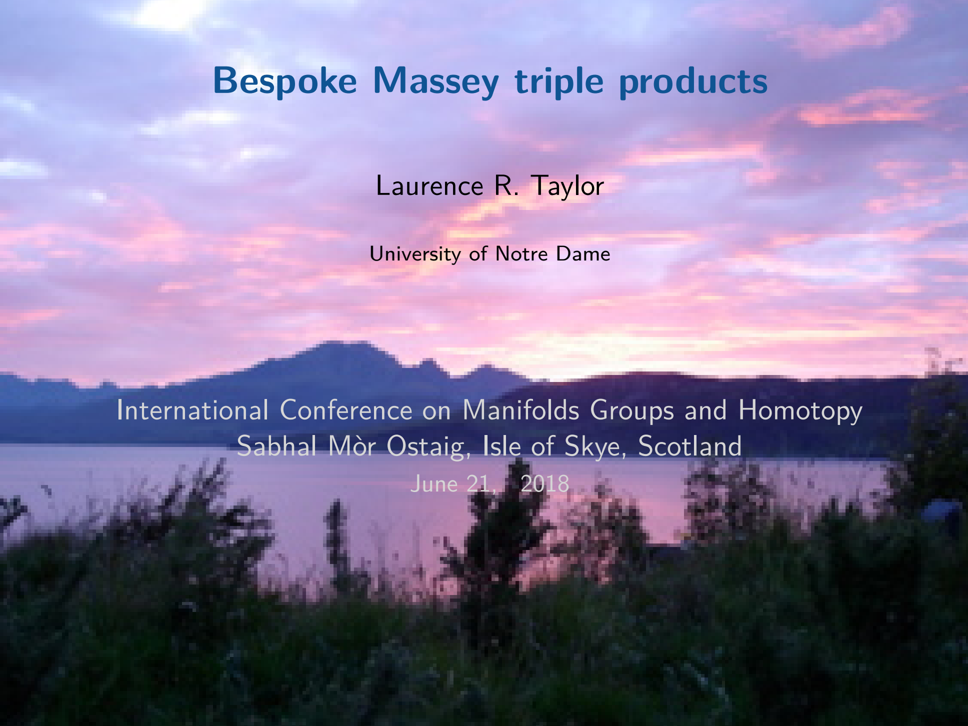

Bespoke Massey triple products

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Outline

Basic setup and notation

Definition/Recall of the products

Discussion of the products

Symmetry

The Jacobite diversion

Pairings

Duality

Examples

Because of visibility problems at the back of the room there is a large amount of space at the bottom of many slides.

Basic setup and notation

Fix a set with four elements $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$ and a PID Λ. Let $\mathbf{R}$ denote the following basic data. There is a $\mathbb{Z}$-graded cochain complex R of Λ modules with differential d of degree $+1$. Require an associative multiplication and a $\cup_1$ product satisfying Hirsch's formula. For each subset $K \subset \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$ there is to be a subcomplex R_K such that each R_K is a two sided ideal for the product and the $\cup_1$ product. Require $R_\emptyset = R$ and $R_{K_1 \cup K_2} = R_{K_1} \cap R_{K_2}$. Write HR_K for the cohomology of R_K.

Cochains in R_K will always be homogeneous. Write $|x|$ for the grading of x in R (and hence in any R_K) and write $x \in HR_K^{|x|}$.
Given two sets of cohomology classes $\{x_1, \cdots, x_k\}$, $x_i \in HR_{K_i}^{|x_i|}$, and $\{y_1, \cdots, y_\ell\}$, $y_i \in HR_{L_i}^{|y_i|}$, write $\{x_1, \cdots, x_k\} \cup \{y_1, \cdots, y_\ell\} = 0$ if all the $x_i \cup y_j = 0 \in HR_{K_i \cup L_j}^{|x_i|+|y_j|}$.

