Fibrations, cofibrations and related results

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Configuration spaces, braids and applications

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The goal is to introduce two approaches to understanding the homotopy type of configuration spaces.

Before explaining these two methods, let us introduce some notation that will make formulas to follow easier to write out. Let S be a finite set. Define

$$Conf\,(X, S)$$

to be the space of injective functions $S \to X$, topologized as a subset of the product space X^S. (If S has cardinality k, we often write $S = \{1, \cdots, k\}$ and write a point in $Conf\,(X, S)$ as $(x_1, \cdots, x_k)$ with $x_i \neq x_j$ for $i \neq j$.)

Using arbitrary finite sets simplifies notation. For instance, if $\phi\colon S_2 \to S_1$ is injective, composition clearly induces a map

$$\mathrm{p}_\phi\colon \mathit{Conf}\,(X, S_1) \to \mathit{Conf}\,(X, S_2)$$

The symmetric group action, Σ_S, on $\mathit{Conf}\,(X, S)$ is just composition of permutations so the action is a right action. These conventions require that the multiplication of cycles in a symmetric group is the composition multiplication:
$(123)(12) =?\ (123)(12) = (13)$ NOT (23).

The fibration method

Fix a subset $S_2 \subset S_1$ and consider the induced map

$$\mathrm{p}\colon \mathit{Conf}\,(X, S_1) \to \mathit{Conf}\,(X, S_2)$$

An arbitrary point in $\mathit{Conf}\,(X, S_2)$ is an injection $\iota\colon S_2 \to X$. Let $Q_\iota \subset X$ be the image of ι. Then the fibre over ι is

$$\mathrm{p}^{-1}(\iota) = \mathit{Conf}\,(X - Q_\iota, S_1 - S_2)$$

Theorem (Faddel and Neuwirth, 1962)

If X is a paracompact, finite dimensional manifold, all the $p^{-1}(\iota)$ are homeomorphic and p is a fibre bundle.

To compute homology or cohomology, one can apply the Serre spectral sequence. This requires knowledge of the monodromy of the fibration. At its lowest level, the monodromy of a fibration $F \to E \to B$ is a homomorphism

$$\pi_1(B, b) \to Aut\,(F)$$

where F is the fibre over B and $Aut\,(F)$ is the group of homotopy classes of automorphisms of F. If $E \to B$ has a section, the monodromy can be promoted to a homomorphism

$$\pi_1(B, b) \to Aut_*\,(F)$$

In general the monodromy is non-trivial. One important case is the case in which S_2 has cardinality one less than the cardinality of S_1.

In this case the fibre is $X - Q_\iota$, which is just the original manifold punctured $|S_2|$-times.

If X is non-compact then

$$X - Q_\iota = X \vee_{|S_2|} S^{n-1}$$

and if X is compact, X minus a point is non-compact.
If $|S_2| > 2$ then for homology/cohomology with coefficients in a ring R, the coefficients are non-trivial unless X is orientable for cohomology with coefficients in a ring R. If X is orientable, then the coefficients in this spectral sequence are trivial if $H^1(X; R) = 0$. If $H^1(X; R) \neq 0$, and X is compact, then the coefficients are non-trivial.

In particular, the spectral sequence has trivial coefficients for n-dimensional Euclidean space, $\mathbb{R}^n$. Moreover $\mathbb{R}^n$ punctured $|S_2|$-times is a wedge of $|S_2|$ $(n-1)$-dimensional spheres. There are no differentials and the additive structure of $H^*(Conf\,(\mathbb{R}^n, S_2)\,; \mathbb{Z})$ is easily worked out.

Additional structure will be described later.

A different approach to the Fadell-Neuwirth theorem observes that $Top(X)$ acts diagonally on $Conf\,(X, S)$ and if X is a paracompact manifold and the dimension $n \geqslant 2$ then the action is transitive. The isotropy subgroup of the point $\iota \in Conf\,(X, S)$ is $Top(X, Q_\iota)$, the subgroup of homeomorphisms fixing ι. Hence

$$Conf\,(X, S) \cong Top(X)/Top(X, Q_\iota)$$

and there is a fibre bundle

$$Conf\,(X, S) \to BTop(X, Q_\iota) \to BTop(X)$$

Hence information on configuration spaces informs on the difference between the group of homeomorphisms and the subgroup which fixes a finite set of points.

