Surgery Spaces: Formulae and Structure

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In the Fall of 1977 it became clear to us that, in our work on local surgery, we could get better theorems if we had formulae for computing surgery obstructions of problems over closed manifolds. Wall's paper [9] had come out but there was rumored to be an error and the corrigendum [10] had not yet appeared.

It seemed philosophically clear that such formulae must be contained in Ranicki's work [5] and we set out to find them from this point of view. We were of course helped by the fact that Morgan and Sullivan had worked out the answer in the important special case of the trivial group [2], and by the belief that Wall [9] could not be too far off. (He wasn't.)

The formulae herein will contain no surprises for the experts but we hope that having them explicitly written out in the literature may prove useful to others. More surprising perhaps is that we completely determine the homotopy type of Ranicki's spaces and spectra modulo our ignorance of their homotopy groups. There are geometric problems whose solution involves Ranicki's spaces, not just their homotopy groups, so the above analysis should be useful.

As an example of this last statement, Quinn [4] has shown that the obstruction to deforming a map between manifolds to a block-bundle projection has a piece involving his spaces, which are

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homotopy equivalent to Ranicki's.

The guiding principle behind this work is that all the deep mathematics should be pushed onto others: notably Ranicki [5], Morgan and Sullivan [2], and Brumfiel and Morgan [1]. What remains is, we hope, a pleasant, if energetic romp through the stable category which is nevertheless not devoid of interest.

We conclude this section with our thanks to both John Morgan and Andrew Ranicki for conversations, correspondence, and preprints.

§1. Statement of results.

A summary of the relevant part of Ranicki's work is our first order of business. In §15 of [5], Ranicki constructs semi-simplical monoids $\mathbb{L}^m(A,\varepsilon)$, $\mathbb{L}_m(A,\varepsilon)$, and $\hat{\mathbb{L}}^m(A,\varepsilon)$ and related spectra $\mathbb{L}^\circ(A,\varepsilon)$, $\mathbb{L}_\circ(A,\varepsilon)$ and $\hat{\mathbb{L}}(A,\varepsilon)$. The k-simplices of the monoids are just

$$(m+k)\text{-dimensional} \begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \varepsilon\text{-hyperquadratic} \end{cases} \text{Poincaré } (k+2)\text{-ads over } A \text{ with}$$

some additional restrictions.

To simplify the notation, let $\mathbb{L}^\circ$ denote $\mathbb{L}^\circ(Z,1)$; $\mathbb{L}_\circ$ denote $\mathbb{L}_\circ(Z,1)$; and $\hat{\mathbb{L}}$ denote $\hat{\mathbb{L}}(Z,1)$. Further let $\mathbb{L}^\circ(\pi)$ denote $\mathbb{L}^\circ(Z\pi,1)$; $\mathbb{L}_\circ(\pi)$ denote $\mathbb{L}_\circ(Z\pi,1)$; etc.

Tensor product of chain complexes induces numerous pairings. The spectrum $\mathbb{L}^\circ$ becomes a commutative ring spectrum, and every other spectrum $\mathbb{L}^\circ(A,\varepsilon)$, $\mathbb{L}_\circ(A,\varepsilon)$, or $\hat{\mathbb{L}}(A,\varepsilon)$ becomes a module spectrum over $\mathbb{L}^\circ$. The spectrum $\hat{\mathbb{L}}$ is a commutative ring spectrum and $\hat{\mathbb{L}}(A,\varepsilon)$ is a module spectrum over it.

There are also maps between these spectra. There is a symmeterization map $(1+T)\colon \mathbb{L}_\circ(A,\varepsilon) \to \mathbb{L}^\circ(A,\varepsilon)$ whose cofibre is $\hat{\mathbb{L}}(A,\varepsilon)$. The resulting long cofibration sequence is a sequence of $\mathbb{L}^\circ$ module spectra. The map $\mathbb{L}^\circ \to \hat{\mathbb{L}}$ is even a map of ring spectra. There is also a map of $\mathbb{L}^\circ$ module spectra $e_8\colon \mathbb{L}^\circ(A,\varepsilon) \to \mathbb{L}_\circ(A,\varepsilon)$ which is given by tensor product with the even form of index 8 often denoted E_8.

Ranicki also defines two geometric maps:

$$\sigma^*\colon \mathrm{MSTOP} \to \mathbb{L}^\circ \quad \text{and} \quad A_\pi\colon K(\pi,1)^+ \to \mathbb{L}^\circ(\pi).$$

The map σ^* is a map of ring spectra and the map A_π gives the assembly maps:

i) $\mathrm{MSTOP} \wedge (K(\pi,1)^+) \xrightarrow{\sigma^* \wedge A_\pi} \mathbb{L}^\circ \wedge \mathbb{L}^\circ(\pi) \to \mathbb{L}^\circ(\pi)$

induces the symmetric signature map, while

ii) $\mathrm{MSTOP} \wedge \mathbb{L}_\circ \wedge (K(\pi,1)^+) \xrightarrow{\sigma^* \wedge 1 \wedge A_\pi} \mathbb{L}^\circ \wedge \mathbb{L}_\circ \wedge \mathbb{L}^\circ(\pi) \to \mathbb{L}_\circ(\pi)$

induces the surgery obstruction.

In particular, $\pi_i(\mathbb{L}_\circ(\pi))$ is the i^{th} Wall surgery group for oriented problems with fundamental group π (for $i \geq 0$); $\mathbb{L}_\circ$ is essentially $Z \times G/TOP$; and the above map is the old Sullivan-Wall map $\Omega_*(G/TOP \times K(\pi,1)) \to L_*(\pi)$ ([8] p.176, 13B.3).

With these definitions fixed we can state our first theorem.

<u>Theorem A</u>: The spectra $\mathbb{L}^\circ(A,\varepsilon)$ and $\mathbb{L}_\circ(A,\varepsilon)$ are generalized Eilenberg-MacLane spectra when localized at 2, and, when localized away from 2, are both $bo\Lambda_0 \vee \Sigma^1 bo\Lambda_1 \vee \Sigma^2 bo\Lambda_2 \vee \Sigma^3 bo\Lambda_3$ where $bo\Lambda_i$ denotes connective KO theory with coefficients in the group Λ_i. In

our case, A_i is $\pi_i(\mathbb{L}^\circ(A,\varepsilon))\otimes Z[\frac{1}{2}]$.

The spectrum $\hat{\mathbb{L}}(A,\varepsilon)$ is a generalized Eilenberg-MacLane spectrum.

While Theorem A is nice one should not read too much into it. It is true that the map $A_\pi\colon K(\pi,1)^+ \to \mathbb{L}^\circ(\pi)$ is determined by some cohomology classes and some elements in KO-theory with coefficients, but this is not much use in understanding the assembly maps unless one knows the $\mathbb{L}^\circ$ module structure of $\mathbb{L}^\circ(\pi)$ and $\mathbb{L}_\circ(\pi)$.

