Fibrations, cofibrations and related results, II

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Configuration spaces, braids and applications

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More results in the $M \times \mathbb{R}$ case

To describe the various Thom spaces which go into the decomposition of $\Sigma Conf (M \times \mathbb{R}, S)$, begin by discussing 1-dimensional CW complexes. Given a finite set $S$, an ordered 1-complex $\Gamma$ is a CW complex with vertex set $S$ and a set of edges $\mathcal{E}(\Gamma)$. Each edge is oriented and the set of edges is ordered.

Given an edge $e \in \mathcal{E}(\Gamma)$ define $A_e = A_{s_2,s_1}$ where $e$ starts at vertex $s_1$ and ends at vertex $s_2$. Define $A_\Gamma = A_{e_1} \cdots A_{e_k}$ where $e_1, \ldots, e_k$ are the edges of $\Gamma$ in order. These conventions set up a bijection between products of the $A$’s and ordered 1-complexes. It can be shown that

$$A_\Gamma \neq 0 \text{ if and only if } H_1(\Gamma) = 0.$$ 

Hence $A_\Gamma \neq 0$ if and only if each path component of $\Gamma$ is a tree or a single vertex. If we continue the arboreal theme by calling components with single vertices seeds, then $A_\Gamma \neq 0$ if and only if $\Gamma$ is a forest.
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![Diagram](image)

Draw a new edge from $s_1$ to $s_2$, provided $e_1 < e_2$, to get the triangle on the next page.
The three-term relation says that a combination of three ordered 1-complexes is 0. They are obtained by combining the three ways of deleting an edge from the triangle, and reordering an edge or two.

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\[ A_{\Gamma_3} + A_{\Gamma_1} - A_{\Gamma_2} = 0 \]
Theorem

Given a vertex which supports a three-term relation then for the three graphs described above

\[ H_*(\Gamma_3) \cong H_*(\Gamma_2) \cong H_*(\Gamma_1) \]

A graph partitions its set of vertices by saying two are equivalent if and only if they lie in the same path component. All three graphs yield the same partition.
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Proof.
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One basis is given by the *admissible forests*. To define when an forest is admissible, it is first necessary to order $S$. Then we can orient an edge by starting at the smaller vertex and going to the larger. We can order the edges using lexicographical order. A forest is *admissible* provided no vertex supports an incoming three-term relation using the above orientations and ordering.

**Theorem**

*If $\mathcal{A}(S)$ is the set of admissible forests on the ordered vertex set $S$ then the elements $A_\Gamma$ for all $\Gamma \in \mathcal{A}(S)$ are an additive basis for $H^*(\text{Conf}(\mathbb{R}^n, S); \mathbb{Z})$, $n \geq 2$.***
For any forest $\Gamma$ there is a diagonal

$$\Delta_\Gamma : X^{\pi_0(\Gamma)} \to X^S$$

defined by $(\Delta_\Gamma(\iota))(s) = \iota([s])$ where $[s] \in \pi_0(\Gamma)$ is the path component of $\Gamma$ containing $s$. 
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Remark: The admissible basis has an additional property that there is an algorithm for writing any forest as a linear combination of admissible forests.
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The top representation

The sub-group of $H^\ast(Conf(\mathbb{R}^n, S); \mathbb{Z})$ generated by all $A_\Gamma$ with the associated partition fixed form a subgroup of $H^{(n-1)(|S|-r)}(Conf(\mathbb{R}^n, S); \mathbb{Z})$ where $r$ is number of path components of $\Gamma$, which is also the number of elements in the partition.
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Classes $A_\Gamma$ in this dimension come from forests which are connected and vice versa. From this one sees that $H^*(Conf(\mathbb{R}^n, S); \mathbb{Z})$ is built up out of tensor products of top dimensional groups for various subsets of $S$. 

Example Let $S = \{1, 2, 3, 4, 5\}$ and let $\{\{1, 2, 3\}, \{4, 5\}\}$ be a partition. Then a summand of $H^3((n-1)(|S| - r))(Conf(\mathbb{R}^n, S); \mathbb{Z})$ is a tensor product of the top group for 3 points tensor the top group for 2 points.
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Recall every forest partitions the set $S$ and by taking the cardinality of each set in the partition, we get a partition of the integer $|S|$. 

Leher & Solomon wrote down the Poincaré character for the rational representation on $H^*(\text{Conf}(\mathbb{R}^n, S); \mathbb{Q})$. Fred & I worked out the representation over $\mathbb{Z}$ as a sum of tensor products of representations induced from Young subgroups: i.e. subgroups of the form $\Sigma_{S_1} \times \cdots \times \Sigma_{S_k} \subset \Sigma_{S_1} \perp \cdots \perp S_k$. 

