Bilinear forms and Wu-like cosets.

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Introduction

The squaring map associated to a symmetric bilinear form over a field of characteristic 2 is a linear map. If the form is non-degenerate, there is characteristic class. The middle Wu class of an even dimensional manifold is an example and additional examples will be considered here.

- Bilinear forms
- Linked bilinear forms
- Left/right characteristic cosets
- Examples

Bilinear forms

Basic definitions

Let V and W be vector spaces over $\mathbb{F}_2$. If finite dimension is required, it will be mentioned explicitly.
A *bilinear form* is a function

$$\mu\colon V \times W \to \mathbb{F}_2$$

which is linear in each variable separately.

$$V^\perp \subset W = \left\{ w \in W \;\middle|\; \mu(v,w) = 0 \text{ for all } v \in V \right\}$$

The subspace $W^\perp \subset V$ is defined similarly.

The form is W- *surjective* provided the composition

$$W \xrightarrow{\text{adjoint}} V^* \to \left(V/W^\perp\right)^*$$

is surjective.

Linked bilinear forms

Definitions

A *linked bilinear form* is a pair consisting of a bilinear form $\mu\colon V \times W \to \mathbb{F}_2$ and a linear map $\mathfrak{u}\colon V \to W$.

The *associated bilinear form* is the function

$$A_{\mu,\mathfrak{u}}(v_1, v_2) = \mu\big(v_1, \mathfrak{u}(v_2)\big)\colon V \times V \xrightarrow{1_V \times \mathfrak{u}} V \times W \xrightarrow{\mu} \mathbb{F}_2$$

A linked bilinear form is *symmetric* provided $A_{\mu,\mathfrak{u}}$ is symmetric.

If a group G acts on V and W, μ is *equivariant* provided $\mu(gv, gw) = \mu(v, w)$ for all $v \in V$, $w \in W$ and $g \in G$.

A linked form is *equivariant* provided μ and $\mathfrak{u}$ are. It follows that $A_{\mu,\mathfrak{u}}$ is an equivariant form.

Linked bilinear forms

Remarks

Proposition

If $(\mu, \mathfrak{u})$ *is a symmetric linked bilinear form, then* $\mathfrak{u}(W^{\perp}) \subset V^{\perp}$.

Proposition

If $(\mu, \mathfrak{u})$ *is a symmetric linked bilinear form, then* $K = K_{\mu,\mathfrak{u}} = \mathfrak{u}^{-1}(V^{\perp})$ *is the perpendicular subspace for* V *and the form* $A_{\mu,\mathfrak{u}}$ *on either side.*

The subspace K is also called the annihilator of the form.

Linked bilinear forms

Isometries

An *isometry of linked forms* $(\mu_1, \mathfrak{u}_1)$ to $(\mu_2, \mathfrak{u}_2)$ is a pair of linear maps $\iota_V \colon V_1 \to V_2$ and $\iota_W \colon W_2 \to W_1$ such that

$$\begin{tikzcd}
V_1 \arrow[r, "\mathfrak{u}_1"] \arrow[d, "\iota_V"] & W_1 \\
V_2 \arrow[r, "\mathfrak{u}_2"] & W_2 \arrow[u, "\iota_W"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& V_1 \times W_1 \arrow[dr, "\mu_1"] & \\
V_1 \times W_2 \arrow[ur, "1_{V_1} \times \iota_W"] \arrow[dr, "\iota_V \times 1_{W_2}"'] & & \mathbb{F}_2 \\
& V_2 \times W_2 \arrow[ur, "\mu_2"'] &
\end{tikzcd}$$

commute.

The map $\iota_V \colon V_1 \to V_2$ is an isometry from the associated form $A_{\mu_1, \mathfrak{u}_1}$ to $A_{\mu_2, \mathfrak{u}_2}$.

Linked bilinear forms

Induced isometries

The right hand diagram on the previous page commutes if and only if

$$\mu_1\big(v_1, \iota_W(w_2)\big) = \mu_2\big(\iota_V(v_1), w_2\big) \text{ for all } (v_1, w_2) \in V_1 \times W_2$$

Let (ι_V, ι_W) be an isometry of linked forms $(\mu_1, \mathfrak{u}_1)$ to $(\mu_2, \mathfrak{u}_2)$.

Theorem

$\iota_W(V_2^\perp) \subset V_1^\perp$ *and* $\iota_V(W_1^\perp) \subset W_2^\perp$.

