorphism $R \cong (0,1)$, we find easily that γ_r may be taken as the map which sends

$$\langle (t_1, z_1), \ldots, (t_r, z_r)\rangle \quad \text{to} \quad \langle (b(t_1), z_1), \ldots, (b(t_r), z_r), (1, 0)\rangle,$$

where $t_i \in R$, $z_i \in R^{n-1}$, $0 \in R^{n-1}$ is taken as its basepoint, and $b\colon R \to (-\infty, 0)$ is an increasing homeomorphism which, as a map into R, is homotopic to the identity through increasing maps. Since λ_r sends

$$\langle (t_1, z_1), \ldots, (t_r, z_r)\rangle \quad \text{to} \quad \langle (t_1, z_1), \ldots, (t_r, z_r), (1+\max t_i, 0)\rangle,$$

it is a simple exercise to construct the required homotopy.

Making use of (4), 1·9(iii) and 2·5, to handle parameter spaces, we deduce the following analog of (4), 8·4.

COROLLARY 3·2. *For all based spaces P and connected based spaces X and all $n \geqslant 2$ (including $n = \infty$, when $\Omega_0^\infty \Sigma^\infty(X^+)$ is to be interpreted as $Q_0 X^+$), there are isomorphisms in the stable category*

$$Q_\infty(P \wedge \Omega_0^n \Sigma^n(X^+)) \cong Q_\infty(P \wedge \bar{C}(R^n, X)) \to \bigvee_{q \geqslant 1} Q_\infty(P \wedge \bar{D}_q(R^n, X)).$$

Moreover, these isomorphisms are compatible as n varies.

Of course, $S^0 = \{1\}^+$ and the corollary specializes to give a splitting of $\Omega_0^n S^n$ for all $n \geqslant 2$. If $\eta\colon S^3 \to S^2$ is the Hopf map, then $\Omega^2\eta\colon \Omega^2 S^3 \to \Omega_0^2 S^2$ is clearly a homotopy equivalence. The following result compares the stable splitting $\Omega^2 S^3 \simeq \bigvee_q D_q(R^2, S^1)$ of (4) with the stable splitting

$$\Omega_0^2 S^2 \simeq \bigvee_q \bar{D}_q(R^2, \{1\})$$

obtained here.

PROPOSITION 3·3. *Stably, $\bar{D}_{2q+1}(R^2, \{1\})$ is trivial and the composite*

$$D_q(R^2, S^1) \to \Omega^2 S^3 \xrightarrow{\Omega^2\eta} \Omega_0^2 S^2 \to \bar{D}_{2q}(R^2, \{1\})$$

is an equivalence for all q.

Proof. Since we are working stably, it suffices to prove that $\tilde{H}_* \bar{D}_{2q+1}(R^2, \{1\}) = 0$ and that the displayed composite induces an isomorphism on mod p homology for all primes p.

(i) $p = 2$. $H_* \Omega^2 S^3 = P\{x_n | n \geqslant 0\}$, where x_0 is the fundamental class of $H_* S^1$ and $x_n = Q^{2^n} x_{n-1}$ for $n \geqslant 1$. $H_* \Omega_0^2 S^2 = P\{x'_n | n \geqslant 0\}$, where $x'_0 = Q^1[1] * [-2]$ and

$$x'_n = Q^{2^n} x'_{n-1} \quad \text{for} \quad n \geqslant 1.$$

See (3). As a second loop map, $\Omega^2\eta$ preserves operations. It therefore sends x_n to x'_n. Since x_n has filtration 2^n in $H_* C_2 S^1$ while x'_n has filtration 2^{n+1} in $H_* \bar{C}_2 S^0$, the desired conclusions hold at the prime 2.

(ii) $p > 2$.

$$H_* \Omega^2 S^3 = E\{x_n | n \geqslant 0\} \otimes P\{\beta x_n | n \geqslant 1\},$$

where x_0 is the fundamental class of $H_* S^1$ and $x_n = Q^{p^{n-1}} x_{n-1}$ for $n \geqslant 1$.

$$H_* \Omega_0^2 S^2 = E\{x'_n | n \geqslant 0\} \otimes P\{\beta x'_n | n \geqslant 1\},$$

where $x'_0 = \lambda_1([1], [1]) * [-2]$ and $x'_n = Q^{p^{n-1}} x'_{n-1}$. See (3). Up to non-zero constant, $\Omega^2 \eta$ sends x_n to x'_n. Again, x_n has filtration p^n in $H_* C_2 S^1$ while x'_n has filtration $2p^n$ in $H_* \bar{C}_2 S^0$.

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