Fibrations, cofibrations and related results

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Configuration spaces, braids and applications
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The fibration method

Fix a subset $S_2 \subset S_1$ and consider the induced map $p: \text{Conf}(X, S_1) \to \text{Conf}(X, S_2)$.

An arbitrary point in $\text{Conf}(X, S_2)$ is an injection $\iota: S_2 \to X$.

Let $Q_\iota \subset X$ be the image of $\iota$.

Then the fibre over $\iota$ is $p^{-1}(\iota) = \text{Conf}(X - Q_\iota, S_1 - S_2)$.

Theorem (Fadell and Neuwirth, 1962)

If $X$ is a paracompact, finite dimensional manifold, all the $p^{-1}(\iota)$ are homeomorphic and $p$ is a fibre bundle.
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If $X$ is a paracompact, finite dimensional manifold, all the $p^{-1}(\iota)$ are homeomorphic and $p$ is a fibre bundle.
To compute homology or cohomology, one can apply the Serre spectral sequence. This requires knowledge of the monodromy of the fibration. At its lowest level, the monodromy of a fibration $F \to E \to B$ is a homomorphism

$$\pi_1(B, b) \to \text{Aut}(F)$$

where $F$ is the fibre over $B$ and $\text{Aut}(F)$ is the group of homotopy classes of automorphisms of $F$. 
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In general the monodromy is non-trivial. One important case is the case in which $S_2$ has cardinality one less than the cardinality of $S_1$.

In this case the fibre is $X - Q_\ell$, which is just the original manifold punctured $|S_2|$-times.
If $X$ is non-compact then

$$X - Q_t = X \lor_{|S_2|} S^{n-1}$$

and if $X$ is compact, $X$ minus a point is non-compact.
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If $|S_2| > 2$ then for homology/cohomology with coefficients in a ring $R$, the coefficients are non-trivial unless $X$ is orientable for cohomology with coefficients in a ring $R$. 

If $H_1(X; R) \neq 0$, and $X$ is compact, then the coefficients are non-trivial. In particular, the spectral sequence has trivial coefficients for $n$-dimensional Euclidean space, $\mathbb{R}^n$. Moreover $\mathbb{R}^n$ punctured $|S_2|$-times is a wedge of $|S_2| (n-1)$-dimensional spheres.

There are no differentials and the additive structure of $H^*(\text{Conf}(\mathbb{R}^n, S_2); \mathbb{Z})$ is easily worked out. Additional structure will be described later.
If $X$ is non-compact then

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A different approach to the Fadell-Neuwirth theorem observes that $\text{Top}(X)$ acts diagonally on $\text{Conf}(X, S)$ and if $X$ is a paracompact manifold and the dimension $n \geq 2$ then the action is transitive.

The isotropy subgroup of the point $\iota \in \text{Conf}(X, S)$ is $\text{Top}(X, Q \iota)$, the subgroup of homeomorphisms fixing $\iota$. Hence $\text{Conf}(X, S) \sim \text{Top}(X)/\text{Top}(X, Q \iota)$ and there is a fibre bundle $\text{Conf}(X, S) \to B\text{Top}(X, Q \iota) \to B\text{Top}(X)$.

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Hence information on configuration spaces informs on the difference between the group of homeomorphisms and the subgroup which fixes a finite set of points.
In the cofibration approach, one starts with a finite set $S_1$ and a subset $S_2$ with one fewer elements.
The cofibration method

In the cofibration approach, one starts with a finite set $S_1$ and a subset $S_2$ with one fewer elements. Let $\mathcal{W}(X, S_1, S_2)$ be the subspace of all $\iota: S_1 \to X$ which are injective when restricted to $S_2$. 
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The inclusion $Conf(X, S_1) \subset X^{S_1}$ lands in $W(X, S_1, S_2)$ and is an open subset if $X$ is Hausdorff. To understand the inclusion $Conf(X, S_1) \subset W(X, S_1, S_2)$ we will construct a Mayer-Vietoris sequence.
For each $t \in S_2$ there is a diagonal map 
$\Delta_t: \text{Conf} (X, S_2) \to \mathcal{W} (X, S_1, S_2)$ induced from the diagonal map 
$\Delta_t: X^{S_2} \to X^{S_1}$ defined by 

$$\Delta_t(t)(s) = \begin{cases} 
    t(s) & s \in S_2 \\
    t(t) & s \in S_1 - S_2
\end{cases}$$