Left/right characteristic cosets

Definitions

Let $f\colon V \to \mathbb{F}_2$ be a linear map and let $(\mu, \mathfrak{u})$ be a symmetric linked form. The *right characteristic coset for* f, $\mathcal{R}_f$, is the set of all $w \in W$ such that $f(v) = \mu(v, w)$ for all $v \in V$. The name is a slight misnomer as $\mathcal{R}_f$ may be empty, but if it is non-empty, it is a coset of $V^\perp \subset W$.

The *left characteristic coset for* f, $\mathcal{L}_f$, is the set of all $\ell \in V$ such that $f(v) = A_{\mu,\mathfrak{u}}(v, \ell)$ for all $v \in V$.
Note the change in form. If $\mathcal{L}_f$ is non-empty, then it is a coset of $K = \mathfrak{u}^{-1}(V^\perp)$.

Left/right characteristic cosets

Naturality

Theorem

Let (ι_V, ι_W) *be an isometry from* $(\mu_1, \mathfrak{u}_1) \to (\mu_2, \mathfrak{u}_2)$. *Let* $f\colon V_2 \to \mathbb{F}_2$ *be a linear map. Then*

$$\iota_W(\mathcal{R}_f) \subset \mathcal{R}_{f\circ\iota_V}$$

Proof.

Let $x \in \mathcal{R}_f$ so $f(v_2) = \mu_2(v_2, x)$ for all $v_2 \in V_2$. For $v_1 \in V_2$, $\iota_V(v_1) \in V_2$ so $f\big(\iota_V(v_1)\big) = \mu_2\big(\iota_V(v_1), x\big)$ for all $v_1 \in V_1$. Then $(f \circ \iota_V)(v_1) = \mu_1\big(v_1, \iota_W(x)\big)$ for all $v_1 \in V_1$, so $\iota_W(x) \in \mathcal{R}_{f\circ\iota_V}$. $\square$

Left/right characteristic cosets

Additional properties

Theorem

Let $(\mu, \mathfrak{u})$ be a linked form and let $f\colon V \to \mathbb{F}_2$ be a linear map. Then $\mathfrak{u}(\mathcal{L}_f) \subset \mathcal{R}_f$.

A classic method of producing linear maps is to use the squaring map associated to a symmetric bilinear form.

Let $(\mu, \mathfrak{u})$ be a symmetric linked form and let $\mathbf{S}_{\mu,\mathfrak{u},}\colon V \to \mathbb{F}_2$ be the squaring map for the associated form.

To simplify notation, define $\mathcal{L}_{\mu,\mathfrak{u}} = \mathcal{L}_{\mathbf{S}_{\mu,\mathfrak{u}}}$ and $\mathcal{R}_{\mu,\mathfrak{u}} = \mathcal{R}_{\mathbf{S}_{\mu,\mathfrak{u}}}$.

These will be called the *left/right characteristic cosets of the symmetric linked form.*

Left/right characteristic cosets

Summary

For W-surjective, symmetric, linked bilinear forms, the following hold.

- $\mathcal{R}_{\mu,\mathfrak{u}}$ is always non-empty.
- If $\dim V/K < \infty$, $\mathcal{L}_{\mu,\mathfrak{u}}$ is non-empty.
- $\mathfrak{u}(\mathcal{L}_{\mu,\mathfrak{u}}) \subset \mathcal{R}_{\mu,\mathfrak{u}}$.
- If there exists $x \in V$ such that $\mathfrak{u}(x) \in \mathcal{R}_{\mu,\mathfrak{u}}$, then $x \in \mathcal{L}_{\mu,\mathfrak{u}}$.
- If (ι_V, ι_W) is an isometry from $(\mu_1, \mathfrak{u}_1)$ to $(\mu_2, \mathfrak{u}_2)$, $\iota_W(\mathcal{R}_{\mu_2,\mathfrak{u}_2}) \subset \mathcal{R}_{\mu_1,\mathfrak{u}_1}$.
- If $\dim V/K < \infty$, $\mu(\mathcal{L}_{\mu,\mathfrak{u}}, \mathcal{R}_{\mu,\mathfrak{u}})$ is the dimension of V/K mod 2.

Examples

Basic set up

Let M^n be a smooth, paracompact, Hausdorff manifold without boundary of dimension n. The examples here all start with the bilinear form, $V = H^k_c(M;\mathbb{F}_2)$, $W = H^{n-k}(M;\mathbb{F}_2)$ and μ is the cup product evaluated on the fundamental class. For $v \in V$ and $w \in W$, write $v \bullet w$ for $\mu(v, w)$.