Definition/Recall of the products

To define a Massey triple product start with elements $x_i \in HR_{\mathbf{i}}^{|x_i|}$, $i \in \{1, 2, 3\}$ such that $\{x_2\} \cup \{x_1, x_3\} = 0$. Pick representative cocycles $\hat{x}_i \in R_{\mathbf{i}}$ and cochains $X_{12} \in R_{\mathbf{12}}$ and $X_{23} \in R_{\mathbf{23}}$ such that $dX_{ij} = \hat{x}_i \cup \hat{x}_j$. Let $m = |x_1| + |x_2| + |x_3| - 1$. Check

$$X_{12} \cup \hat{x}_3 + (-1)^{|x_1|+1} \hat{x}_1 \cup X_{23}$$

is a cocycle in $R_{\mathbf{123}}^m$. The Massey triple product $\langle x_1, x_2, x_3 \rangle \subset HR_{\mathbf{123}}^m$ consists of all elements arising from this construction. It is a coset of the submodule $\mathcal{J}_{x_1,x_3}^m = x_1 \cup HR_{\mathbf{23}}^{m-|x_1|} + HR_{\mathbf{12}}^{m-|x_3|} \cup x_3$.

To define the four-fold product start with elements $x_i \in HR_{\mathbf{i}}^{|x_i|}$, $i \in \{0,1,2,3\}$ such that $\{x_0, x_2\} \cup \{x_1, x_3\} = 0$. Define

$$[x_0, x_1, x_2, x_3] = x_0 \cup \langle x_1, x_2, x_3 \rangle \in HR_{\mathbf{0123}}^{m+|x_0|}$$

Note that $[x_0, x_1, x_2, x_3]$ is a single element.

Discussion of the products

- Say $\langle x_1, x_2, x_3\rangle$ and $[x_0, x_1, x_2, x_3]$ are *defined* if the requisite cup products are 0.
- Say $\langle x_1, x_2, x_3\rangle$ is the *triple product associated to the four-fold product* $[x_0, x_1, x_2, x_3]$.
- Theorem. *If* $[x_0, x_1, x_2, x_3] \neq 0$ *then* $\langle x_1, x_2, x_3\rangle \neq 0$.
- Say x_0 is an *alibi witness* for the associated triple product if $[x_0, x_1, x_2, x_3]$ is defined and not 0.
- Alibis often exist. In a compact manifold, oriented for a field Λ, any non-trivial triple product has an alibi x_0 so that $[x_0, x_1, x_2, x_3]$ lands in the top dimension. Not immediately obvious but more later.

Naturality. *If* $f^*\colon \mathbf{R} \to \mathbf{S}$ *is a map of basic data, then*

$$f^*\big(\langle x_1, x_2, x_3\rangle\big) \subset \langle f^*(x_1), f^*(x_2), f^*(x_3)\rangle \tag{1}$$

$$f^*\big([x_0, x_1, x_2, x_3]\big) = \big[f^*(x_0), f^*(x_1), f^*(x_2), f^*(x_3)\big] \tag{2}$$

Symmetry

The symmetric group on $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$ acts on basic data sets. For $\sigma \in \mathcal{S}_4$, let $\mathbf{R}^\sigma$ have the same $R_\emptyset$ as $\mathbf{R}$. Let $\mathbf{R}^\sigma_{\mathbf{i}} = \mathbf{R}_{(\mathbf{i})\sigma}$. If $[a, b, c, d]$ and $\sigma \in \mathcal{S}_4$ define $[a, b, c, d]_\sigma$ by writing $a = x_{\mathbf{0}}$, $b = x_{\mathbf{1}}$, $c = x_{\mathbf{2}}$, $d = x_{\mathbf{3}}$ and defining

$$[a, b, c, d]_\sigma = \left[x_{(\mathbf{0})\sigma}, x_{(\mathbf{1})\sigma}, x_{(\mathbf{2})\sigma}, x_{(\mathbf{3})\sigma}\right]$$

The elements (0123) and (13) generate a dihedral subgroup $\mathcal{D}_8$ of $\mathcal{S}_4$ such that, if $\sigma \in \mathcal{D}_8$ then $[x_0, x_1, x_2, x_3]_\sigma$ is defined if and only if $[x_0, x_1, x_2, x_3]$ is defined.