If $X$ is Hausdorff, the image of $\Delta_t$ is closed and restricted to 
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metric ANR, so are all the other spaces under consideration and 
so there are disjoint open sets $U_t \subset \mathcal{W} (X, S_1, S_2)$ which are mapped into one another by the evident $\Sigma_{S_2}$ action with $U_t$ a 
neighborhood of the image of $\Delta_t$. Let $\partial U_t$ denote $U_t$ minus the 
image of $\Delta_t$. 
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Then
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\bigcap_{t \in S_2} \partial U_t \quad \subset \quad \bigcap_{t \in S_2} U_t
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is a Mayer-Vietoris square. This means that \(\mathcal{W} (X, S_1, S_2)\) is homotopy equivalent to the double mapping cylinder of the green inclusions.
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\[ \mathcal{W}(X, S_1, S_2) - \Delta_t(Conf \,(X, S_2)) \subset \mathcal{W}(X, S_1, S_2) \]

and let \(\rho_t: \mathcal{W}(X, S_1, S_2) \to \mathcal{C}_t\) be the standard map.
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and let \( \rho_t : \mathcal{W} (X, S_1, S_2) \to \mathcal{C}_t \) be the standard map. For each \( s \in S_2 \) there is an induced map from the mapping cone of \( \partial U_s \subset U_s \) to \( \mathcal{C}_t \). If \( s = t \) excision says this induced map is a homotopy equivalence. If \( s \neq t \) then the induced map is null-homotopic.
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\bigcap_{t \in S_2} \partial U_t \subset \bigcap_{t \in S_2} U_t
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\text{Conf} (X, S_1) \subset \mathcal{W} (X, S_1, S_2)
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Hence
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\text{Conf} (X, S_1) \subset \mathcal{W} (X, S_1, S_2) \xrightarrow{\rho} \bigvee_{s \in S_2} \mathcal{C}_s
\]

is a homotopy cofibration sequence.
\[
\bigoplus_{t \in S_2} \partial U_t \subset \bigcup_{t \in S_2} U_t
\]

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is a homotopy cofibration sequence. The map \(\rho\) followed by a projection \(\bigvee_{s \in S_2} \mathcal{C}_s \rightarrow \mathcal{C}_t\) is the map \(\rho_t\).
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is a homotopy cofibration sequence. The map \(\rho\) followed by a projection \(\bigvee_{s \in S_2} \mathcal{C}_s \rightarrow \mathcal{C}_t\) is the map \(\rho_t\).

**Theorem**

If \(X\) is a finite-dimensional paracompact manifold without boundary, then \(\mathcal{C}_t\) is the Thom complex of the tangent bundle of \(X\), pulled back to \(\text{Conf} (X, S_2)\) via the projection onto the \(t\) coordinate.
\[ Conf (X, S_1) \subset \mathcal{W} (X, S_1, S_2) \xrightarrow{\rho} \bigvee_{s \in S_2} \mathfrak{C}_t \]

In cohomology with coefficients in a ring \( R \), the above sequence is a sequence of \( H^* (\mathcal{W} (X, S_1, S_2) ; R) \)-modules so the map \( \rho \) is completely determined in cohomology by the images of the Thom classes, \( \rho^*_t (U_t) \).
\[ \text{Conf} (X, S_1) \subset \mathcal{W} (X, S_1, S_2) \xrightarrow{\rho} \bigvee_{s \in S_2} \mathcal{C}_s \]