Since the assembly maps are basically unkown, we agree to write A_* for any of the following maps:

$$(1.1)\quad \bigoplus_i H_{*-4i}(\pi;Z_{(2)}) \oplus H_{*-(4i+1)}(\pi;Z/2) \to \pi_*(\mathbb{L}^\circ(\pi))_{(2)}$$

$$(1.2)\quad KO_* \quad (K(\pi,1)) \to \pi_*(\mathbb{L}^\circ(\pi))_{(odd)}$$

$$(1.3)\quad \bigoplus_i H_{*-4i}(\pi;Z_{(2)}) \oplus H_{*-(4i+2)}(\pi;Z/2) \to \pi_*(\mathbb{L}_\circ(\pi))_{(2)}$$

$$(1.4)\quad KO_* \quad (K(\pi,1)) \to \pi_*(\mathbb{L}_\circ(\pi))_{(odd)}$$

where 1.1 and 1.2 are induced from A_π and the pairing $\mathbb{L}^\circ \wedge \mathbb{L}^\circ(\pi) \to \mathbb{L}^\circ(\pi)$ and 1.3 and 1.4 are induced from A_π and the pairing $\mathbb{L}_\circ \wedge \mathbb{L}^\circ(\pi) \to \mathbb{L}_\circ(\pi)$.

Given an oriented topological manifold M with $\pi_1(M)=\pi$, we wish to give a formula for the symmetric signature of M, $\sigma^*(M)$. We first fix some notation.

We let $\mathcal{L}\in H^{4i}(BSTOP;Z_{(2)})$ denote the class defined by Morgan and

Sullivan [2] §7. Let $V \in H^{2i}(BSTOP;Z/2)$ denote the total Wu class. Here and below we only list a typical group in which a graded cohomology class like $\mathcal{L}$ or V lies. We have

Theorem B. Let $g\colon M \to K(\pi,1)$ classify the universal cover, and let $\nu\colon M \to BSTOP$ classify the normal bundle. Then, at 2, we have

$$(1.5)\quad \sigma^*(M)_{(2)} = A_* \, g_*(\nu^*(\mathcal{L} + V\, Sq^1 V) \cap [M]) \ .$$

Away from 2 M has a bo-orientation and hence a fundamental class $[M]_K$. We have

$$(1.6)\quad \sigma^*(M)_{(odd)} = A_* \, g_* \, [M]_K .$$

For the surgery obstruction we have the following formulae due to Wall [9].

Theorem C. Let g and ν be as above, and let $f\colon M \to \mathbb{L}_0$ classify some surgery problem. Then

$$(1.7)\quad \sigma_*(f)_{(2)} = A_* \, g_* \, ((\nu^*(\mathcal{L}) \cup f^*(\ell) + \nu^*(\mathcal{L}) \cup f^*(k) + \delta^*(\nu^*(V\, Sq^1 V) \cup f^*(k))) \cap [M]),$$

where δ^* denotes the integral bockstein. Furthermore we have

$$(1.8)\quad \sigma_*(f)_{(odd)} = A_* \, g_* \, (f^*(\Delta) \cap [M]_K)$$

The classes $\ell \in H^{4i}(\mathbb{L}_\circ; Z_{(2)})$ and $k \in H^{4i+2}(\mathbb{L}_\circ; Z/2)$ are defined below and $\Delta \in KO^0(\mathbb{L}_\circ; Z[\frac{1}{2}])$ is the equivalence from Theorem A.

Remark: Theorems B and C follow easily from 1.9 and 1.13 below. To actually carry out the proof of 1.5 and 1.7, one needs to remember that slant product gives the equivalence between $\pi_i(K(G,n) \wedge X^+)$ and $H_{n-i}(X;G)$. The formulae follow from well-known properties of the cap, cup, and slant products. Formula 1.6 is a tautology and 1.8 follows from the naturality of the cap product.

To prove our results we need to analyze $\mathbb{L}^\circ$, $\mathbb{L}_\circ$ and the various maps and pairings between them. In §3 we shall construct cohomology classes $L \in H^{4i}(\mathbb{L}^\circ; Z_{(2)})$; $r \in H^{4i+1}(\mathbb{L}^\circ; Z/2)$; $\ell \in H^{4i}(\mathbb{L}_\circ; Z_{(2)})$; and $k \in H^{4i+2}(\mathbb{L}_\circ; Z/2)$ such that these classes exhibit $\mathbb{L}^\circ$ and $\mathbb{L}_\circ$ as generalized Eilenberg-MacLane spectra at 2.

Moreover, the map σ^*: MSTOP $\to \mathbb{L}^\circ$ is determined at 2 by the formulae

(1.9) $\sigma^*(L) = \mathfrak{L} \cdot U$ and $\sigma^*(r) = (V\ Sq^1 V) \cdot U$

where U is the Thom class.

The map $(1+T)$: $\mathbb{L}_\circ \to \mathbb{L}^\circ$ is determined at 2 by the formulae

(1.10) $(1+T)^*(L) = 8\ell$ and $(1+T)^*(r) = 0$.

The map e_8: $\mathbb{L}^\circ \to \mathbb{L}_\circ$ is determined at 2 by the formulae

(1.11) $e_8^*(\ell) = L$ and $e_8^*(k) = 0$.

The ring map $\rho\colon \mathbb{L}^\circ \wedge \mathbb{L}^\circ \to \mathbb{L}^\circ$ is determined at 2 by the formulae

$$(1.12)\quad \rho^*(L) = L \wedge L \quad\text{and}\quad \rho^*(r) = r \wedge L + L \wedge r.$$

The module map $\mathfrak{m}\colon \mathbb{L}^\circ \wedge \mathbb{L}_\circ \to \mathbb{L}_\circ$ is determined at 2 by

$$(1.13)\quad \mathfrak{m}^*(\ell) = L \wedge \ell + \delta^*(r \wedge k) \quad\text{and}\quad \mathfrak{m}^*(k) = L \wedge k.$$

There remains the spectrum $\hat{\mathbb{L}}$. We have a map $\hat{\mathbb{L}} \to \Sigma\mathbb{L}_\circ$ and so we get cohomology classes $\Sigma\ell \in H^{4i+1}(\hat{\mathbb{L}};Z_{(2)})$ and $\Sigma k \in H^{4i+3}(\hat{\mathbb{L}};Z/2)$. There are classes $\hat{L} \in H^{4i}(\hat{\mathbb{L}};Z/8)$ and $\hat{r} \in H^{4i+1}(\hat{\mathbb{L}};Z/2)$. The classes $\hat{L}$, $\hat{r}$, and Σk exhibit $\hat{\mathbb{L}}$ as a generalized Eilenberg-MacLane spectrum. Moreover we have the following formulae.

The classes $\hat{L}$ and $\Sigma\ell$ are related by β_8, the mod 8 bockstein, via

$$\beta_8{}^*(\hat{L}) = \Sigma\ell\ .$$

The map $\mathbb{L}^\circ \to \hat{\mathbb{L}}$ is determined by the formulae

$$(1.14)\quad \hat{r} \to r\,; \quad \hat{L} \to \text{mod 8 reduction of } L; \text{ and } \quad \Sigma k \to 0.$$

The map $\hat{\mathbb{L}} \to \Sigma\,\mathbb{L}_\circ$ has been discussed.