The cofibration method

In the cofibration approach, one starts with a finite set S_1 and a subset S_2 with one fewer elements. Let $\mathcal{W}(X, S_1, S_2)$ be the subspace of all $\iota\colon S_1 \to X$ which are injective when restricted to S_2. The space $\mathcal{W}(X, S_1, S_2)$ is homeomorphic to $X \times \mathit{Conf}(X, S_2)$ and so by induction can be assumed understood.

The inclusion $\mathit{Conf}(X, S_1) \subset X^{S_1}$ lands in $\mathcal{W}(X, S_1, S_2)$ and is an open subset if X is Hausdorff. To understand the inclusion $\mathit{Conf}(X, S_1) \subset \mathcal{W}(X, S_1, S_2)$ we will construct a Mayer-Vietoris sequence.

For each $t \in S_2$ there is a diagonal map $\Delta_t \colon \mathit{Conf}\,(X, S_2) \to \mathcal{W}\,(X, S_1, S_2)$ induced from the diagonal map $\Delta_t \colon X^{S_2} \to X^{S_1}$ defined by

$$\Delta_t(\iota)(s) = \begin{cases} \iota(s) & s \in S_2 \\ \iota(t) & s \in S_1 - S_2 \end{cases}$$

If X is Hausdorff, the image of Δ_t is closed and restricted to $\mathit{Conf}\,(X, S_2)$ the different images are disjoint. Assuming X is a metric ANR, so are all the other spaces under consideration and so there are disjoint open sets $U_t \subset \mathcal{W}\,(X, S_1, S_2)$ which are mapped into one another by the evident Σ_{S_2} action with U_t a neighborhood of the image of Δ_t. Let ∂U_t denote U_t minus the image of Δ_t.

Then

$$
\begin{array}{ccc}
\coprod_{t \in S_2} \partial U_t & \subset & \coprod_{t \in S_2} U_t \\
\cap & & \cap \\
Conf\,(X, S_1) & \subset & \mathcal{W}\,(X, S_1, S_2)
\end{array}
$$

is a Mayer-Vietoris square. This means that $\mathcal{W}\,(X, S_1, S_2)$ is homotopy equivalent to the double mapping cylinder of the green inclusions. It further follows that the mapping cone of the horizontal green inclusion is homotopy equivalent to the mapping cone of the black horizontal inclusion. Let $\mathfrak{C}_t$ be the mapping cone of the inclusion

$$\mathcal{W}\,(X, S_1, S_2) - \Delta_t\big(Conf\,(X, S_2)\big) \subset \mathcal{W}\,(X, S_1, S_2)$$

and let $\rho_t \colon \mathcal{W}\,(X, S_1, S_2) \to \mathfrak{C}_t$ be the standard map.
For each $s \in S_2$ there is an induced map from the mapping cone of $\partial U_s \subset U_s$ to $\mathfrak{C}_t$. If $s = t$ excision says this induced map is a homotopy equivalence. If $s \neq t$ then the induced map is null-homotopic.

$$\begin{array}{ccc} \coprod_{t \in S_2} \partial U_t & \subset & \coprod_{t \in S_2} U_t \\ \cap & & \cap \\ Conf\,(X, S_1) & \subset & \mathcal{W}\,(X, S_1, S_2) \end{array}$$

Hence

$$Conf\,(X, S_1) \subset \mathcal{W}\,(X, S_1, S_2) \xrightarrow{\rho} \bigvee_{s \in S_2} \mathfrak{C}_s$$

is a homotopy cofibration sequence. The map ρ followed by a projection $\bigvee_{s \in S_2} \mathfrak{C}_s \to \mathfrak{C}_t$ is the map ρ_t.