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If \(X\) is closed, compact and oriented, and if \(R\) is a field \(\mathbb{F}\), Milnor-Stasheff, Thm. 11.11, p. 128, work out this image, \(\Delta \in H^* (X^{\{1,2\}}, \mathbb{F})\).
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\[ \rho_s^* (U_s) = \iota_{s, x}^* (\Delta) \]

where \( \{x\} = S_1 - S_2 \). For any \( s_1, s_2 \in S_1 \) let \( \Delta_{s_2, s_1} = \iota_{s_2, s_1}^* (\Delta) \).
The case $X = M \times \mathbb{R}^2$
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The map $Conf(X, \{1, 2\}) \subset W(X, S_1, S_2)$ has a retraction $r: W(X, S_1, S_2) \to Conf(X, \{1, 2\})$ such that the composition

$$W(X, S_1, S_2) \xrightarrow{r} Conf(X, \{1, 2\}) \subset W(X, S_1, S_2)$$

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$$r(\iota)(s) = \begin{cases} 
  e_1(\iota(s)) & s \in S_2 \\
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Note any cohomology class $U$ with $r^*(U) = 0$ comes from a unique class in the mapping cone. In the manifold case, the mapping cone is a Thom space. If $X$ is also orientable of dimension $n$, then there exists a unique element $U \in H^{n-1} \left( \text{Conf} \left( X, \{1, 2\} \right); \mathbb{Z} \right)$ such that $r^*(U) = 0$ and $U$ comes from the Thom class.
The case $X = M \times \mathbb{R}^2$

For every ordered pair of elements $(s_1, s_2)$ with $s_1, s_2 \in S$ define $\iota_{s_2, s_1}: \{1, 2\} \to S$ by $\iota_{s_2, s_1}(j) = s_j$, $j = 1, 2$. Let $\iota_{s_2, s_1}$ also denote the induced map of configuration spaces. Define

$$A_{s_2, s_1} \in H^{n-1}(\text{Conf}(X, S) ; \mathbb{Z})$$

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With field coefficients $H^*(Conf(X, S); \mathbb{F})$ is the quotient of $S[A_{s_2, s_1}] \otimes H^*(M; \mathbb{F}) \otimes S$ modulo the relations below. Here $S[A_{s_2, s_1}]$ is the graded symmetric algebra on the elements $A_{s_2, s_1}$, $s_1, s_2 \in S$. 

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4. Given $m \in H^r(M; \mathbb{F})$ and $s \in S$ let $[m]_s$ denote the element $1 \otimes \cdots \otimes m \otimes \cdots \otimes 1 \in H^r(M; \mathbb{F}) \otimes S$ which has a 1 in every position but the $s^{th}$ where it is $m$. 
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[m]_{s_2} A_{s_2, s_1} = [m]_{s_1} A_{s_2, s_1}
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A decomposition of the suspension of $Conf(M \times \mathbb{R}, S_1)$

The inclusion $Conf(M \times \mathbb{R}, S) \subset W(M \times \mathbb{R}, S_1, S_2)$ is also split.
A decomposition of the suspension of $\text{Conf} (M \times \mathbb{R}, S_1)$

The inclusion $\text{Conf} (M \times \mathbb{R}, S_1) \subset \mathcal{W} (M \times \mathbb{R}, S_1, S_2)$ is also split. The cofibre of this inclusion is the Thom complex of the tangent bundle to $M \times \mathbb{R}$ pulled back to $\text{Conf} (M \times \mathbb{R}, S_2)$,
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Denote the Thom complex of the tangent bundle to $\mathbb{M}$ pulled back to $Conf(\mathbb{M} \times \mathbb{R}, S_2)$ by $T_s(\tau \mathbb{M})$. It follows that $\Sigma Conf(\mathbb{M} \times \mathbb{R}, S_1) \sim \Sigma \mathcal{W}(\mathbb{M} \times \mathbb{R}, S_1, S_2) \vee \bigvee_{s \in S_2} \Sigma T_s(\tau \mathbb{M})$.

One can apply the cofibration method to bundles over $\mathbb{X}$ and eventually see that $\Sigma Conf(\mathbb{M} \times \mathbb{R}, S_1)$ is a wedge to Thom complexes of sums of pulled back tangent bundles over products of copies of $\mathbb{M}$. In particular $\Sigma Conf(\mathbb{R}^n, S_1)$ is a wedge of spheres.