The pairing $\hat{\rho}\colon \hat{\mathbb{L}} \wedge \hat{\mathbb{L}} \to \hat{\mathbb{L}}$ is determined by the formulae

$$(1.15)\quad \begin{aligned} &\hat{\rho}^*(\hat{L}) = \hat{L} \wedge \hat{L} + i_*(\hat{r} \wedge \Sigma k + \Sigma k \wedge \hat{r})\,; \quad \hat{\rho}^*(\hat{r}) = \hat{r} \wedge \hat{L} + \hat{L} \wedge \hat{r}\,; \\ &\hat{\rho}^*(\Sigma k) = \Sigma k \wedge \hat{L} + \hat{L} \wedge \Sigma k\ ; \text{ and } \hat{\rho}^*(\Sigma\ell) = \Sigma\ell \wedge \hat{L} + \hat{L} \wedge \Sigma\ell \end{aligned}$$

where $i\colon Z/2 \to Z/8$ is monic.

Ranicki's hyperquadratic signature map $MSG \to \hat{\mathbb{L}}$ is a map of ring spectra. We shall take care that our class $\hat{L}$ pulls back to the Brumfiel-Morgan [1] class in $H^{4i}(BSG;Z/8)$ times the Thom class. We do not understand the pull backs of $\hat{r}$, Σk, and $\Sigma\ell$ although there are obvious guesses based on Brumfiel's and Morgan's work.

The specfic formulae above yield some nice results on the surgery assembly map 1.3. It is clear that $\bigoplus_i H_{*-(4i+2)}(\pi;Z/2) \to \pi_*(\mathbb{L}_\circ(\pi))_{(2)}$ factors through the natural map $\pi_{*+1}(\hat{\mathbb{L}}(\pi)) \to \pi_*(\mathbb{L}_\circ(\pi))$ and that there are commutative diagrams

$$(1.16)\qquad \begin{array}{ccc} \oplus H_{*+1-4i}(\pi;Z/8) & \to & \pi_{*+1}(\hat{\mathbb{L}}(\pi)) \\ \Big\downarrow (\beta_8)_* & & \Big\downarrow \\ \oplus H_{*-4i}(\pi;Z_{(2)}) & \to & \pi_*(\mathbb{L}_\circ(\pi))_{(2)} \end{array}$$

and

$$(1.17)\qquad \begin{array}{ccc} \oplus H_{*-4i}(\pi;Z_{(2)}) & \to & \pi_*(\mathbb{L}^\circ(\pi))_{(2)} \\ \Big\downarrow \cong & & \Big\downarrow (e_8)_* \\ \oplus H_{*-4i}(\pi;Z_{(2)}) & \to & \pi_*(\mathbb{L}_\circ(\pi))_{(2)} \end{array}$$

The first diagram follows from 1.15; the second from 1.11.

If π is a finite group whose 2 Sylow group is abelian or generalized quaternion, Stein [6] shows that the map $\oplus \tilde{H}_{*-4i}(\pi;Z_{(2)}) \to \pi_*(\mathbb{L}^\circ(\pi))_{(2)} \to \pi_*(\mathbb{L}_\circ(\pi))_{(2)}$ is trivial. Hence

Corollary (Stein[6]): If π is a finite group whose 2 Sylow group is abelian or generalized quaternion, then, for the surgery assembly map, we have

$$\oplus \tilde{H}_{*-4i}(\pi;Z_{(2)}) \to \pi_*(\mathbb{L}_\circ(\pi))_{(2)} \quad \text{is trivial.}$$

In particular, a surgery problem over a closed manifold M with $\pi_1(M) = \pi$ as in the corollary is solvable if its index obstruction is zero and if the associated Kervaire classes k_{4i+2} are zero. This is related to some results of Morgan and Pardon [3].

We conclude this section with a remark on notation. Henceforth we will suppress Thom classes in our formulae and write 1.9 for example as $\sigma^*(L) = \mathcal{L}$, etc.

§2. MSO-module spectra.

An MSO-module spectrum is a spectrum, E, equipped with a map $\mu\colon MSO \wedge E \to E$ such that

(2.1) the composite $S^0 \wedge E \xrightarrow{u \wedge 1} MSO \wedge E \xrightarrow{\mu} E$

is the identity, where $u\colon S^0 \to MSO$ is the unit, and

(2.2) the diagram
$$\begin{tikzcd} MSO \wedge MSO \wedge E \arrow[r, "1\wedge\mu"] \arrow[d, "m\wedge 1"] & MSO \wedge E \arrow[d, "\mu"] \\ MSO \wedge E \arrow[r, "\mu"] & MSO \end{tikzcd}$$

commutes, where $m\colon MSO \wedge MSO \to MSO$ is the multiplication.

Such spectra are common. All of the Ranicki spectra discused in §1 and bordism theories like MSTOP are examples. Our first result is a proof of

Theorem $A_{(2)}$: Any module spectrum over MSO becomes a generalized Eilenberg-MacLane spectrum after localizing at 2.

Proof: Let E denote the spectrum in question, and let $K(\pi_* E_{(2)})$ be a product of Eilenberg-MacLane spectra such that

$\pi_i(K(\pi_* E_{(2)})) = \pi_i(E_{(2)})$. A map $\varphi\colon E \to K(\pi_* E_{(2)})$ determines a homomorphism $\hat{\varphi}_i\colon H_i(E;Z) \to \pi_i(E_{(2)})$ for each i, and conversely, any collection of such homomorphisms can be realized by some φ (probably several). There are projection maps $\rho_i\colon K(\pi_* E_{(2)}) \to K(\pi_i E_{(2)}, i)$ and the induced map $H_i(E;Z) \xrightarrow{\varphi_*} H_i(K(\pi_* E_{(2)});Z) \xrightarrow{(\rho_i)_*} H_i(K(\pi_i E_{(2)}, i);Z)$

is just $\hat{\varphi}_i$ followed by the Hurewicz map.

Now consider $\mu\colon MSO \wedge E \to E$. We can localize to get $\mu_{(2)}\colon MSO_{(2)} \wedge E \to E_{(2)}$ such that $S^0 \wedge E \to MSO_{(2)} \wedge E \to E_{(2)}$ is the localization map. But $u\colon S^0 \to MSO_{(2)}$ factors as $S^0 \to K(Z,0) \to MSO_{(2)}$ (since $MSO_{(2)}$ is a product of Eilenberg-MacLane spectra) so we get a map $\hat{\varphi}_i\colon H_i(E;Z) \to \pi_i E_{(2)}$ such that the composite $\pi_i(E) \xrightarrow{\text{Hurewicz}} H_i(E;Z) \xrightarrow{\hat{\varphi}_i} \pi_i E_{(2)}$ is localization. It is easy to see that any associated $\varphi\colon E \to K(\pi_* E_{(2)})$ is a 2-local equivalence. //

The next step in our understanding of the MSO-module spectrum E is to actually describe cohomology classes which give the equivalence between E and a generalized Eilenberg-MacLane spectrum. Morgan and Sullivan [2] have given a good way to describe 2-local cohomology classes of E: one gives homomorphisms with certain properties out of the bordism of E. Since the bordism of E is just the homotopy of $MSO \wedge E$, one way is to give homomorphisms out of the homotopy groups of E and then use μ. This procedure gives the homomorphisms studied by Morgan and Sullivan and we wish to study it in general.

A certain amount of finiteness seems necessary, so we say that E has finite type if $\bigoplus_{-\infty<i<r} \pi_i(E)$ is a finitely generated group for each $r \in Z$.