Theorem

If X is a finite-dimensional paracompact manifold without boundary, then $\mathfrak{C}_t$ is the Thom complex of the tangent bundle of X, pulled back to $Conf\,(X, S_2)$ via the projection onto the t coordinate.

$$Conf\,(X, S_1) \subset \mathcal{W}\,(X, S_1, S_2) \xrightarrow{\;\rho\;} \bigvee_{s \in S_2} \mathfrak{C}_t$$

In cohomology with coefficients in a ring R, the above sequence is a sequence of $H^*(\mathcal{W}\,(X, S_1, S_2)\,; R)$-modules so the map ρ is completely determined in cohomology by the images of the Thom classes, $\rho_t^*(\,U_t)$.
If X is closed, compact and oriented, and if R is a field $\mathbb{F}$, Milnor-Stasheff, Thm. 11.11, p. 128, work out this image, $\Delta \in H^*(X^{\{1,2\}}, \mathbb{F})$. Once there is a reason to do so, working out the general case is not difficult. In any case, naturality shows

$$\rho_t^*(\,U_t) = \iota_{t,x}^*(\Delta)$$

where $\{x\} = S_1 - S_2$. For any s_1, $s_2 \in S_1$ let $\Delta_{s_2,s_1} = \iota_{s_2,s_1}^*(\Delta)$.

The case $X = M \times \mathbb{R}^2$

The map $\mathit{Conf}\,(X, S_1) \subset \mathcal{W}\,(X, S_1, S_2)$ has a retraction $r\colon \mathcal{W}\,(X, S_1, S_2) \to \mathit{Conf}\,(X, S_1)$ such that the composition

$$\mathcal{W}\,(X, S_1, S_2) \xrightarrow{r} \mathit{Conf}\,(X, S_1) \subset \mathcal{W}\,(X, S_1, S_2)$$

is homotopic to the identity. It comes from an orientation-preserving embedding $e_1 \amalg e_2 \colon \mathbb{R}^2 \amalg \mathbb{R}^2 \subset \mathbb{R}^2$:

$$r(\iota)(s) = \begin{cases} e_1\big(\iota(s)\big) & s \in S_2 \\ e_2\big(\iota(s)\big) & \text{otherwise} \end{cases}$$

Note any cohomology class U with $r^*(U) = 0$ comes from a unique class in the mapping cone. In the manifold case, the mapping cone is a Thom space. If X is also orientable of dimension n, then there exists a unique element $U \in H^{n-1}\big(\mathit{Conf}\,(X, \{1,2\})\,; \mathbb{Z}\big)$ such that $r^*(U) = 0$ and U comes from the Thom class.

The case $X = M \times \mathbb{R}^2$

For every ordered pair of elements (s_1, s_2) with $s_1,\ s_2 \in S$ define $\iota_{s_2,s_1} : \{1, 2\} \to S$ by $\iota_{s_2,s_1}(j) = s_j$, $j = 1,\ 2$. Let ι_{s_2,s_1} also denote the induced map of configuration spaces. Define

$$A_{s_2,s_1} \in H^{n-1}(\mathit{Conf}\,(X, S)\,; \mathbb{Z})$$

as $\iota^*_{s_2,s_1}(U)$.

With field coefficients $H^*(\mathit{Conf}\,(X, S)\,; \mathbb{F})$ is the quotient of $\mathbf{S}[A_{s_2,s_1}] \otimes H^*(M; \mathbb{F})^{\otimes S}$ modulo the relations below. Here $\mathbf{S}[A_{s_2,s_1}]$ is the graded symmetric algebra on the elements A_{s_2,s_1}, $s_1,\ s_2 \in S$. The relations are

- $A_{s_2,s_1} = (-1)^{n-1} A_{s_1,s_2}$
- $A^2_{s_2,s_1} = 0$
- $A_{s_2,s_1} A_{s_3,s_1} = A_{s_2,s_1}(A_{s_3,s_1} - A_{s_3,s_2})$
- Given $m \in H^r(M; \mathbb{F})$ and $s \in S$ let $[m]_s$ denote the element $1 \otimes \cdots \otimes m \otimes \cdots \otimes 1 \in H^r(M; \mathbb{F})^{\otimes S}$ which has a 1 in every position but the s^{th} where it is m. Then $[m]_{s_2} A_{s_2,s_1} = [m]_{s_1} A_{s_2,s_1}$