We will return to the question of enumerating these Thom spaces later.
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The Totaro spectral sequence for manifolds

Given any graded commutative ring, $B$, we can form a new graded commutative ring $S[A_{s_2,s_1}] \otimes B \otimes S$ modulo the four relations

- $A_{s_2,s_1} = (-1)^{n-1}A_{s_1,s_2}$
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Theorem (Totaro 1993-6)

Let $M$ be a manifold of dimension $n \geq 2$. The Leray spectral sequence for the inclusion $\text{Conf}(M,S) \subset M^S$ with field coefficients $\mathbb{F}$ has $E^2$ term given by applying the construction above with $B = H^*(M; \mathbb{F})$. The $d_2$ differential is determined by $d_2(A_{s_2,s_1}) = \Delta_{s_2,s_1}$ where $\Delta_{s_2,s_1} = i_{s_2,s_1}^* (\Delta)$ and where $\Delta$ is the diagonal class $\Delta \in H^n(M^{\{1,2\}}; \mathbb{F})$. 
The Totaro spectral sequence for manifolds

Given any graded commutative ring, $B$, we can form a new graded commutative ring $S[A_{s_2,s_1}] \otimes B^{\otimes S}$ modulo the four relations

- $A_{s_2,s_1} = (-1)^{n-1} A_{s_1,s_2}$
- $A_{s_2,s_1}^2 = 0$
- $A_{s_2,s_1} A_{s_3,s_1} = A_{s_2,s_1} (A_{s_3,s_1} - A_{s_3,s_2})$
- Given $m \in B$ and $s \in S$, $[m]_{s_2} A_{s_2,s_1} = [m]_{s_1} A_{s_2,s_1}$

Theorem (Totaro 1993-6)

Let $M$ be a manifold of dimension $n \geq 2$. The Leray spectral sequence for the inclusion $\text{Conf} (M, S) \subset M^S$ with field coefficients $\mathbb{F}$ has $E^2$ term given by applying the construction above with $B = H^*(M; \mathbb{F})$. The $d_2$ differential is determined by $d_2(A_{s_2,s_1}) = \Delta_{s_2,s_1}$ where $\Delta_{s_2,s_1} = i_{s_2,s_1}^* (\Delta)$ and where $\Delta$ is the diagonal class $\Delta \in H^n(M^{\{1,2\}}; \mathbb{F})$. The spectral sequence converges as a $\Sigma_S$-algebra to $H^*(\text{Conf} (M, S); \mathbb{F})$. 
Remark: Totaro shows that if $M$ is a smooth complex projective variety and $\mathbb{F}$ has characteristic zero, $d_2$ is the only differential. Felix and Thomas prove the same result for any formal manifold, which extends the result to spaces like products of spheres.

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Remark: Aouina & Klein and Cohen & Taylor show that homotopy equivalent manifolds give stably-homotopy equivalent configuration spaces. In particular, their Totaro spectral sequences have the same $E_2$, the same $d_2$ and the same $\oplus_{p+q=r} E_{\infty}^{p,q}$ for each $r$.

It is tempting to conjecture that the spectral sequences are isomorphic even though there is no obvious map between them.
More results in the $M \times \mathbb{R}$ case (START HERE)

To describe the various Thom spaces which go into the decomposition of $\Sigma \text{Conf} (M, S)$, begin by discussing 1-dimensional CW complexes.

Given a finite set $S$, an ordered 1-complex $\Gamma$ is a CW complex with vertex set $S$ and a set of edges $E(\Gamma)$. Each edge is oriented and the set of edges is ordered. Given an edge $e \in E(\Gamma)$ define $A_e = A_{s_2} \cdot \cdots \cdot A_{s_1}$ where $e$ starts at vertex $s_1$ and ends at vertex $s_2$. Define $A_{\Gamma} = A_{e_1} \cdots A_{e_k}$ where $e_1$, ..., $e_k$ are the edges of $\Gamma$ in order. These conventions set up a bijection between products of the $A$'s and ordered 1-complexes.