In what follows, R denotes either $Z_{(2)}$ or $Z/2^j$. Our basic data is a collection of homomorphisms $\Psi_R\colon \bigoplus_{-\infty<i<\infty} \pi_i(E;R) \to R$ such that

$$(2.3)\qquad \begin{array}{ccc} \oplus\, \pi_*(E;Z_{(2)}) & \to & Z_{(2)} \\ \downarrow & & \downarrow \\ \oplus\, \pi_*(E;Z/2^j) & \to & Z/2^j \end{array} \quad \text{and} \quad \begin{array}{ccc} \oplus\, \pi_*(E;Z/2^j) & \to & Z/2^j \\ \downarrow & & \downarrow \\ \oplus\, \pi_*(E;/2^{j+1}) & \to & Z/2^{j+1} \end{array}$$

commute.

The composites $\Psi_*\colon \oplus\, \pi_*(MSO \wedge E;R) \xrightarrow{\mu_*} \oplus\, \pi_*(E;R) \xrightarrow{\Psi_R} R$ are compatible homomorphisms in the Morgan-Sullivan sense. The requirement that they be multiplicative with respect to the index may be phrased as follows. The composite

$$\pi_p(MSO;R) \otimes \pi_q(MSO \wedge E;R) \to \pi_{p+q}(MSO \wedge MSO \wedge E;R) \xrightarrow{(m\wedge 1)_*} \pi_{p+q}(MSO \wedge E;R) \to R$$

is also the composite $\pi_p(MSO;R) \otimes \pi_q(MSO \wedge E;R) \xrightarrow{(\mathrm{Index})\otimes\Psi_*} R \otimes R \to R$, where Index : $\pi_p(MSO;R) \to R$ is just the mod R index homomorphism ([2] p. 473). The particular form of our Ψ_* permits us to get away with an apparently weaker statement. We have

<u>Proposition 1</u>: Suppose given an MSO-module spectrum E of finite type and a collection of homomorphisms $\Psi_R\colon \pi_*(E;R) \to R$ satisfying 2.3. Then, if

$$(2.4)\qquad \begin{array}{ccc} \pi_p(MSO;R) \otimes \pi_q(E;R) & \xrightarrow{(\mathrm{Index})\otimes\Psi_R} & R\otimes R \\ \downarrow & & \downarrow \\ \pi_{p+q}(MSO \wedge E;R) \xrightarrow{\mu_*} \pi_{p+q}(E;R) & \xrightarrow{\Psi_R} & R \end{array}$$

commutes for all $p, q \in Z$, we have a unique cohomology class $\gamma \in H^*(E; Z_{(2)})$ such that, for any $c \in \pi_*(MSO \wedge E; R)$, the equation

$$(2.5) \qquad \Psi_R\, \mu_*(c) = \langle \mathcal{L} \wedge \gamma, c \rangle$$

holds.

The omitted proof follows easily from the work of Morgan and Sullivan [2] §4 and 2.2.

There is also a mod 2 version of Proposition 1. Given one homomorphism $\Psi: \pi_*(E; Z/2) \to Z/2$ such that 2.4 commutes, we get a unique class $\gamma \in H^*(E; Z/2)$ such that 2.5 holds. Sullivan proves this in [7]. There is even a $Z/2^r$ version in Brumfiel-Morgan [1]. We state it so as to require that the Ψ_R satisfy 2.3 and 2.4 for all $Z/2^j$ with $j \leqq r$.

Addendum to Proposition 1: Suppose $\Psi_R: \pi_i(E; R) \to R$ is zero for all R under consideration. Then diagram 2.4 shows that $\Psi_R: \pi_{i+4k}(E; R) \to R$ is also zero for all negative integers k. The proof that γ is unique shows more. It shows that the components of γ in dimensions $i+4k$ must be zero for all non-postive integers k.

The classes that Morgan and Sullivan built by this method satisfy $\mu^*(\gamma) = \mathcal{L} \wedge \gamma$. Our next goal is to give a general explanation for this formula. Suppose that we have three MSO-module spectra, E_i, μ_i for $i=1,2,3$, of finite type, and three sets of homomorphisms $\Psi_i: \pi_*(E_i; R) \to R$ $i=1,2,3$ satisfying 2.3 and 2.4. We get three classes $\gamma_i \in H^*(E_i; Z_{(2)})$ (or in $H^*(E_i; Z/2^r)$).

With the above notation fixed, let us further suppose that we have a pairing $\nu\colon E_1 \wedge E_2 \to E_3$ such that

$$(2.6)\qquad \begin{tikzcd} MSO \wedge E_1 \wedge MSO \wedge E_2 \arrow[r, "1\wedge T\wedge 1"] \arrow[d, "\mu_1\wedge\mu_2"] & MSO \wedge MSO \wedge E_1 \wedge E_2 \arrow[r, "m\wedge\nu"] & MSO \wedge E_3 \arrow[d, "\mu_3"] \\ E_1 \wedge E_2 \arrow[rr, "\nu"] & & E_3 \end{tikzcd}$$

commutes. Then we have

<u>Theorem 1</u>: The diagram

$$\begin{tikzcd} \pi_p(E_1;R) \otimes \pi_q(E_2;R) \arrow[r] \arrow[d, "\Psi_1\otimes\Psi_2"] & \pi_{p+q}(E_1 \wedge E_2;R) \arrow[r, "\nu_*"] & \pi_{p+q}(E_3;R) \arrow[d, "\Psi_3"] \\ R \otimes R \arrow[rr] & & R \end{tikzcd}$$

commutes for all $p,q \in Z$ and all R under consideration

$$\text{iff} \qquad \nu^*(\gamma_3) = \gamma_1 \wedge \gamma_2 \quad .$$

Remark: By applying the theorem to the map $\mu_i\colon MSO \wedge E_i \to E_i$ one easily sees that $\mu_i^*(\gamma_i) = \mathcal{L} \wedge \gamma_i$.

This remark can be amplified slightly. Let $x \in H^*(E;Z_{(2)})$ (or $H^*(E;Z/2^r)$) be any cohomology class. Then $\langle x, \ \rangle\colon \pi_*(E;R) \to R$ is a set of homomorphisms satisfying 2.3. If $\mu^*(x) = \mathcal{L} \wedge x$, then they satisfy 2.4 as well, and the cohomology class determined by Proposition 1 is just x again. We have

<u>Corollary 1</u>: Given two classes $x_1, x_2 \in H^*(E;Z_{(2)})$ such that $\mu^*(x_i) = \mathcal{L} \wedge x_i$ $i=1,2$, then $x_1 = x_2$ iff $\langle x_1, \ \rangle = \langle x_2, \ \rangle\colon \pi_*(E;R) \to R$.

Proof of Theorem 1: See that the diagram commutes if $\nu^*(\gamma_3) = \gamma_1 \wedge \gamma_2$ by using 2.1 to map $E_1 \wedge E_2$ to $MSO \wedge E_1 \wedge MSO \wedge E_2$ and chasing the resulting diagrams. We concentrate on the converse.