A decomposition of the suspension of $Conf\,(M \times \mathbb{R}, S_1)$

The inclusion $Conf\,(M \times \mathbb{R}, S) \subset \mathcal{W}\,(M \times \mathbb{R}, S_1, S_2)$ is also split. The cofibre of this inclusion is the Thom complex of the tangent bundle to $M \times \mathbb{R}$ pulled back to $Conf\,(M \times \mathbb{R}, S_2)$, which is the suspension of the Thom complex of the tangent bundle to M pulled back to $Conf\,(M \times \mathbb{R}, S_2)$. Denote the Thom complex of the tangent bundle to M pulled back to $Conf\,(M \times \mathbb{R}, S_2)$ by $T_s(\tau_M)$. It follows that

$$\Sigma Conf\,(M \times \mathbb{R}, S_1) \cong \Sigma \mathcal{W}\,(M \times \mathbb{R}, S_1, S_2) \vee \bigvee_{s \in S_2} \Sigma T_s(\tau_M)$$

One can apply the cofibration method to bundles over X^S and eventually see that $\Sigma Conf\,(M \times \mathbb{R}, S_1)$ is a wedge to Thom complexes of sums of pulled back tangent bundles over products of copies of M.

In particular $\Sigma Conf\,(\mathbb{R}^n, S_1)$ is a wedge of spheres. We will return to the question of enumerating these Thom spaces later.

The Totaro spectral sequence for manifolds

Given any graded commutative ring, B, we can form a new graded commutative ring $\mathbf{S}[A_{s_2,s_1}] \otimes B^{\otimes S}$ modulo the four relations

- $A_{s_2,s_1} = (-1)^{n-1} A_{s_1,s_2}$
- $A_{s_2,s_1}^2 = 0$
- $A_{s_2,s_1} A_{s_3,s_1} = A_{s_2,s_1} (A_{s_3,s_1} - A_{s_3,s_2})$
- Given $m \in B$ and $s \in S$ $[m]_{s_2} A_{s_2,s_1} = [m]_{s_1} A_{s_2,s_1}$

Theorem (Totaro 1993-6)

Let M be a manifold of dimension $n \geqslant 2$. The Leray spectral sequence for the inclusion $Conf(M,S) \subset M^S$ with field coefficients $\mathbb{F}$ has E^2 term given by applying the construction above with $B = H^*(M;\mathbb{F})$. The d_2 differential is determined by $d_2(A_{s_2,s_1}) = \Delta_{s_2,s_1}$ where $\Delta_{s_2,s_1} = \iota^*_{s_2,s_1}(\Delta)$ and where Δ is the diagonal class $\Delta \in H^n(M^{\{1,2\}};\mathbb{F})$. The spectral sequence converges as a Σ_S-algebra to $H^*(Conf(M,S);\mathbb{F})$.

The Totaro spectral sequence (continued)

Remark: Totaro shows that if M is a smooth complex projective variety and $\mathbb{F}$ has characteristic zero, d_2 is the only differential. Felix and Thomas prove the same result for any formal manifold, which extends the result to spaces like products of spheres.

Remark: If $\Delta = 0$, the Totaro spectral sequence collapses.

Remark: If $\mathbb{F}$ has characteristic zero and if M has non-trivial Massey triple products, Felix and Thomas show that there may be additional differentials. An example is the tangent sphere bundle to $S^2 \times S^2$.

Remark: Longoni and Salvatore show that even though the lens spaces $L_{7,1}$ and $L_{7,2}$ are homotopy equivalent, their configuration spaces are not whenever $|S| \geqslant 2$.

The Totaro spectral sequence (continued)

Remark: Aouina & Klein and Cohen & Taylor show that homotopy equivalent manifolds give stably-homotopy equivalent configuration spaces. In particular, their Totaro spectral sequences have the same E_2, the same d_2 and the same $\bigoplus_{p+q=r} E_\infty^{p,q}$ for each r.

It is tempting to conjecture that the spectral sequences are isomorphic even though there is no obvious map between them.

More results in the $M \times \mathbb{R}$ case (START HERE)

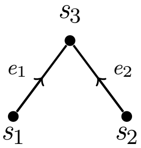
To describe the various Thom spaces which go into the decomposition of $\Sigma Conf\,(M, S)$, begin by discussing 1-dimensional CW complexes. Given a finite set S, an *ordered 1-complex* Γ is a CW complex with vertex set S and a set of edges $\mathcal{E}(\Gamma)$. Each edge is oriented and the set of edges is ordered.