The examples come from various choices of subspaces $V_M \subset H^k_c(M;\mathbb{F}_2)$ and linking map $\mathfrak{u}\colon V_M \to H^{n-k}(M;\mathbb{F}_2)$. Here the linking map is always of the form $V_M \xrightarrow{\hat{\mathfrak{u}}} H^{n-k}_c(M;\mathbb{F}_2) \xrightarrow{f} H^k_c(M;\mathbb{F}_2)$ where f forgets that the class has compact support. Since the manifold M will determine the linked form in each example write $\mathcal{R}_M$ and $\mathcal{L}_M$ for the right and left characteristic cosets.

Since this form is W-surjective, $\mathcal{R}_M$ always exists.

Examples

Naturality

The tangent bundle map makes both $H^*(M;\mathbb{F}_2)$ and $H_c^*(M;\mathbb{F}_2)$ into $H^*(BO;\mathbb{F}_2)$ algebras.

If $\kappa\colon U \subset M$ is a codimension zero embedding, there are induced maps $\kappa^*\colon H^{n-k}(M;\mathbb{F}_2) \to H^{n-k}(U;\mathbb{F}_2)$ and $\kappa_!\colon H_c^k(U;\mathbb{F}_2) \to H_c^k(M;\mathbb{F}_2)$. The pair $(\kappa_!, \kappa^*)$ is an isometry and both $\kappa_!$ and κ^* are $H^*(BO;\mathbb{F}_2)$ module maps.

Suppose a linked form is defined for each object in some category of manifolds and codimension zero embeddings. The linked form is *natural for codimension zero embeddings* provided $\kappa_!(V_U) \subset V_M$ and

$$\begin{tikzcd} V_U \arrow[r, "\kappa_!"] \arrow[d, "\mathfrak{u}_U"'] & V_M \arrow[d, "\mathfrak{u}_M"'] \\ H^{n-k}(U;\mathbb{F}_2) & H^{n-k}(M;\mathbb{F}_2) \arrow[l, "\kappa^*"] \end{tikzcd}$$

commutes.

Examples

Wu's result

Suppose M has dimension $n = 2k$. Let $V = H_c^k(M; \mathbb{F}_2)$, $W = H^k(M; \mathbb{F}_2)$ and $\hat{\mathfrak{u}}$ is the identity. This is a natural family for all manifolds.

If M is compact, it is a classic result of Wu's that the right characteristic coset of this linked form is $\mathrm{v}_k(M)$ where $\mathrm{v}_k(M)$ is the k^{th} Wu class of the tangent bundle.

Examples

A general result on natural families

Fix some category of manifolds and codimension zero embeddings. Suppose that for every element $x \in H_{n-k}(M;\mathbb{F}_2)$ there exist codimension zero embeddings (in the category) $\kappa_x \colon U_x \subset M$ and $\kappa'_x \colon U_x \subset N$ with N compact such that there exists $y \in H_{n-k}(U_x;\mathbb{F}_2)$ with $(\kappa_x)_*(y) = x$. Call such a category *compactly determined.*

Suppose given two families of elements $a_M, b_M \in H^{n-k}(M;\mathbb{F}_2)$ which are natural for codimension zero embeddings in a compactly determined category of manifolds. If $a_M = b_M$ for all compact M then $a_M = b_M$ for all M.

Examples

Wu's result

Returning to Wu's example, it follows from the previous slide that

- $\mathcal{R}_{\mu,\mathfrak{u}} = \mathrm{v}_k(M)$ for all paracompact manifolds M.
- If M is compact, $\mathcal{L}_{\mu,\mathfrak{u}} = \mathrm{v}_k(M)$.
- If the image of $H^k_c(M;\mathbb{F}_2)$ in $H^k(M;\mathbb{F}_2)$ has finite dimension ℓ then $\mathcal{L}_{\mu,\mathfrak{u}}$ is non-empty and $\mathcal{L}_{\mu,\mathfrak{u}} \bullet \mathrm{v}_k$ is ℓ mod 2.
- If M is a countable connected sum of $\mathbb{CP}^2$'s, then $\mathcal{L}_{\mu,\mathfrak{u}}$ is empty.