Definition. Given an ordered set of four integers, $e_{\mathbf{0}}$, $e_{\mathbf{1}}$, $e_{\mathbf{2}}$ and $e_{\mathbf{3}}$, each $e_{\mathbf{i}}$ determines a parity $p_{\mathbf{i}} \in \{0,1\}$. These four parities can be arranged in a *parity vector*, $p_{\mathbf{0}}p_{\mathbf{1}}p_{\mathbf{2}}p_{\mathbf{3}}$ which can be read as a binary integer between 0 and 15. Conveniently these numbers are represented by a single hex digit between 0 and F. Let $\mathbf{H}$ denote the set of single hex digits. It is a four dimensional $\mathbb{Z}/2\mathbb{Z}$ vector space with addition given by bitwise-exclusive-or.

Define the *parity vector function*, $\mathfrak{p}(e_{\mathbf{0}}, e_{\mathbf{1}}, e_{\mathbf{2}}, e_{\mathbf{3}}) \in \mathbf{H}$ to be the single hex digit just described. Given four cohomology classes x_i, $i \in \{0,1,2,3\}$ define

$$\mathfrak{p}(x_0, x_1, x_2, x_3) \in \mathbf{H} \qquad \text{using } e_{\mathbf{i}} = |x_i|$$

Theorem 1. There is a function $s\colon \mathcal{D}_8 \times \mathbf{H} \to \{0,1\}$

$$[x_0, x_1, x_2, x_3]_\sigma = (-1)^{s_\sigma(\mathfrak{p}(x_0,x_1,x_2,x_3))}\,[x_0, x_1, x_2, x_3]$$

If $\mathcal{D}_8$ acts on $\mathbf{H}$ by permuting the binary digits, s is an example of a crossed-homomorphism.

Table 1. $s_\sigma(\mathbf{h})$

$\mathbf{h}$ \ σ	()	(0123)	(02)(13)	(0321)	(13)	(03)(12)	(02)	(01)(23)
0	0	1	0	1	1	0	1	0
1	0	0	0	1	1	1	1	0
2	0	1	1	1	1	0	0	0
3	0	1	1	1	0	0	0	1
4	0	1	0	0	1	0	1	1
5	0	1	1	0	0	0	1	1
6	0	1	0	0	0	1	1	1
7	0	0	1	0	0	0	1	0
8	0	0	1	0	1	1	0	1
9	0	0	0	1	1	1	0	1
A	0	0	1	1	1	0	0	1
B	0	1	0	0	0	1	0	0
C	0	0	1	0	1	0	1	1
D	0	1	1	1	0	1	1	1
E	0	0	0	1	0	0	0	1
F	0	0	0	0	0	0	0	0

$$[x_0, x_1, x_2, x_3]_\sigma = (-1)^{s_\sigma(\mathfrak{p}(x_0,x_1,x_2,x_3))}\,[x_0, x_1, x_2, x_3]$$

Note that the triple products associated to these permutations of $[x_0, x_1, x_2, x_3]$ are often different so if one four-fold product is non-zero several different triple products will be non-zero. Since there exist elements of $\mathcal{D}_8$ which move position $\mathbf{i}$, $i \in \{1, 2, 3\}$ to $\mathbf{0}$, other consequences include the following.

Theorem 2. $[x_0, x_1, x_2, x_3]$ is linear in each variable separately.