Given an edge $e \in \mathcal{E}(\Gamma)$ define $A_e = A_{s_2,s_1}$ where e starts at vertex s_1 and ends at vertex s_2. Define $A_\Gamma = A_{e_1} \cdots A_{e_k}$ where $e_1, \ldots, e_k$ are the edges of Γ in order. These conventions set up a bijection between products of the A's and ordered 1-complexes. It can be shown that

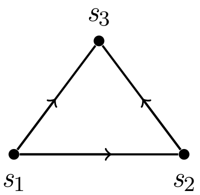
$$A_\Gamma \neq 0 \text{ if and only if } H_1(\Gamma) = 0.$$

Hence $A_\Gamma \neq 0$ if and only if each path component of Γ is a tree or a single vertex. If we continue the arboreal theme by calling components with single vertices *seeds*, then $A_\Gamma \neq 0$ if and only if Γ is a *forest*.

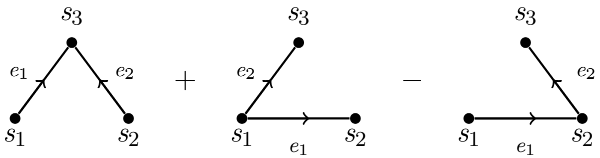
The key to the proof of the previous result is the graphical version of the three-term relation which can be described using ordered 1-complexes. Say that a vertex s_3 *supports an incoming three-term relation* provided there are at least two edges which have s_3 as an incoming end. There may well be additional vertices and edges which are not drawn in the picture.



Draw a new edge from s_1 to s_2, provided $e_1 < e_2$, to get the triangle on the next page.



The three-term relation says that a combination of three ordered 1-complexes is 0. They are obtained by combining the three ways of deleting an edge from the triangle, and reordering an edge or two.



$$A_{\Gamma_3} \quad + \quad A_{\Gamma_1} \quad - \quad A_{\Gamma_2} \quad = \quad 0$$

Theorem

Given a vertex which supports a three-term relation then for the three graphs described above

$$H_*(\Gamma_3) \cong H_*(\Gamma_2) \cong H_*(\Gamma_1)$$

A graph partitions its set of vertices by saying two are equivalent if and only if they lie in the same path component. All three graphs yield the same partition.

Certain collections of ordered 1-complexes give a basis for $H^*(\mathit{Conf}(\mathbb{R}^n, S); \mathbb{Z})$. Clearly the ordered 1-complexes in a basis must be a forest, but there are more forests than basis elements whenever $|S| > 2$.

One basis is given by the admissible forests. To define when an forest is admissible, it is first necessary to order S. Then we can orient an edge by starting at the smaller vertex and going to the larger. We can order the edges using lexicographical order. A forest is *admissible* provided no vertex supports an incoming three-term relation using the above orientations and ordering.

Theorem

If $\mathcal{A}(S)$ is the set of admissible forests on the ordered vertex set S then the elements A_Γ for all $\Gamma \in \mathcal{A}(S)$ are an additive basis for $H^*(Conf(\mathbb{R}^n, S); \mathbb{Z})$, $n \geqslant 2$.

For any forest Γ there is a diagonal

$$\Delta_\Gamma \colon X^{\pi_0(\Gamma)} \to X^S$$

defined by $(\Delta_\Gamma(\iota))(s) = \iota([s])$ where $[s] \in \pi_0(\Gamma)$ is the path component of Γ containing s. If X is a manifold, let ν_γ be the normal bundle of $X^{\pi_0(\Gamma)}$ in X^S. Note it is a sum of various tangent bundles of X pulled back to $X^{\pi_0(\Gamma)}$.

Let $\mathfrak{A}(S)$ be a set of forests such that the collection A_Γ, $\Gamma \in \mathfrak{A}(S)$ is a basis for $H^*\big(Conf\,(\mathbb{R}^n, S)\,;\mathbb{Z}\big)$.
Then

$$\Sigma\, Conf\left(M \times \mathbb{R}^1, S\right) \cong \bigvee_{\Gamma \in \mathfrak{A}(S)} \Sigma\, T(\nu_\Gamma)$$

Remark: The admissible basis has an additional property that there is an algorithm for writing any forest as a linear combination of admissible forests.