It can be shown that $A_{\Gamma} \neq 0$ if and only if $H_1(\Gamma) = 0$. Hence $A_{\Gamma} \neq 0$ if and only if each path component of $\Gamma$ is a tree or a single vertex. If we continue the arboreal theme by calling components with single vertices seeds, then $A_{\Gamma} \neq 0$ if and only if $\Gamma$ is a forest.
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\[ s_1 \quad e_1 \quad e_2 \quad s_2 \]

Draw a new edge from $s_1$ to $s_2$, provided $e_1 < e_2$, to get the triangle on the next page.
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![Diagram](image)

Draw a new edge from \( s_1 \) to \( s_2 \), provided \( e_1 < e_2 \), to get the triangle on the next page.
The three-term relation says that a combination of three ordered 1-complexes is 0. They are obtained by combining the three ways of deleting an edge from the triangle, and reordering an edge or two.

\[
\begin{align*}
\lambda_1 &+ \lambda_2 + \lambda_3 = 0
\end{align*}
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\[
A_{\Gamma_3} + A_{\Gamma_1} - A_{\Gamma_2} = 0
\]
Theorem

Given a vertex which supports a three-term relation then for the three graphs described above

\[ H_\ast(\Gamma_3) \cong H_\ast(\Gamma_2) \cong H_\ast(\Gamma_1) \]

A graph partitions its set of vertices by saying two are equivalent if and only if they lie in the same path component. All three graphs yield the same partition.
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Certain collections of ordered 1-complexes give a basis for \( H^*(Conf(\mathbb{R}^n, S) ; \mathbb{Z}) \). Clearly the ordered 1-complexes in a basis must be a forest, but there are more forests than basis elements whenever \(|S| > 2|\).
One basis is given by the admissible forests. To define when an forest is admissible, it is first necessary to order $S$. Then we can orient an edge by starting at the smaller vertex and going to the larger. We can order the edges using lexicographical order. A forest is \textit{admissible} provided no vertex supports an incoming three-term relation using the above orientations and ordering.

\textbf{Theorem}

If $\mathcal{A}(S)$ is the set of admissible forests on the ordered vertex set $S$ then the elements $A_\Gamma$ for all $\Gamma \in \mathcal{A}(S)$ are an additive basis for $H^*(\text{Conf} (\mathbb{R}^n, S);\mathbb{Z})$, $n \geq 2$. 
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**Theorem**

*If $A(S)$ is the set of admissible forests on the ordered vertex set $S$ then the elements $A_\Gamma$ for all $\Gamma \in A(S)$ are an additive basis for $H^\ast(Conf(\mathbb{R}^n, S); \mathbb{Z})$, $n \geq 2$.*

For any forest $\Gamma$ there is a diagonal

$$\Delta_\Gamma : X^{\pi_0(\Gamma)} \rightarrow X^S$$

defined by $(\Delta_\Gamma(\iota))(s) = \iota([s])$ where $[s] \in \pi_0(\Gamma)$ is the path component of $\Gamma$ containing $s$. If $X$ is a manifold, let $\nu_\gamma$ be the normal bundle of $X^{\pi_0(\Gamma)}$ in $X^S$. Note it is a sum of various tangent bundles of $X$ pulled back to $X^{\pi_0(\Gamma)}$. 
Let \( \mathfrak{A}(S) \) be a set of forests such that the collection \( A_\Gamma, \Gamma \in \mathfrak{A}(S) \) is a basis for \( H^*(Conf(\mathbb{R}^n, S); \mathbb{Z}) \).

Then

\[
\sum Conf(M \times \mathbb{R}^1, S) \cong \bigvee_{\Gamma \in \mathfrak{A}(S)} \sum T(\nu_\Gamma)
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**Remark:** The admissible basis has an additional property that there is an algorithm for writing any forest as a linear combination of admissible forests.
The top representation

The sub-group of $H^*(Conf(\mathbb{R}^n, S); \mathbb{Z})$ generated by all $A_\Gamma$ with the associated partition fixed form a subgroup of $H^{(n-1)(|S|-r)}(Conf(M,S); \mathbb{Z})$ where $r$ is number of path components of $\Gamma$, which is also the number of elements in the partition.
Hence the highest non-trivial cohomology group of $Conf(\mathbb{R}^n, S)$ is in dimension $(n-1)(|S|-1)$. Classes $A_\Gamma$ in this dimension come from forests which are connected and vice versa.
From this one sees that $H^*(Conf(\mathbb{R}^n, S); \mathbb{Z})$ is built up out of tensor products of top dimensional groups for various subsets of $S$.