Theorem 3. Suppose given $v_i \in HR_{\mathbf{i}}$, $i \in \{0,1,2,3\}$. Let the parity vectors be $P_x = \mathfrak{p}(x_0, x_1, x_2, x_3)$ and $P_v = \mathfrak{p}(v_0, v_1, v_2, v_3)$. Then there is a function $\varepsilon = \varepsilon(P_v, P_x) \colon \mathbf{H} \times \mathbf{H} \to \{0,1\}$ such that

$$v_0 \cup v_1 \cup v_2 \cup v_3 \cup [x_0, x_1, x_2, x_3] = (-1)^{\varepsilon} [v_0 \cup x_0, v_1 \cup x_1, v_2 \cup x_2, v_3 \cup x_3]$$

TABLE 2. $\varepsilon(\mathbf{h_v}, \mathbf{h_x})$

$\mathbf{h_v}$ \ $\mathbf{h_x}$	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0 ; 8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1 ; 9	1	1	0	0	0	0	1	1	0	0	1	1	1	1	0	0
2 ; A	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
3 ; B	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
4 ; C	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
5 ; D	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0
6 ; E	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
7 ; F	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1

Jacking up Massey products. Let M and N be a Λ-oriented, closed, compact, manifolds of dimensions m and n respectively. Suppose given classes v_i, $i \in \{0, 1, 2, 3\}$ with

$$v_0 \cup v_1 \cup v_2 \cup v_3 = u \in H^n(N; \Lambda)$$

Then, if $[x_0, x_1, x_2, x_3] \in H^k(M; \Lambda)$ is defined,
$[v_0 \cup x_0, v_1 \cup x_1, v_2 \cup x_2, v_3 \cup x_3] \in H^{n+k}(N \times M; \Lambda)$ is defined.
Moreover, under the split injection
$H^k(M; \Lambda) = H^n(N; \Lambda) \otimes H^k(M; \Lambda) \to H^{n+k}(N \times M; \Lambda)$,

$$u \otimes [x_0, x_1, x_2, x_3] = \pm [v_0 \cup x_0, v_1 \cup x_1, v_2 \cup x_2, v_3 \cup x_3]$$

The Jacobite diversion

If all three triple products below are defined they fit into a Jacobi relation $\pm\langle x_1, x_2, x_3\rangle \pm \langle x_1, x_2, x_3\rangle_{(123)} \pm \langle x_1, x_2, x_3\rangle_{(132)} = 0$.
If the three corresponding four-fold products are defined, they fit into a Jacobi relation. Additionally, under these hypotheses, the action of any $\sigma \in \mathcal{S}_4$ on a defined $[x_0, x_1, x_2, x_3]$ is defined. There are four subgroups of order 3 in $\sigma \in \mathcal{S}_4$ determined by which position is fixed. This gives rise to four Jacobi relations.

For each Jacobi relation there is a choice of 3-cycle $\tau_{\mathbf{i}}$ and three functions, $\mathfrak{j}_{\mathbf{i},k} \colon \mathbf{H} \to \{0,1\}$ which, when evaluated on the parity vector $\mathfrak{p}(x_0, x_1, x_2, x_3)$ satisfy

$$(-1)^{\mathfrak{j}_{\mathbf{i},0}(\mathfrak{p}(x_0,x_1,x_2,x_3))}\,[x_0, x_1, x_2, x_3] \;\; + \\ (-1)^{\mathfrak{j}_{\mathbf{i},1}(\mathfrak{p}(x_0,x_1,x_2,x_3))}\,[x_0, x_1, x_2, x_3]_{\tau_{\mathbf{i}}} + \\ (-1)^{\mathfrak{j}_{\mathbf{i},2}(\mathfrak{p}(x_0,x_1,x_2,x_3))}\,[x_0, x_1, x_2, x_3]_{\tau_{\mathbf{i}}^2} = 0$$