Examples

Lusztig-Milnor-Peterson form

This example starts with M^n where $n = 2k+1$; $V = H_c^k(M;\mathbb{F}_2)$, $W = H^{k+1}(M;\mathbb{F}_2)$ and $\hat{\mathfrak{u}} = Sq^1\colon H_c^k(M;\mathbb{F}_2) \to H_c^{k+1}(M;\mathbb{F}_2)$. The associated form is $A_{\mu,Sq^1}(v_1, v_2) = v_1 \bullet Sq^1(v_2)$. This linked form is not always symmetric but if M is orientable, it is. It is also natural for orientable manifolds and all codimension zero embeddings.

Examples

Lusztig-Milnor-Peterson form

Lusztig, Milnor and Peterson showed that if $k = 2\ell$, and M is compact, $\mathcal{R}_M = Sq^1(\mathrm{v}_{2\ell})$. This follows from the formula $\beta\mathcal{P}_{2k}(x) = Sq^{2\ell}Sq^1(x) + x \cup Sq^1(x)$ where β is a Bockstein and $\mathcal{P}_{2k}$ is the Pontryagin square. In the compact case it further follows that $\mathrm{v}_{2\ell} \in \mathcal{L}_M$.

In the case in which M is compact, the characteristic number $v_{2k}Sq^1(v_{2k})$ is the dimension mod 2 of the form on $H^k(M;\mathbb{F}_2)\big/Sq^1\big(H^{k-1}(M;\mathbb{F}_2)\big)$.
With more work Lusztig, Milnor and Peterson identify this with the de Rham invariant.

Examples

Lusztig-Milnor-Peterson form

In case $k = 2\ell + 1$ Ed Miller observed, in the language here, that $\mathcal{R}_M$ was a class which is natural for codimension zero embeddings. The example of $\mathbb{RP}^3$ shows that $\mathcal{R}_M$ is not determined by the tangent bundle.

Examples

Lusztig-Milnor-Peterson form

For non-orientable manifolds the Lusztig-Milnor-Peterson form need not be symmetric: for example $M = \mathbb{RP}^{2k} \times S^1$.

One way to restore symmetry is to take
$W_M = H^{k+1}(M; \mathbb{F}_2)$ and
$V_M = \left\{ v \in H^k_c(M; \mathbb{F}_2) \;\middle|\; v \cup \mathrm{w}_1(M) = 0 \in H^{k+1}_c(M; \mathbb{F}_2) \right\}$
where $\mathrm{w}_1(M)$ is the first Stiefel-Whitney class. The associated form is symmetric.

However $V_M^\perp = \big(\mathrm{w}_1(M)\big) H^k(M; \mathbb{F}_2)$ so in general $\mathcal{R}_M$ is a proper coset.

The coset $\mathcal{R}_M$ is natural for codimension zero embeddings.

Examples

Lusztig-Milnor-Peterson form

This gives a different partial answer to Ed Miller's question on the existence of natural characteristic classes which are not determined by the tangent bundle. The Lusztig-Milnor-Peterson coset is defined on all manifolds of odd dimension. In dimensions $4\ell + 3$ the coset is not determined by the tangent bundle.

Examples

Lusztig-Milnor-Peterson form

A simple 3-manifold theorem

Let M^3 be a 3-manifold with $0 \in \mathcal{R}_M$ and let $e\colon \mathbb{RP}^2 \to M$ be an embedding. Then $e(\mathbb{RP}^2)$ is 2-sided.

Corollary

Let M^3 be an orientable 3-manifold with $0 = \mathcal{R}_M$. Then there is no embedding $e\colon \mathbb{RP}^2 \to M$.

Examples

Landweber-Stong form

If M^n is a Spin manifold of dimension $n = 2k+2$, Landweber and Stong observed that $\hat{\mathfrak{u}} = Sq^2\colon H_c^k(M;\mathbb{F}_2) \to H_c^{k+2}(M;\mathbb{F}_2)$ gives a symmetric linked form.
In dimension $8k+2$ they show that the right characteristic element does not come from any element in $H^{4k+2}(BO;\mathbb{F}_2)$ but restricted to the kernel of Sq^1, the right characteristic coset contains $Sq^2(v_{4k})$.
Restricting to the kernel of $\cup\,\mathrm{v}_2$ and $\cup\,\mathrm{v}_1$ extends the form to all manifolds and has appropriate characteristic cosets.