We begin by rephrasing lemma 7.1 ([2] p. 533) for spectra of finite type as

Lemma 1: Two sets of homomorphisms

$$\psi_i\colon \pi_*(\,MSO \wedge E_1 \wedge E_2;R) \rightarrow R \qquad i=1,2$$

satisfying 2.3 are equal if the composites

$$\pi_p(MSO \wedge E_1;R) \otimes \pi_q(MSO \wedge E_2;R) \rightarrow \pi_{p+q}(MSO \wedge E_1 \wedge MSO \wedge E_2;R) \xrightarrow{(m\wedge 1\wedge 1)_*\,(1\wedge T\wedge 1)_*} \pi_{p+q}(MSO \wedge E_1 \wedge E_2;R) \xrightarrow{\psi_i} R$$ for $i=1,2$ are equal

for all $p,q \in Z$ and all R under consideration.

Using this result we proceed to use the uniqueness part of the Morgan-Sullivan description of cohomology classes. We must show that $\langle \mathcal{L}\wedge\nu^*(\gamma_3),c\rangle = \langle \mathcal{L}\wedge\gamma_1\wedge\gamma_2,c\rangle$ for all $c \in \pi_*(MSO \wedge E_1 \wedge E_2;R)$ which, by the lemma above, are of the form $c = (m\wedge 1\wedge 1)_*\,(1\wedge T\wedge 1)_*\,(c_1\wedge c_2)$.

So

$$\langle \mathcal{L}\wedge\nu^*(\gamma_3),c\rangle = \langle \mathcal{L}\wedge\gamma_3,(1\wedge\nu)_*(c)\rangle = \Psi_3\,(\mu_3)_*\,(1\wedge\nu)_*\,(c) \quad \text{by 2.5.}$$

Now $(\mu_3)_*\,(1\wedge\nu)_*\,(c) = \nu_*\,(\mu_1\wedge\mu_2)_*\,(c_1\wedge c_2)$ as a diagram chase using 2.6 shows. But Ψ_3 of this is $\Psi_1(c_1)\cdot\Psi_2(c_2)$ since our diagram commutes. We finish by showing $\langle \mathcal{L}\wedge\gamma_1\wedge\gamma_2,c\rangle = \Psi_1(c_1)\cdot\Psi_2(c_2)$ also.

$$\begin{aligned}
\langle \mathcal{L}\wedge\gamma_1\wedge\gamma_2,c\rangle &= \langle \mathcal{L}\wedge\gamma_1\wedge\gamma_2,(m\wedge 1\wedge 1)_*\,(1\wedge T\wedge 1)_*\,(c_1\wedge c_2)\rangle \\
&= \langle \mathcal{L}\wedge\gamma_1\wedge\mathcal{L}\wedge\gamma_2,c_1\wedge c_2\rangle \quad \text{since } m^*\mathcal{L} = \mathcal{L}\wedge\mathcal{L} \\
&= \langle \mathcal{L}\wedge\gamma_1,c_1\rangle\,\langle \mathcal{L}\wedge\gamma_2,c_2\rangle = \Psi_1(c_1)\cdot\Psi_2(c_2) \quad \text{by 2.5. //}
\end{aligned}$$

When it works, the above discussion is quite satisfactory. There are however natural homomorphisms $\Psi_R\colon \pi_*(E;R) \to R$ such as the deRham invariant or the surgery obstruction for which diagram 2.4 fails to commute. A more general treatment seems necessary.

We fix some class $\beta \in H^*(MSO \wedge E; Z_{(2)})$ (or $H^*(MSO \wedge E; Z/2^r)$) such that the homomorphisms $\Psi_R\, \mu_* + \langle \beta\, ,\ \rangle\colon \pi_*(MSO \wedge E;R) \to R$ are compatible and multiplicative with respect to the index. With this data fixed, and for E of finite type, we have

<u>Proposition 2</u>: There exists a unique cohomology class $\ell \in H^*(E;Z_{(2)})$ (or $H^*(E;Z/2^r)$) such that, for any $c \in \pi_*(MSO \wedge E;R)$

$$(2.7) \quad \langle \mathcal{L} \wedge \ell, c\rangle = \Psi_R\, \mu_*(c) + \langle \beta, c\rangle\ .$$

Note that Proposition 1 follows from Proposition 2 using $\beta=0$, but we preferred to write out the easy, natural case first. We now need the analogue of Theorem 1, so we return to our three spectra, E_i, our pairing $\mathbf{v}$, and our commutative diagram 2.6. Our homomorphisms $\Psi_i\colon \pi_*(E_i;R) \to R$ satisfy 2.3 but not necessarily 2.4. We have classes $\beta_i \in H^*(MSO \wedge E_i; Z_{(2)})$ (or $H^*(MSO \wedge E_i; Z/2^r)$) as in Proposition 2.

To complicate matters still further, the diagram in Theorem 1 will not commute in cases of interest to us. Hence we fix a class $\alpha \in H^*(E_1 \wedge E_2; Z_{(2)})$ (or $H^*(E_1 \wedge E_2; Z/2^r)$) such that

$$\begin{array}{ccc} \pi_p(E_1;R) \otimes \pi_q(E_2;R) & \rightarrow & \pi_{p+q}(E_1 \wedge E_2;R) \\ \downarrow \Psi_1 \otimes \Psi_2 & & \downarrow \Psi_3 \quad \nu_*(\ ) + \langle \alpha,\ \rangle \\ R \otimes R & \longrightarrow & R \end{array}$$

commutes. With all of these hypotheses we have

Theorem 2: There exists a class $\mathfrak{s} \in H^*(E_1 \wedge E_2; Z_{(2)})$ (or $H^*(E_1 \wedge E_2; Z/2^r)$) such that

i) the mod R reductions of $(1 \wedge T \wedge 1)^* \ (m \wedge 1 \wedge 1)^* \ (1 \wedge \nu)^*(\beta_3)$ and $\mathfrak{L} \wedge \ell_1 \wedge \beta_2 + \beta_1 \wedge \mathfrak{L} \wedge \ell_2 - \beta_1 \wedge \beta_2 + (\mu_1 \wedge \mu_2)^*(\alpha) + (1 \wedge T \wedge 1)^*(\mathfrak{L} \wedge \mathfrak{L} \wedge \mathfrak{s})$ evaluate the same on $\pi_*(MSO \wedge E_1 \wedge MSO \wedge E_2; R)$;

ii) $\nu^*(\ell_3) = \ell_1 \wedge \ell_2 + \mathfrak{s}$

both hold. Either condition determines $\mathfrak{s}$ uniquely.

The analogue of Corollary 1 is

Corollary 2: Given two cohomology classes x_1, $x_2 \in H^*(E; Z_{(2)})$ such that $\mu^*(x_i) = \mathfrak{L} \wedge x_i + \beta$ i=1,2, then

$x_1 = x_2$ iff $\langle x_1,\ \rangle = \langle x_2,\ \rangle : \pi_*(E;R) \rightarrow R$.

Proof of Theorem 2: We begin with the uniqueness statement. Clearly ii) determines $\mathfrak{s}$ uniquely. The Morgan-Sullivan uniqueness result shows that condition i) can be satisfied by at most one $\mathfrak{s}$.

Let us define $\mathfrak{s}$ so that condition ii) holds. Then we need to verify that i) is satisfied. By Lemma 1, it suffices to do this for all c of the form $c = c_1 \wedge c_2$, where $c_i \in \pi_*(MSO \wedge E_i; R)$.