Example

Let $S = \{1, 2, 3, 4, 5\}$ and let $\{\{1, 2, 3\}, \{4, 5\}\}$ be a partition. Then a sumand of $H^{3(n-1)}(Conf(\mathbb{R}^n, S); \mathbb{Z})$ is a tensor product of the top group for 3 points tensor the top group for 2 points.
The top representation (continued)

Recall every forest partitions the set $S$ and by taking the cardinality of each set in the partition, we get a partition of the integer $|S|$. Given any two forests with the same integer partition, there are permutations of $S$ which take one to the other.

Hence under the action of the symmetric group, the cohomology decomposes into summands corresponding to integer partitions of $|S|$. The cohomological degree of the corresponding $A_\Gamma$ can be determined from the integer partition.

Leher & Solomon wrote down the Poincaré character for the rational representation on $H^*(Conf(\mathbb{R}^n, S); \mathbb{Q})$.

Fred & I worked out the representation over $\mathbb{Z}$ as a sum of tensor products of representations induced from Young subgroups: i.e. subgroups of the form

$$\Sigma S_1 \times \cdots \times \Sigma S_k \subset \Sigma S_1 \perp \cdots \perp S_k$$
The top representation comes from the partition with one subset (or one integer). For example, one basis for this group consists of the admissible trees. Notice however that a permutation applied to an admissible tree is often not admissible.
Here is an admissible tree on \( \{a, b, c, d, e, f\} \) ordered alphabetically.
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![Tree Diagram]

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The permutation \( (abcdef) \) applied to the above admissible tree gives the tree which is no longer admissible: the orientation on the edge between \( b \) and \( c \) has the “wrong” orientation.

If we reorient this edge “correctly”, then \( c \) supports an incoming three-term relation.
There are other bases for this top rep which are useful. A *rooted tree* is a tree with a distinguished vertex. A tree is called *linear* provided every vertex has valence 1 or 2. There must be exactly two vertices of valence 1. Fix one of the vertices of valence one, say \( v \).

A *linear rooted tree with root* \( v \) is a linear tree with vertex set \( S \) with one vertex of valence 1 being \( v \). Direct the edges so that you start at \( v \) and just keep going. Number the edges in the order in which they appear along the tree starting at the root. Here are the two rooted trees on \( \{1, 2, 3\} \):

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\bullet & \bullet & \bullet \\
\end{array}
\quad
\begin{array}{ccc}
1 & 3 & 2 \\
\bullet & \bullet & \bullet \\
\end{array}
\]

This shows that the top rep as an integral representation of the symmetric subgroup of \( \Sigma_S \) fixing \( v \) is free.
$C(M, X)$

Fix a countably infinite set, say $\mathbb{N}$. Given a space $M$ and a based space $(X, \ast)$ consider the space of maps $f: \mathbb{N} \to M \times X$. Define the support of $f$ to be the subset of $\mathbb{N}$ such that $f(s) \neq \ast$. Let $E(M, X)$ be the subspace of functions whose support is a finite subset, say $S$, and such that the composition $S \xrightarrow{f} M \times X \to M$ is injective.

Define

$$C(M, X) = E(M, X) / \approx$$

where $\approx$ is the equivalence relation generated by the following two types of relations:

1. $f_1 \approx f_2$ if $f_1$ and $f_2$ have the same support and are equal when restricted to that support

2. $f_1 \approx f_2$ if there exists a bijection $\phi: \mathbb{N} \to \mathbb{N}$ such that $f_1 \circ \phi = f_2$
$C(M, X)$ (canonical identifications)

Since any two countably infinite sets are bijectively equivalent, the choice of set $\mathbb{N}$ is not usually important: if it is we will write $C_N(M, X)$. 