Table 3. Jacobi relations

	0	1	2	3	4	5	6	7
0	0, 0, 0	0, 0, 0	0, 0, 0	0, 0, 1	0, 0, 0	1, 0, 0	0, 1, 0	1, 1, 1
1	0, 0, 0	0, 1, 1	0, 1, 1	0, 0, 1	0, 0, 0	1, 0, 0	0, 1, 1	1, 1, 0
2	0, 0, 0	0, 0, 0	0, 0, 0	0, 0, 0	0, 0, 1	1, 0, 0	0, 1, 0	1, 1, 1
3	0, 0, 0	0, 0, 0	0, 1, 1	0, 0, 0	0, 1, 1	1, 1, 1	0, 1, 0	1, 0, 1
	8	9	A	B	C	D	E	F
0	0, 0, 0	0, 0, 0	0, 0, 0	0, 0, 1	0, 0, 0	1, 0, 0	0, 1, 0	1, 1, 1
1	0, 0, 0	0, 1, 0	0, 0, 0	0, 1, 1	0, 1, 1	1, 1, 0	0, 1, 1	1, 1, 1
2	0, 0, 1	0, 0, 0	0, 0, 0	0, 0, 1	0, 0, 1	1, 0, 1	0, 1, 1	1, 1, 1
3	0, 0, 0	0, 1, 1	0, 1, 1	0, 1, 1	0, 1, 0	1, 0, 1	0, 1, 1	1, 1, 1

If $x^2 = 0$ then $\langle x, x, x \rangle$ is defined and $3 \langle x, x, x \rangle = 0$. (If $\Lambda = \mathbb{Z}/3\mathbb{Z}$, Kraines identifies $\langle x, x, x \rangle$ as a Steenrod operation.) Symmetry then implies the next result.

Theorem 4. If $[x_0, x_1, x_2, x_3]$ is defined and if any three of the x_i are equal, then $3 [x_0, x_1, x_2, x_3] = 0$.

Pairings

Fix x_1 and x_3 and define a submodule of $HR^k_{\mathbf{2}}$ by

$$\boxed{x_2 \in A^k_{x_1,x_3}(R_{\mathbf{2}}) \text{ if and only if } \{x_2\} \cup \{x_1, x_3\} = 0.}$$

$$\boxed{x_2 \in A^k_{x_1,x_3}(R_{\mathbf{2}}) \text{ if and only if } \{x_2\} \cup \{x_1, x_3\} = 0.}$$

Similarly x_1 and x_3 show up in

$$\boxed{\mathcal{J}^k_{x_1,x_3}(R_{\mathbf{123}}) = x_1 \cup HR_{\mathbf{23}}^{k-|x_1|} + x_3 \cup HR_{\mathbf{12}}^{k-|x_3|} \subset HR^k_{\mathbf{123}}}$$

$$\boxed{x_2 \in A^k_{x_1,x_3}(R_{\mathbf{2}}) \text{ if and only if } \{x_2\} \cup \{x_1, x_3\} = 0.}$$

$$\boxed{\mathcal{J}^k_{x_1,x_3}(R_{\mathbf{123}}) = x_1 \cup HR_{\mathbf{23}}^{k-|x_1|} + x_3 \cup HR_{\mathbf{12}}^{k-|x_3|} \subset HR^k_{\mathbf{123}}}$$

Massey triple products are then single-valued in the quotient group $HR^k_{\mathbf{123}}/\mathcal{J}^k_{x_1,x_3}(R_{\mathbf{123}})$. Define

$$\boxed{\mathcal{M}^k_{x_1,x_3}(R_{\mathbf{123}}) \subset HR^k_{\mathbf{123}}/\mathcal{J}^k_{x_1,x_3}(R_{\mathbf{123}})}$$

to be the submodule of all Massey products.

There are similar definitions with $\mathbf{2}$ replaced by $\mathbf{0}$.