Examples

Some remarks on $H^*(BO; \mathbb{F}_2)$

For each integer $k \geqslant 0$, there exists a linear transformation

$$\mathfrak{l}^k \colon H^\ell(BO; \mathbb{F}_2) \to H^{k+\ell}(BO; \mathbb{F}_2)$$

defined for any $\omega \in H^\ell(BO; \mathbb{F}_2)$ by

$$\mathfrak{l}^k(\omega) = \sum_{i=0}^{k} \mathrm{v}_i \cup \chi(Sq^{k-i})(\omega)$$

Cultural remark: $\mathfrak{l}^k(\omega)$ vanishes for all M^n, $n < 2k + \ell$ by Brown and Peterson's work on vanishing characteristic classes.

Examples

Wu shifted by Stiefel-Whitney

If M has dimension $n = 2k + \ell$ and if $\hat{\mathfrak{u}}(x) = x \cup \omega$

$$x \bullet (x \cup \omega) = x \bullet \mathfrak{l}^k(\omega)$$

Note $\mathfrak{l}^k(1) = \mathrm{v}_k$ so this is a generalization of Wu's result.

Examples

Lusztig-Milnor-Peterson shifted by Stiefel-Whitney

Suppose the dimension of M is $n = 2k + \ell + 1$.

Note for any space X and class $x \in H^k(X; \mathbb{F}_2)$,
$\mathfrak{l}^1(x) = (\mathrm{w}_1 \cup x) + Sq^1(x)$ is Greenblatt's twisted Bockstein.
Hence $\mathfrak{l}^1\big(\mathfrak{l}^1(x)\big) = 0$.

If $\mathfrak{l}^1(\omega) = 0$, $x \bullet (\omega \cup Sq^1(x))$ is symmetric.

Since $H^k(BO; \mathbb{Z}^{\mathrm{w}_1})$ is a $\mathbb{F}_2$ vector space, $\mathfrak{l}^1(\omega) = 0$ if and only if there exist some class κ_ω such that $\mathfrak{l}^1(\kappa_\omega) = \omega$. Then

$$x \bullet (\omega \cup Sq^1(x)) = x \bullet \mathfrak{l}^1\big(\mathfrak{l}^{k+1}(\kappa_\omega)\big)$$

Examples

Equivariant Wu

Suppose a finite 2 group G acts on a manifold M of dimension $n = 2k + \ell$. For any $x \in H^r(M; \mathbb{F}_2)$, define

$$\mathbf{N}(x) = \sum_{g \in G} g^*(x)$$

Fix $\omega \in H^\ell(BO; \mathbb{F}_2)$.

Theorem

Let G act freely on M. Let $K = \ker(\mathbf{N} \cup \omega) \colon H_c^k(M; \mathbb{F}_2) \to H_c^{k+\ell}(M; \mathbb{F}_2)$ and suppose $H_c^k(M; \mathbb{F}_2)/K$ has finite dimension. Then there exist $\eta \in H_c^k(M; \mathbb{F}_2)$ such that

$$\mathfrak{l}^k(\omega) = \mathbf{N}(\eta \cup \omega)$$

Corollary

Suppose given the hypotheses of the theorem.

If $\mathfrak{l}^k(\omega) \neq 0 \in H^{k+\ell}(M; \mathbb{F}_2)$, $\mathbb{F}_2[G] \subset H^{k+\ell}(M; \mathbb{F}_2)$.

Examples

Equivariant Wu

Let $M_1 = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \# (S^1 \times S^3) \# (S^1 \times S^3)$ and let $M_2 = S^2 \times S^1 \times S^1$. The manifolds M_1 and M_2 have the same homology and the same signatures. The manifold M_1 has a free $\mathbb{Z}/2\mathbb{Z}$ action and M_2 has free cyclic group actions for all orders. Since $\mathrm{v}_2(M_1) \neq 0$, M_1 has no free action by any group of order 4.

Examples

Equivariant Wu

The proof of the theorem proceeds by considering the linking map

$$\hat{\mathfrak{u}}\colon H_c^k(M;\mathbb{F}_2) \xrightarrow{\mathbf{N}} H_c^k(M;\mathbb{F}_2) \xrightarrow{\cup\,\omega} H_c^{k+\ell}(M;\mathbb{F}_2)$$

It also uses a result of Bredon's that for free involutions τ and $x \in H_c^k(M;\mathbb{F}_2)$, $x \cup \tau^*(x) = p^*(y)$ for some $y \in H_c^{2k}(M/\tau;\mathbb{F}_2)$.

For η take any element in $\mathcal{L}_M$.