The top representation

The sub-group of $H^*(Conf\,(\mathbb{R}^n, S)\,; \mathbb{Z})$ generated by all A_Γ with the associated partition fixed form a subgroup of $H^{(n-1)(|S|-r)}(Conf\,(M, S)); \mathbb{Z})$ where r is number of path components of Γ, which is also the number of elements in the partition.
Hence the highest non-trivial cohomology group of $Conf\,(\mathbb{R}^n, S)$ is in dimension $(n-1)(|S|-1)$. Classes A_Γ in this dimension come from forests which are connected and vice versa.
From this one sees that $H^*(Conf\,(\mathbb{R}^n, S)\,; \mathbb{Z})$ is built up out of tensor products of top dimensional groups for various subsets of S.

Example

Let $S = \{1, 2, 3, 4, 5\}$ and let $\{\{1, 2, 3\}, \{4, 5\}\}$ be a partition. Then a sumand of $H^{3(n-1)}(Conf\,(\mathbb{R}^n, S)\,; Z)$ is a tensor product of the top group for 3 points tensor the top group for 2 points.

The top representation (continued)

Recall every forest partitions the set S and by taking the cardinality of each set in the partition, we get a partition of the integer $|S|$. Given any two forests with the same integer partition, there are permutations of S which take one to the other.

Hence under the action of the symmetric group, the cohomology decomposes into summands corresponding to integer partitions of $|S|$. The cohomological degree of the corresponding A_Γ can be determined from the integer partition.

Leher & Solomon wrote down the Poincaré character for the rational representation on $H^*(\mathit{Conf}\,(\mathbb{R}^n, S)\,;\mathbb{Q})$.

Fred & I worked out the representation over $\mathbb{Z}$ as a sum of tensor products of representations induced from Young subgroups: i.e. subgroups of the form

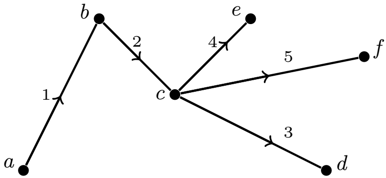
$$\Sigma_{S_1} \times \cdots \times \Sigma_{S_k} \subset \Sigma_{S_1 \sqcup \cdots \sqcup S_k}$$

The top representation (continued)

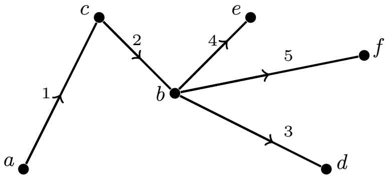
The top representation comes from the partition with one subset (or one integer). For example, one basis for this group consists of the admissible trees.

Notice however that a permutation applied to an admissible tree is often not admissible.

Here is an admissible tree on $\{a, b, c, d, e, f\}$ ordered alphabetically.



The permutation $\begin{pmatrix} abcdef \\ acbdef \end{pmatrix}$ applied to the above admissible tree gives the tree



which is no longer admissible: the orientation on the edge between b and c has the “wrong” orientation.
If we reorient this edge “correctly”, then c supports an incoming three-term relation.

There are other bases for this top rep which are useful. A *rooted tree* is a tree with a distinguished vertex. A tree is called *linear* provided every vertex has valence 1 or 2. There must be exactly two vertices of valence 1. Fix one of the vertices of valence one, say $\mathbf{v}$.

A *linear rooted tree with root* $\mathbf{v}$ is a linear tree with vertex set S with one vertex of valence 1 being $\mathbf{v}$. Direct the edges so that you start at $\mathbf{v}$ and just keep going. Number the edges in the order in which they appear along the tree starting at the root. Here are the two rooted trees on $\{1, 2, 3\}$:

This shows that the top rep as an integral representation of the of the symmetric subgroup of Σ_S fixing $\mathbf{v}$ is free.

$C(M, X)$

Fix a countably infinite set, say $\mathbf{N}$. Given a space M and a based space $(X, *)$ consider the space of maps $f\colon \mathbf{N} \to M \times X$. Define the support of f to be the subset of $\mathbf{N}$ such that $f(s) \neq *$. Let $E(M, X)$ be the subspace of functions whose support is a finite subset, say S, and such that the composition $S \xrightarrow{f} M \times X \to M$ is injective.