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Here is an admissible tree on \( \{a, b, c, d, e, f\} \) ordered alphabetically.

The permutation \((abcfde)\) applied to the above admissible tree gives the tree which is no longer admissible: the orientation on the edge between \(b\) and \(c\) has the "wrong" orientation. If we reorient this edge "correctly", then \(c\) supports an incoming three-term relation.
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Theorem: The set of linear trees with root $v$ is a basis for $H(n - 1)(|S| - 1)$ $(\text{Conf}(R_n, S); \mathbb{Z})$. This shows that the top rep as an integral representation of the symmetric subgroup of $\Sigma$ fixing $v$ is free.
There are other bases for this top rep which are useful. A rooted tree is a tree with a distinguished vertex. A tree is called linear provided every vertex has valence 1 or 2. There must be exactly two vertices of valence 1. Fix one of the vertices of valence one, say v.
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A linear rooted tree with root v is a linear tree with vertex set S with one vertex of valence 1 being v. Direct the edges so that you start at v and just keep going. Number the edges in the order in which they appear along the tree starting at the root.
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\begin{array}{ccc}
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\bullet & \bullet & \bullet \\
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Fix a countably infinite set, say $N$. Given a space $M$ and a based space $(X, \ast)$ consider the space of maps $f : N \to M \times X$. Define the support of $f$ to be the subset of $N$ such that $f(s) \neq \ast$. Let $E(M, X)$ be the subspace of functions whose support is a finite subset, say $S$, and such that the composition $Sf \to M \times X \to M$ is injective. Define $C(M, X) = E(M, X)$ where $\approx$ is the equivalence relation generated by the following two types of relations:

1. $f_1 \approx f_2$ if $f_1$ and $f_2$ have the same support and are equal when restricted to that support
2. $f_1 \approx f_2$ if there exists a bijection $\varphi : N \to N$ such that $f_1 \circ \varphi = f_2$
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where $\sim$ is the equivalence relation generated by the following two types of relations:

1. $f_1 \sim f_2$ if $f_1$ and $f_2$ have the same support and are equal when restricted to that support
2. $f_1 \sim f_2$ if there exists a bijection $\phi : N \to N$ such that $f_1 \circ \phi = f_2$
Since any two countably infinite sets are bijectively equivalent, the choice of set $\mathbb{N}$ is not usually important: if it is we will write $C_{\mathbb{N}}(M, X)$.

$C(M, X)$ (canonical identifications)

Define the braid space $B(M, S)$ to be $\text{Conf}(M, S) / \Sigma S$.

Given another finite set $T$ of the same cardinality, any choice of bijection $\phi: T \to S$ induces a homeomorphism $\phi: \text{Conf}(M, S) \to \text{Conf}(M, T)$ which descends to a homeomorphism $B(M, S) \to B(M, T)$.

The remark is that two different $\phi$’s induced the same map on the braid spaces so they may be canonically identified.

In the sequel we will write $B_k(M)$ whenever the index set has cardinality $k$. 
$C(M, X)$ (canonical identifications)

Since any two countably infinite sets are bijectively equivalent, the choice of set $\mathbb{N}$ is not usually important: if it is we will write $C_{\mathbb{N}_1}(M, X)$. Any bijection $\phi: \mathbb{N}_1 \to \mathbb{N}_2$ induces a homeomorphism $C_{\mathbb{N}_2}(M, X) \to C_{\mathbb{N}_1}(M, X)$. 

In a few pages we will also need the following related remark. Define the braid space $B(M, S)$ to be $\text{Conf}(M, T) / \Sigma T$. Given another finite set $T$ of the same cardinality, any choice of bijection $\phi: T \to S$ induces a homeomorphism $\phi: \text{Conf}(M, S) \to \text{Conf}(M, T)$ which descends to a homeomorphism $B(M, S) \to B(M, T)$. The remark is that two different $\phi$'s induced the same map on the braid spaces so they may be canonically identified. In the sequel we will write $B_k(M)$ whenever the index set has cardinality $k$. 
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The remark is that two different \( \phi \)'s induced the same map on the braid spaces so they may be canonically identified. In the sequel we will write \( B_k (M) \) whenever the index set has cardinality \( k \).
Filter $C(M, X)$ by letting $F_k(M, X) \subset C(M, X)$ be the image of all functions in $E(M, X)$ whose support has at most $k$ elements.
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Define $D_k(M, X)$ to be the cofibre of the inclusion $F_{k-1}(M, X) \subset F_k(M, X)$.
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Define $D_k(M, X)$ to be the cofibre of the inclusion $F_{k-1}(M, X) \subset F_k(M, X)$. If $(X, \ast)$ is an NDR pair then so is $(F_k(M, X), F_{k-1}(M, X))$ and we can identify the cofibre.
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Therefore map $Conf(M, S) \times_{\Sigma_S} X^S \to D_k(M, X)$ is onto and if $F\Delta \subset X^S$ is the set of points with at least one coordinate the base point, then

$$(Conf(M, S) \times_{\Sigma_S} X^S \mathbin/ (Conf(M, S) \times_{\Sigma_S} F\Delta) \to D_k(M, X)$$