Define the braid space $B(M, S)$ to be $\text{Conf}(M, S)/\Sigma S$. Given another finite set $T$ of the same cardinality, any choice of bijection $\phi: T \to S$ induces a homeomorphism $\phi: \text{Conf}(M, S) \to \text{Conf}(M, T)$ which descends to a homeomorphism $B(M, S) \to B(M, T)$. The remark is that two different $\phi$'s induced the same map on the braid spaces so they may be canonically identified. In the sequel we will write $B_k(M)$ whenever the index set has cardinality $k$. 
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Since any two countably infinite sets are bijectively equivalent, the choice of set $\mathbb{N}$ is not usually important: if it is we will write $C_{\mathbb{N}_1}(M, X)$.

Any bijection $\phi: \mathbb{N}_1 \rightarrow \mathbb{N}_2$ induces a homeomorphism $C_{\mathbb{N}_2}(M, X) \rightarrow C_{\mathbb{N}_1}(M, X)$. Thanks to relation (2), any two $\phi$ induce identical maps. In particular, any two versions of this construction can be canonically identified.
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In a few pages we will also need the following related remark. Define the braid space $B(M, S)$ to be $Conf(M, T)/\Sigma_T$. Given another finite set $T$ of the same cardinality, any choice of bijection $\phi: T \to S$ induces a homeomorphism $\phi: Conf(M, S) \to Conf(M, T)$ which descends to a homeomorphism $B(M, S) \to B(M, T)$.

The remark is that two different $\phi$’s induced the same map on the braid spaces so they may be canonically identified. In the sequel we will write $B_k(M)$ whenever the index set has cardinality $k$. 
Filter \( C(M, X) \) by letting \( F_k(M, X) \subset C(M, X) \) be the image of all functions in \( E(M, X) \) whose support has at most \( k \) elements. Notice both relations (1) and (2) preserve the cardinality of the support of the functions.

Define \( D_k(M, X) \) to be the cofibre of the inclusion \( F_{k-1}(M, X) \subset F_k(M, X) \). If \((X, \ast)\) is an NDR pair then so is \((F_k(M, X), F_{k-1}(M, X))\) and we can identify the cofibre. Fix a finite set of cardinality \( k \), \( S \subset \mathbb{N} \). The composition \( \text{Conf}(M, S) \times X^S \rightarrow F_k(M, X) \) is onto and factors through the orbit space \( \text{Conf}(M, S) \times \Sigma_S X^S \).

The map \( \text{Conf}(M, S) \times \Sigma_S X^S \rightarrow D_k(M, X) \) is onto and if \( F\Delta \subset X^S \) is the set of points with at least one coordinate the base point, then

\[
\text{Conf}(M, S) \times \Sigma_S X^S / \text{Conf}(M, S) \times \Sigma_S F\Delta \rightarrow D_k(M, X)
\]

is a homeomorphism.
Any other choice of finite set of cardinality $k$ gives a similar identification and any choice of bijection induces the same map. With a bit of fiddling, one can rewrite $D_k(M, X)$ as

$$D_k(M, X) = \text{Conf}(M, S) \ltimes_{\Sigma S} X^[[S]]$$

where $X^[[S]]$ denotes the $S$-fold smash product.
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We would like to extend the natural map $f_k : F_k (M, X) \to D_k (M, X)$ to a map $C (M, X) \to D_k (M, X)$ but this is not usually possible.
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It is however possible to do so stably.
Stable splitting of $C(M, X)$

To describe the extension, first try the most naive thing you can think of: given $f$ with support $S$ of cardinality bigger than $k$, just restrict to some subset of cardinality $k$ and pass to the quotient $D_k(M, X)$. The obvious problem is which subset to take. The solution in situations like this where there is no natural choice is to take all choices.