$<(1\wedge T\wedge 1)^* \ (m\wedge 1\wedge 1)^* \ (1\wedge\nu)^*(\beta_3), c> = <\beta_3, (1\wedge\nu)_* \ (m\wedge 1\wedge 1)_* \ (1\wedge T\wedge 1)_*(c)>$.

This, by 2.7, is just

$<\mathcal{L}\wedge\ell_3, (1\wedge\nu)_* \ (m\wedge 1\wedge 1)_* \ (1\wedge T\wedge 1)_*(c)> -\Psi_3 \ (\mu_3)_*(1\wedge\nu)_*(m\wedge 1\wedge 1)_*(1\wedge T\wedge 1)_*(c) =$

$<\mathcal{L}\wedge\mathcal{L}\wedge\nu^*\ell_3, (1\wedge T\wedge 1)_*(c)> - \Psi_3 \ \nu_* \ (\mu_1\wedge\mu_2)_*(c_1\wedge c_2)$ by 2.6. This in turn is

$<\mathcal{L}\wedge\ell_1\wedge\mathcal{L}\wedge\ell_2, c_1\wedge c_2> - \Psi_1(c_1)\cdot\Psi_2(c_2) + <\alpha, (\mu_1\wedge\mu_2)_*(c_1\wedge c_2)> +$

$$<(1\wedge T\wedge 1)^* \ (\mathcal{L}\wedge\mathcal{L}\wedge s), c_1\wedge c_2>$$

by condition ii) and the definition of α.

This in its turn is equal, by 2.7, to

$<\mathcal{L}\wedge\ell_1\wedge\mathcal{L}\wedge\ell_2, c_1\wedge c_2> - (<\mathcal{L}\wedge\ell_1, c_1> - <\beta_1, c_1>) \ (<\mathcal{L}\wedge\ell_2, c_2> - <\beta_2, c_2>) \ +$

$$<\alpha, (\mu_1\wedge\mu_2)_*(c_1\wedge c_2)> + <(1\wedge T\wedge 1)^* \ (\mathcal{L}\wedge\mathcal{L}\wedge s), \ c_1\wedge c_2> \ .$$

Multiplying out and simplifying we get

$<\mathcal{L}\wedge\ell_1\wedge\beta_2 + \beta_1\wedge\mathcal{L}\wedge\ell_2 - \beta_1\wedge\beta_2 + (\mu_1\wedge\mu_2)^*(\alpha) + (1\wedge T\wedge 1)^* \ (\mathcal{L}\wedge\mathcal{L}\wedge s), c_1\wedge c_2>$

which really is what we wanted to get. //

§3. The spectra $\mathbb{L}^\circ$, $\mathbb{L}_\circ$, and $\hat{\mathbb{L}}$ at 2.

The goal of this section is to use the techniques and results of the previous section to analyze the Ranicki spectra $\mathbb{L}^\circ$, $\mathbb{L}_\circ$ and the maps ρ: $\mathbb{L}^\circ\wedge\mathbb{L}^\circ \to \mathbb{L}^\circ$; $\mathbf{m}$: $\mathbb{L}^\circ\wedge\mathbb{L}_\circ \to \mathbb{L}_\circ$; σ^*: MSTOP $\to \mathbb{L}^\circ$; $(1+T)$: $\mathbb{L}_\circ \to \mathbb{L}^\circ$; and e_8: $\mathbb{L}^\circ \to \mathbb{L}_\circ$.

The map ρ makes $\mathbb{L}^\circ$ into a ring spectrum, and the composite MSO $\to$ MSTOP $\to \mathbb{L}^\circ$ is a map of ring spectra. The map $\mathbf{m}$ makes $\mathbb{L}_\circ$ into an $\mathbb{L}^\circ$ module spectrum, so $\mathbb{L}^\circ$ and $\mathbb{L}_\circ$ (and indeed $\mathbb{L}^\circ(A,\varepsilon)$ and $\mathbb{L}_\circ(A,\varepsilon)$)

are all module spectra over MSO so the theory in §2 applies. We let μ°: $MSO \wedge \mathbb{L}^\circ \to \mathbb{L}^\circ$ and $\mu_\circ$: $MSO \wedge \mathbb{L}_\circ \to \mathbb{L}_\circ$ denote the structure maps.

We consider four homomorphisms. We have the index $\pi_*(\mathbb{L}^\circ) \to Z_{(2)}$; the deRham invariant $\pi_*(\mathbb{L}^\circ) \to Z/2$; the surgery obstruction $\pi_*(\mathbb{L}_\circ) \to Z_{(2)}$; and the Kervaire invariant $\pi_*(\mathbb{L}_\circ) \to Z/2$.

By the universal coefficients theorem and a bit of luck, $\pi_{4i}(\mathbb{L}^\circ;R) = \pi_{4i}(\mathbb{L}^\circ) \otimes R$ and $\pi_{4i}(\mathbb{L}_\circ;R) = \pi_{4i}(\mathbb{L}_\circ) \otimes R$. The two homomorphisms into $Z_{(2)}$ are zero unless $* \equiv 0 \pmod 4$ and extend uniquely to homomorphisms $\pi_*(\mathbb{L}^\circ;R) \to R$ and $\pi_*(\mathbb{L}_\circ;R) \to R$ which are zero unless $* \equiv 0 \pmod 4$. Both homomorphisms satisfy 2.3.

The index homomorphism satisfies 2.4. To see this, note that $\pi_*(MSO;R) \to \pi_*(\mathbb{L}^\circ;R)$ is onto except when $*=1$ or 2. Proposition 6.6 of Morgan-Sullivan [2] shows that diagram 2.4 commutes, at least if q is not 1 or 2. Since the map $\pi_4(\mathbb{L}^\circ;R) \otimes \pi_i(\mathbb{L}^\circ;R) \to \pi_{i+4}(\mathbb{L}^\circ \wedge \mathbb{L}^\circ;R) \xrightarrow{\rho_*} \pi_{i+4}(\mathbb{L}^\circ;R)$ is an isomorphism by Ranicki [5] § 7 , diagram 2.4 must commute even if $q=1$, or 2. Hence we get a unique class $L \in H^{4i}(\mathbb{L}^\circ;Z_{(2)})$ satisfying 2.5.

The Kervaire invariant $\pi_*(\mathbb{L}_\circ;Z/2) \to Z/2$ is determined by the map $\pi_*(\mathbb{L}_\circ) \to Z/2$ and does satisfy 2.4. For this, use the map constructed by Ranicki [5] (Proposition 15.5) from $Z \times G/TOP$ to the 0^{th} space in the Ω-spectrum for $\mathbb{L}_\circ$ which is a homotopy equivalence. One can then use Sullivan's result [7] that the Kervaire invariant is multiplicative with respect to the index to see that diagram 2.4 commutes. Let $k \in H^{4i+2}(\mathbb{L}_\circ;Z/2)$ denote the class promised by the mod 2 version of Proposition 1.

The deRham invariant $\pi_*(\mathbb{L}^\circ;Z/2) \to Z/2$ definitely does not satisfy 2.4. However, Morgan and Sullivan [2] (Proposition 6.6 and Lemma 8.2) show that "deRham" + $\langle V\, Sq^1 V \wedge L, \ \rangle$ is multiplicative with respect to the index (at least on $\pi_*(MSO;Z/2)$ and we use our usual trick.) This requires that, under the map $MSO \to \mathbb{L}^\circ$, L pulls back to $\mathcal{L}$, but this is easy to see from Corollary 1. Let $r \in H^{4i+1}(\mathbb{L}^\circ;Z/2)$ denote the class promised by the mod 2 version of Proposition 2.