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We would like to extend the natural map $f_k: F_k(M,X) \to D_k(M,X)$ to a map $C(M,X) \to D_k(M,X)$ but this is not usually possible.
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We would like to extend the natural map $f_k: F_k(M, X) \rightarrow D_k(M, X)$ to a map $C(M, X) \rightarrow D_k(M, X)$ but this is not usually possible. It is however possible to do so stably.
Stable splitting of $C(M, X)$

To describe the extension, first try the most naive thing you (I?) can think of: given $f$ and $T$ any finite set of cardinality $k$, define $f|_T : \mathbb{N} \to M \times X$ by $f|_T(s) = \begin{cases} f(s) & s \in T \\ (m, \ast) & s \notin T \end{cases}$ where $m \in M$ is any point you like.
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Let $\mathbb{N}' = \binom{\mathbb{N}}{k}$ denote the set of all subsets of $\mathbb{N}$ of cardinality $k$. Note $\mathbb{N}'$ is also countably infinite.
Define a map

\[ h_k : C_N(M, X) \rightarrow C_{N'}(B_k(M), D_k(M, X)) \]

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If \( S \) is in the support of \( f \), let \( \langle f \rangle_S \in B_k(M) \) denote the image in \( B_k(M) \) of the point in \( \text{Conf}(M, S) \) given by the composition

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$$h_k(f)(S) = \langle f \rangle_S \times [f]_S$$
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Since \( B_k(M) \) is a manifold, it embeds in \( \mathbb{R}^K \) for some \( K \) and so there is a map

\[
h_k : C_{\mathcal{N}} (M, X) \to C_{\mathcal{N}'} \left( \mathbb{R}^K, D_k(M, X) \right)
\]

Moreover, \( D_k(M, X) \) is path-connected and so a theorem of Peter May supplies a map

\[
C_{\mathcal{N}'} \left( \mathbb{R}^K, D_k(M, X) \right) \to \Omega^K \Sigma^K D_k(M, X)
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$$h_k: C_N(M, X) \to \Omega^K \Sigma^K D_k(M, X)$$

and a commutative diagram
To belabor the point, we could adjoint $h_k$ and note $\Sigma^K F_k(M, X) \subset \Sigma^K C(M, X)$:

\[ F_k(M, X) \xrightarrow{f_k} D_k(M, X) \]

\[ C(M, X) \xrightarrow{h_k} \Omega^K \Sigma^K D_k(M, X) \]

The $K$ certainly increases as $k$ increases, but if we pass to the stable world we get an equivalence $Q(M, X) \sim Q_\infty \lor k=1 D_k(M, X)$.
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\Sigma^K f_k : \Sigma^K F_k (M, X) \subset \Sigma^K C (M, X) \xrightarrow{\text{ad} h_k} \Sigma^K D_k (M, X)
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\[ QC(M, X) \cong Q \bigvee_{k=1}^{\infty} D_k(M, X) \]
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There are many results concerning this construction and its pieces.
Remark: $QC(M, S^0) = Q \bigvee_{k=1}^{\infty} B_k(M)$
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Remark: For an appropriate \( K \) there is a map

\[
\Sigma^{k:K} D_k(M, X) \to D_k\left(M, \Sigma^K X\right)
\]

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This generalizes the case in which \( M = \mathbb{R}^n \) due to Peter May.
Remark: If $X$ is path connected, then a theorem of Bödigheimer says that $C(M, X)$ is weakly-homotopy equivalent to a space of sections of a certain bundle $E \to M$. 

- The bundle is formed from the tangent bundle $T \to M$ as follows. Take the fibre-wise one point compactification of the tangent bundle: take $T \perp M \to M$ and topologize so that each fibre is the one-point compactification of the fibre of $T$. This bundle is denoted $\hat{T}$ and it has a section at infinity. Then $E$ is the reduced fibre-wise smash of $\hat{T}$ with $X$. This a bundle with fibre the reduced suspension $\Sigma^n X$. The reduced suspension has a base point and so $E$ has a section at infinity, $\sigma_\infty$. The support of a section $\sigma: M \to E$ is the set of all points $m \in M$ such that $\sigma(m) \neq \sigma_\infty(m)$. A section has compact support provided the closure of the support is compact.

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Corollary

$C(M, X)$ is a proper homotopy invariant of $M$. 