Let $\mathcal{N}' = \binom{\mathbb{N}}{k}$ denote the set of all subsets of $\mathbb{N}$ of cardinality $k$. Note $\mathcal{N}'$ is also countably infinite.
Define a map

\[ h_k : C_N(M, X) \to C_{N'}(B_k(M), D_k(M, X)) \]

as follows. Recall that a point in \( E(M, X) \) is a map \( f : N \to M \times X \) satisfying some additional conditions. We need to define a map \( h_k(f) : N' \to B_k(M) \times D_k(M, X) \). An element of \( N' \) is a set \( S \subset N \) of cardinality \( k \). Hence \( f \) restricted to \( S \) is a point in \( F_k(M, X) \) and hence a point in \( D_k(M,X) \), denote this point by \( [f]_S \). If \( S \) is not in the support of \( f \), \([f]_S \) is the base point. If \( S \) is in the support of \( f \), let \( \langle f \rangle_S \in B_k(M) \) denote the image in \( B_k(M) \) of the point in \( Conf(M, S) \) given by the composition

\[ S \subset N \xrightarrow{f} M \times X \to M \]

If \( S \) is not in the support of \( f \), define \( \langle f \rangle_S \) to be any point you like in \( B_k(M) \). Define

\[ h_k(f)(S) = \langle f \rangle_S \times [f]_S \]
Check that the support of $h_k(f)$ is the set of all subsets of cardinality $k$ contained in the support of $f$. The additional requirements to be a point in $C_{N'} (B(M, k), D_k (M, X))$ can be checked.

Since $B_k (M)$ is a manifold, it embeds in $\mathbb{R}^K$ for some $K$ and so there is a map

$$h_k: C_N (M, X) \rightarrow C_{N'} \left( \mathbb{R}^K, D_k (M, X) \right)$$

Moreover, $D_k (M, X)$ is path-connected and so a theorem of Peter May supplies a map $C_{N'} \left( \mathbb{R}^K, D_k (M, X) \right) \rightarrow \Omega^K \Sigma^K D_k (M, X)$ and so we get a map

$$h_k: C_N (M, X) \rightarrow \Omega^K \Sigma^K D_k (M, X)$$

and a commutative diagram
To belabor the point, we could adjoint $h_k$ and note

$$
\Sigma^K f_k: \Sigma^K F_k (M, X) \subset \Sigma^K C (M, X) \xrightarrow{\text{adh} h_k} \Sigma^K D_k (M, X)
$$

The $K$ certainly increases as $k$ increases, but if we pass to the stable world we get an equivalence

$$
QC (M, X) \cong Q \underset{k=1}{\bigvee}^{\infty} D_k (M, X)
$$

There are many results concerning this construction and its pieces.
Remark: $QC (M, S^0) = Q \mathop{\vee}_{k=1}^{\infty} B(M, k)$

Remark: For an appropriate $K$ there is a map

$$\Sigma^{k \cdot K} D_k (M, X) \to D_k \left( M, \Sigma^K X \right)$$

which is a homotopy equivalence.

Remark: A theorem of Jeff Caruso says

$$\Omega C (M, \Sigma X) \cong C (M \times \mathbb{R}, X)$$

This generalizes the case in which $M = \mathbb{R}^n$ due to Peter May.
Remark: If $X$ is path connected, then a theorem of Bödigheimer says that $C(M, X)$ is weakly-homotopy equivalent to a space of sections of a certain bundle $E \to M$. The bundle is formed from the tangent bundle $T \to M$ as follows. Take the fibre-wise one point compactification of the tangent bundle: take $T \perp M \to M$ and topologize so that each fibre is the one-point compactification of the fibre of $T$. This bundle is denoted $\hat{T}$ and it has a section at infinity. Then $E$ is the reduced fibre-wise smash of $\hat{T}$ with $X$. This a bundle with fibre the reduced suspension $\Sigma^n X$. The reduced suspension has a base point and so $E$ has a section at infinity, $\sigma_\infty$. A section $\sigma : M \to E$ has support the set of all points $m \in M$ such that $\sigma(m) \neq \sigma_\infty(m)$. A section has compact support provided the closure of the support is compact.

Bödigheimer says that $C(M, X)$ is weakly-homotopy equivalent to the space of sections with compact support of $E \to M$. 
Corollary

$C(M, X)$ is a proper homotopy invariant of $M$.

Of course up to homotopy type $C(M, X)$ only depends on the homotopy type of $X$. 