Finally, the surgery obstruction $\pi_*(\mathbb{L}_\circ;R) \to R$ also does not satisfy 2.4. However "surgery obstruction" + $\langle \delta(V\, Sq^1 V \wedge k), \ \rangle$ is multiplicative with respect to the index, where δ denotes the bockstein associated to the exact sequence $0 \to Z_{(2)} \to Z_{(2)} \to Z/2 \to 0$. This follows by the usual trick from Morgan-Sullivan [2], Proposition 8.6. We let $\ell \in H^{4i}(\mathbb{L}_\circ;Z_{(2)})$ denote the class we get from Proposition 2.

This completes the definitions of the four classes which exhibit $\mathbb{L}^\circ$ and $\mathbb{L}_\circ$ as generalized Eilenberg-MacLane spectra at 2. Our next task is to analyze Ranicki's maps.

Proof of 1.9: We must show $\sigma^*(L) = \mathcal{L}$ and $\sigma^*(r) = V\, Sq^1 V$. Since Morgan and Sullivan used the homomorphism

$$\pi_*(MSO \wedge MSTOP;R) \to \pi_*(MSTOP;R) \to \pi_*(\mathbb{L}^\circ;R) \xrightarrow{\text{Index}} R$$

to define $\mathcal{L}$ ([2] §7) the first equation is Corollary 1. The second follows by a similar argument plus Lemma 8.2 of Morgan-Sullivan [2].

Proof of 1.11: We are to show that $e_8^*(k) = 0$ and $e_8^*(\ell) = L$. Clearly the diagram

$$\begin{tikzcd} \pi_*(\mathbb{L}^\circ;Z/2) \arrow[rr, "(e_8)_*"] \arrow[dr, "0"'] & & \pi_*(\mathbb{L}_\circ;Z/2) \arrow[dl, "\text{Kervaire invariant}"] \\ & Z/2 & \end{tikzcd}$$

commutes, so Corollary 1 shows $e_8^*(k) = 0$. Equally clearly

$$\begin{tikzcd} \pi_*(\mathbb{L}^\circ;R) \arrow[rr, "(e_8)_*"] \arrow[dr, "\text{Index}"'] & & \pi_*(\mathbb{L}_\circ;R) \arrow[dl, "\text{Surgery obstruction}"] \\ & R & \end{tikzcd}$$

commutes. Corollary 2 shows that $e_8{}^*(\ell) = L$.

An analogous argument shows that $(1+T)^*(L) = 8\ell$; $(1+T)^*(r) = 0$, which is 1.10.

Proof of 1.12: We want to show $\rho^*(L) = L \wedge L$ and $\rho^*(r) = r \wedge L + L \wedge r$. First check that diagram 2.6 commutes with $E_1 = E_2 = E_3 = \mathbb{L}^\circ$; $\nu = \rho$; and $\mu_i = \mu^\circ$ $i=1,2,3$. If we let each Ψ_i be the index homomorphism, the trick that we used in showing diagram 2.4 commuted for Ψ_i also shows that the diagram in Theorem 1 commutes. Hence $\rho^*(L) = L \wedge L$.

To get the second equation, we let Ψ_1 be the index homomorphism and let Ψ_2 and Ψ_3 both be the deRham homomorphism. Then $\beta_1 = 0$ and $\beta_2 = \beta_3 = (V\ Sq^1 V) \wedge L$. If we let $\alpha = r \wedge L$, it is easy to check that the necessary diagram commutes, so Theorem 2 applies. Alas we have not yet calculated $(\mu_1 \wedge \mu_2)^*(\alpha)$. The correct answer is easy to guess: $\mu_2{}^*(L) = \mathcal{L} \wedge L$ and $\mu_1{}^*(r) = \mathcal{L} \wedge r + (V\ Sq^1 V) \wedge L$. We accept this answer provisionally and proceed. If we take $\mathfrak{s} = r \wedge L$ it is a laborious calculation to see that i) is satisfied. Hence ii) also holds, so $\rho^*(r) = L \wedge r + r \wedge L$.

The maps μ_1 and μ_2 above are both the map $\mu^\circ\colon MSO \wedge \mathbb{L}^\circ \to \mathbb{L}^\circ$. To justify the above calculations we must analyze this map. Let $E_1 = MSO$, $E_2 = E_3 = \mathbb{L}^\circ$; let $\nu = \mu^\circ$; let Ψ_i be the index homomorphism. Theorem 1 applies, so $(\mu^\circ)^*(L) = \mathcal{L} \wedge L$. To get the other equation, change Ψ_2 and Ψ_3 to be the deRham homomorphism. Then $\beta_1 = 0$ and

$\beta_2 = \beta_3 = (V\ Sq^1 V) \wedge L$. If we take $\alpha = (V\ Sq^1 V) \wedge L$ then Theorem 2 applies. If we take $\mathfrak{s} = (V\ Sq^1 V) \wedge L$ we can calculate that i) is satisfied, so $(\mu^\circ)^*(r) = \mathfrak{L} \wedge r + (V\ Sq^1 V) \wedge L$.

This finishes the proof of 1.12.

Proof of 1.13: To analyze the map $\mathfrak{m}$ take $E_1 = \mathbb{L}^\circ$; $E_2 = E_3 = \mathbb{L}_\circ$; and $\nu = \mathfrak{m}$. The map $\mu_1 = \mu^\circ$ and $\mu_2 = \mu_3 = \mu_\circ$. Diagram 2.6 commutes.

We always take Ψ_1 to be the index homomorphism. To show $\mathfrak{m}^*(k) = L \wedge k$, which is half of 1.13, let us take $\Psi_2 = \Psi_3$ to be the Kervaire invariant. The diagram in Theorem 1 commutes, so the result follows.

To show $\mathfrak{m}^*(\ell) = L \wedge \ell + \delta(r \wedge k)$, which is the remainder of 1.13, let us take $\Psi_2 = \Psi_3$ to be surgery obstruction. Then $\beta_1 = 0$ and $\beta_2 = \beta_3 = \delta((V\ Sq^1 V) \wedge k)$. If we take $\alpha = \delta(r \wedge k)$, Theorem 2 is seen to apply. We have not yet calculated $(\mu_\circ)^*(k)$ so we assume the correct answer, $\mathfrak{L} \wedge k$. Then, with $\mathfrak{s} = \delta(r \wedge k)$ the reader can check that condition i) of Theorem 2 is satisfied, so our result follows.

To calculate $(\mu_\circ)^*(k)$ we apply Theorem 1 with $E_1 = MSO$; $E_2 = E_3 = \mathbb{L}_\circ$; $\nu = \mu_\circ$; $\Psi_1 =$ Index ; $\Psi_2 = \Psi_3 =$ Kervaire.