Define

$$C(M, X) = E(M, X) / \approx$$

where $\approx$ is the equivalence relation generated by the following two types of relations:

1. $f_1 \approx f_2$ if f_1 and f_2 have the same support and are equal when restricted to that support
2. $f_1 \approx f_2$ if there exists a bijection $\phi\colon \mathbf{N} \to \mathbf{N}$ such that $f_1 \circ \phi = f_2$

$C(M, X)$ (canonical identifications)

Since any two countably infinite sets are bijectively equivalent, the choice of set $\mathbf{N}$ is not usually important: if it is we will write $C_{\mathbf{N}}(M, X)$.
Any bijection $\phi\colon \mathbf{N}_1 \to \mathbf{N}_2$ induces a homeomorphism $C_{\mathbf{N}_2}(M, X) \to C_{\mathbf{N}_1}(M, X)$. Thanks to relation (2), any two ϕ induce identical maps. In particular, any two versions of this construction can be canonically identified.

In a few pages we will also need the following related remark. Define the braid space $B(M, S)$ to be $Conf(M, S)/\Sigma_S$. Given another finite set T of the same cardinality, any choice of bijection $\phi\colon T \to S$ induces a homeomorphism $\phi\colon Conf(M, S) \to Conf(M, T)$ which descends to a homeomorphism $B(M, S) \to B(M, T)$
The remark is that two different ϕ's induced the same map on the braid spaces so they may be canonically identified. In the sequel we will write $B_k(M)$ whenever the index set has cardinality k.

$C(M, X)$ (continued)

Filter $C(M, X)$ by letting $F_k(M, X) \subset C(M, X)$ be the image of all functions in $E(M, X)$ whose support has at most k elements. Notice both relations (1) and (2) preserve the cardinality of the support of the functions.
Define $D_k(M, X)$ to be the cofibre of the inclusion $F_{k-1}(M, X) \subset F_k(M, X)$. If $(X, *)$ is an NDR pair then so is $(F_k(M, X), F_{k-1}(M, X))$ and we can identify the cofibre. Fix a finite set of cardinality k, $S \subset \mathbf{N}$. The composition $Conf(M, S) \times X^S \to F_k(M, X)$ is onto and factors through the orbit space $Conf(M, S) \times_{\Sigma_S} X^S$.
The map $Conf(M, S) \times_{\Sigma_S} X^S \to D_k(M, X)$ is onto and if $F\Delta \subset X^S$ is the set of points with at least one coordinate the base point, then

$$Conf(M, S) \times_{\Sigma_S} X^S / Conf(M, S) \times_{\Sigma_S} F\Delta \to D_k(M, X)$$

is a homeomorphism.

$C(M, X)$ (continued)

Any other choice of finite set of cardinality k gives a similar identification and any choice of bijection induces the same map. With a bit of fiddling, one can rewrite $D_k(M, X)$ as

$$D_k(M, X) = Conf(M, S) \ltimes_{\Sigma_S} X^{[S]}$$

where $X^{[S]}$ denotes the S-fold smash product.

We would like to extend the natural map $\mathfrak{f}_k \colon F_k(M, X) \to D_k(M, X)$ to a map $C(M, X) \to D_k(M, X)$ but this is not usually possible.
It is however possible to do so stably.

Stable splitting of $C(M, X)$

To describe the extension, first try the most naive thing you can think of: given f with support S of cardinality bigger than k, just restrict to some subset of cardinality k and pass to the quotient $D_k(M, X)$. The obvious problem is which subset to take. The solution in situations like this where there is no natural choice is to take all choices.

Let $\mathbf{N}' = \binom{\mathbf{N}}{k}$ denote the set of all subsets of $\mathbf{N}$ of cardinality k. Note $\mathbf{N}'$ is also countably infinite.

Define a map

$$h_k\colon C_{\mathbf{N}}(M,X) \to C_{\mathbf{N}'}(B_k(M), D_k(M,X))$$

as follows. Recall that a point in $E(M,X)$ is a map $f\colon \mathbf{N} \to M \times X$ satisfying some additional conditions. We need to define a map $h_k(f)\colon \mathbf{N}' \to B_k(M) \times D_k(M,X)$. An element of $\mathbf{N}'$ is a set $S \subset \mathbf{N}$ of cardinality k. Hence f restricted to S is a point in $F_k(M,X)$ and hence a point in $D_k(M,X)$, denote this point by $[f]_S$. If S is not in the support of f, $[f]_S$ is the base point.