Our analysis of $\hat{\mathbb{L}}$ is less satisfactory. We have a map of $\mathbb{L}^\circ$ module spectra $\hat{\mathbb{L}} \to \Sigma\mathbb{L}_\circ$ so we have perfectly satisfactory classes $\Sigma\ell \in H^{4i+1}(\hat{\mathbb{L}}; Z_{(2)})$ and $\Sigma k \in H^{4i+3}(\hat{\mathbb{L}}; Z/2)$ obtained by pulling back the suspensions of ℓ and k respectively. If $\hat{\mu}: \mathbb{L}^\circ \wedge \hat{\mathbb{L}} \to \hat{\mathbb{L}}$ denotes the module pairing, we have $\hat{\mu}^*(\Sigma k) = L \wedge \Sigma k$. and $\hat{\mu}^*(\Sigma\ell) = L \wedge \Sigma\ell + \delta(r \wedge \Sigma k)$.

The next step in understanding $\hat{\mathbb{L}}$ is to construct the classes $\hat{L}$ and $\hat{r}$, but to do this we need to understand the pairing $\pi_p(\hat{\mathbb{L}};R) \otimes \pi_q(\hat{\mathbb{L}};R) \to \pi_{p+q}(\hat{\mathbb{L}} \wedge \hat{\mathbb{L}};R) \xrightarrow{\hat{\rho}_*} \pi_{p+q}(\hat{\mathbb{L}};R)$. This is accomplished as above except that we use the hyperquadratic signature map $MSG \to \hat{\mathbb{L}}$ and the results of Brumfiel-Morgan [1].

There is a map Index: $\pi_{4i}(\hat{\mathbb{L}};R) \to Z/8 \otimes R$ given by taking the index of the hyperquadratic form associated to the element in $\pi_{4i}(\hat{\mathbb{L}};R)$. Brumfiel and Morgan define an index homomorphism $\pi_{4i}(MSG;R) \to Z/8 \otimes R$. It is not clear that their homomorphism is the composite $\pi_{4i}(MSG;R) \to \pi_{4i}(\hat{\mathbb{L}};R) \to Z/8 \otimes R$, but it is true that we can find a homomorphism $\mathcal{I}$: $\pi_{4i}(\hat{\mathbb{L}};R) \to Z/8 \otimes R$ such that the Brumfiel-Morgan index is $\pi_{4i}(MSG;R) \to \pi_{4i}(\hat{\mathbb{L}};R) \xrightarrow{\mathcal{I}} Z/8 \otimes R$ and such that $\pi_{4i}(\mathbb{L}^\circ;R) \to \pi_{4i}(\hat{\mathbb{L}};R) \xrightarrow{\mathcal{I}} Z/8 \otimes R$ is still the index reduced mod 8.

The deRham invariant of a hyperquadratic form defines a homomorphism $\pi_{4i+1}(\hat{\mathbb{L}};Z/2) \to Z/2$.

The results of Brumfiel and Morgan [1] suffice to determine the pairing $\pi_p(\hat{\mathbb{L}};R) \otimes \pi_q(\hat{\mathbb{L}};R) \to \pi_{p+q}(\hat{\mathbb{L}};R)$. From this it is easy to understand the pairing $\pi_p(\mathbb{L}^\circ;R) \otimes \pi_q(\hat{\mathbb{L}};R) \to \pi_{p+q}(\hat{\mathbb{L}};R)$ induced by the module structure.

One sees that $\mathcal{I}$ does not cause diagram 2.4 to commute, but that $\mathcal{I} + \langle V\, Sq^1 V \wedge \Sigma k, \rangle$ is multiplicative with respect to the index (essentially [1] Theorem 8.4). Let $\hat{L}$ denote the resulting cohomology class.

Likewise the deRham homomorphism does not make diagram 2.4

commute, but by adding $\langle V\, Sq^1 V \wedge \hat{L}\, ,\, \rangle$ we do get a homomorphism that is multiplicative with respect to the index and so defines a class $\hat{r}$.

We can now use Theorem 2 to study $\hat{\mu} : MSO \wedge \hat{\mathbb{L}} \to \hat{\mathbb{L}}$. The results are that $(\hat{\mu})^*(\hat{L}) = \mathcal{L} \wedge \hat{L} + \iota((V\, Sq^1 V) \wedge \Sigma k)$ where $\iota : Z/2 \to Z/8$ is the non-trival map and $(\hat{\mu})^*(\hat{r}) = \mathcal{L} \wedge \hat{r} + (V\, Sq^1 V) \wedge \hat{L}$.

The class Σk can be defined using the only non-trival homomorphism $\pi_{4i+3}(\hat{\mathbb{L}}; Z/2) \to Z/2$. Theorem 1 can be applied to show $(\hat{\mu})^*(\Sigma k) = \mathcal{L} \wedge \Sigma k$.

Now apply Theorem 2 to study the map $\hat{\rho}: \hat{\mathbb{L}} \wedge \hat{\mathbb{L}} \to \hat{\mathbb{L}}$. Equations 1.14 and 1.15 should be clear.

§4. Periodic, connective bo-module spectra.

We say that a connective spectrum E, which is a module spectrum over bo, is periodic if the maps

$$\pi_4(bo) \otimes \pi_q(E) \to \pi_{q+4}(bo \wedge E) \to \pi_{q+4}(E)$$

are isomorphisms for all non-negative q.

Since $\mathbb{L}^\circ$ becomes bo after localization away from 2, one set of examples of connective, periodic bo-module spectra are the spectra $\mathbb{L}^\circ(A,\varepsilon)$ and $\mathbb{L}_\circ(A,\varepsilon)$ after localizing away from 2.

We have

Theorem $A_{(odd)}$: Let E be a connective, periodic bo-module spectrum. Then $E_{(odd)}$ is equivalent to

$$bo\Lambda_0 \vee \Sigma^1 bo\Lambda_1 \vee \Sigma^2 bo\Lambda_2 \vee \Sigma^3 bo\Lambda_3$$

where $bo\Lambda_i$ is bo with coefficients $\Lambda_i = \pi_i(E) \otimes Z[\frac{1}{2}]$.

Proof: Since $\pi_*(bo)$ is odd torsion free, the universal coefficients theorem says that $\pi_*(bo\Lambda_i) = \pi_*(bo) \otimes \Lambda_i$. Let $M(\Lambda_i)$ be the Moore spectrum whose only non-zero homology group is Λ_i in dimension zero. Then $bo\Lambda_i$ is just $bo \wedge M(\Lambda_i)$.

We can map $M(\Lambda_0) \to E_{(odd)}$ so that, on π_0 , the map is an isomorphism. Similarly, we can map $\Sigma^i M(\Lambda_i) \to E_{(odd)}$ so that, on π_i , the map is an isomorphism.

Now periodicity shows that the composite

$$bo \wedge M(\Lambda_0) \to bo \wedge E_{(odd)} \to E_{(odd)}$$

induces an isomorphism on π_{4i} and the zero map on $\pi_{4i+\varepsilon}$ for $\varepsilon=1,2,3$ and all i.

There is a similar statement for $M(\Lambda_1)$, $M(\Lambda_2)$, and for $M(\Lambda_3)$. The theorem follows easily.

Note added in proof: L. Jones has had a proof of Theorem A for $\mathbb{L}_0(Z\pi,\varepsilon)$ for some years: see The non-simply connected characteristic variety theorem, Proc. Symp. Pure Math. Vol. 32 Part I, 131 - 140.

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