If S is in the support of f, let $\langle f\rangle_S \in B_k(M)$ denote the image in $B_k(M)$ of the point in $\mathit{Conf}(M,S)$ given by the composition

$$S \subset \mathbf{N} \xrightarrow{\ f\ } M \times X \to M$$

If S is not in the support of f, define $\langle f\rangle_S$ to be any point you like in $B_k(M)$. Define

$$h_k(f)(S) = \langle f\rangle_S \times [f]_S$$

Check that the support of $h_k(f)$ is the set of all subsets of cardinality k contained in the support of f. The additional requirements to be a point in $C_{\mathbf{N}'}\left(B(M,k), D_k\left(M,X\right)\right)$ can be checked.
Since $B_k\left(M\right)$ is a manifold, it embeds in $\mathbb{R}^K$ for some K and so there is a map

$$h_k\colon C_{\mathbf{N}}\left(M,X\right) \to C_{\mathbf{N}'}\left(\mathbb{R}^K, D_k\left(M,X\right)\right)$$

Moreover, $D_k\left(M,X\right)$ is path-connected and so a theorem of Peter May supplies a map
$C_{\mathbf{N}'}\left(\mathbb{R}^K, D_k\left(M,X\right)\right) \to \Omega^K\Sigma^K D_k\left(M,X\right)$ and so we get a map

$$h_k\colon C_{\mathbf{N}}\left(M,X\right) \to \Omega^K\Sigma^K D_k\left(M,X\right)$$

and a commutative diagram

$$
\begin{array}{ccc}
F_k(M,X) & \xrightarrow{\quad \mathfrak{f}_k \quad} & D_k(M,X) \\
\cap & & \downarrow \\
C(M,X) & \xrightarrow{\quad h_k \quad} & \Omega^K \Sigma^K D_k(M,X)
\end{array}
$$

To belabor the point, we could adjoint h_k and note

$$\Sigma^K \mathfrak{f}_k \colon \Sigma^K F_k(M,X) \subset \Sigma^K C(M,X) \xrightarrow{\mathrm{ad} h_k} \Sigma^K D_k(M,X)$$

The K certainly increases as k increases, but if we pass to the stable world we get an equivalence

$$QC(M,X) \cong Q \bigvee_{k=1}^{\infty} D_k(M,X)$$

There are many results concerning this construction and its pieces.

Remark: $QC\left(M, S^0\right) = Q \bigvee_{k=1}^{\infty} B(M, k)$

Remark: For an appropriate K there is a map

$$\Sigma^{k \cdot K} D_k\left(M, X\right) \rightarrow D_k\left(M, \Sigma^K X\right)$$

which is a homotopy equivalence.

Remark: A theorem of Jeff Caruso says

$$\Omega C\left(M, \Sigma X\right) \cong C\left(M \times \mathbb{R}, X\right)$$

This generalizes the case in which $M = \mathbb{R}^n$ due to Peter May.

Remark: If X is path connected, then a theorem of Bödigheimer says that $C\,(M,X)$ is weakly-homotopy equivalent to a space of sections of a certain bundle $E \to M$. The bundle is formed from the tangent bundle $T \to M$ as follows. Take the fibre-wise one point compactification of the tangent bundle: take $T \amalg M \to M$ and topologize so that each fibre is the one-point compactification of the fibre of T. This bundle is denoted $\hat{T}$ and it has a section at infinity. Then E is the reduced fibre-wise smash of $\hat{T}$ with X. This a bundle with fibre the reduced suspension $\Sigma^n X$. The reduced suspension has a base point and so E has a section at infinity, σ_∞. A section $\sigma\colon M \to E$ has support the set of all points $m \in M$ such that $\sigma(m) \neq \sigma_\infty(m)$. A section has compact support provided the closure of the support is compact.
Bödigheimer says that $C\,(M,X)$ is weakly-homotopy equivalent to the space of sections with compact support of $E \to M$.

Corollary

$C(M, X)$ is a proper homotopy invariant of M.

Of course up to homotopy type $C(M, X)$ only depends on the homotopy type of X.