$A^k_{x_1,x_3}(R_{\mathbf{2}})$ plays two roles. The map

$$\begin{aligned} x_2 \in A^k_{x_1,x_3}(R_{\mathbf{2}}) &\mapsto \langle x_1, x_2, x_3\rangle \\ A^k_{x_1,x_3}(R_{\mathbf{2}}) &\to \mathcal{M}^{k+|x_1|+|x_3|-1}_{x_1,x_3}(R_{\mathbf{123}}) \end{aligned}$$

is a surjective homomorphism so $A^k_{x_1,x_3}(R_{\mathbf{2}})$ *creates* Massey products.

The map

$$
\begin{aligned}
x_2 \in A^k_{x_1,x_3}(R_{\mathbf{2}}), \langle x_1, x_0, x_3\rangle \in & \mathcal{M}^\ell_{x_1,x_3}(R_{\mathbf{013}}) \\
\big(x_2, \langle x_1, x_0, x_3\rangle\big) \mapsto & \,[x_2, x_1, x_0, x_3] \\
A^k_{x_1,x_3}(R_{\mathbf{2}}) \times \mathcal{M}^\ell_{x_1,x_3}(R_{\mathbf{013}}) \to & HR^{k+\ell}_{\mathbf{0123}}
\end{aligned}
$$

is bilinear and *alibis* Massey products.

Duality

Say that the basic data is n-*dually paired* if there exists a homomorphism $\omega\colon HR^n_{\mathbf{0123}} \to \Lambda$ such that, for any partition K_1, K_2 of $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$ which separates $\mathbf{0}$ from $\mathbf{2}$ and any k, the products $HR^{n-k}_{K_1} \otimes HR^k_{K_2} \to HR^n_{\mathbf{0123}} \to \Lambda$ are non-degenerate pairings.

Closed compact manifolds which are Λ-orientable have such a pairing with all the R_K being the singular cochains.

Compact manifolds with boundary which are Λ-orientable have such a pairing as well whenever exactly one of the $R_{\mathbf{i}}$ is the relative cochain complex and the other three are absolute.

If the basic data is n-dually paired, then

$$\boxed{A^{n-k}(R_{\mathbf{0}})_{x_1,x_3} = \left(\mathcal{J}^k_{x_1,x_3}(R_{\mathbf{123}})\right)^{\perp}}$$

$$\boxed{A^{n-k}(R_{\mathbf{2}})_{x_1,x_3} = \left(\mathcal{J}^k_{x_1,x_3}(R_{\mathbf{013}})\right)^{\perp}}$$

Theorem 5. There are non-degenerate pairings

$$A^{n-k}(R_{\mathbf{0}})_{x_1,x_3} \otimes \left(HR^k_{\mathbf{123}}/\mathcal{J}^k_{x_1,x_3}(R_{\mathbf{123}})\right) \to \Lambda$$

$$A^{n-k}(R_{\mathbf{2}})_{x_1,x_3} \otimes \left(HR^k_{\mathbf{013}}/\mathcal{J}^k_{x_1,x_3}(R_{\mathbf{013}})\right) \to \Lambda$$

induced by the cup product. Restricted to the submodule of Massey products, the pairings are given by four-fold products.

There are two sorts of "useless" elements in $A^{n-k}(R_{\mathbf{0}})_{x_1,x_3}$: some give the trivial Massey product; some never alibi anyone. It turns out these two subgroups are the same so . . .

Suppose the basic data is n-dually paired and let $m = n + |x_1| + |x_3| - 1$. Then there is a non-degenerate pairing

$$\boxed{\mathcal{M}^{m-k}_{x_1,x_3}(HR_{\mathbf{013}}) \otimes \mathcal{M}^{k}_{x_1,x_3}(HR_{\mathbf{123}}) \to \Lambda}$$

The pairing sends $\langle x_1, x_0, x_3\rangle \otimes \langle x_1, x_2, x_3\rangle$ to $[x_0, x_1, x_2, x_3]$.

It is occasionally useful to identify “useless” elements in an $A^*(R_{\mathbf{i}})_{x_1,x_3}$. If $x_2 = u \cup v$ with $x_1 \cup u = 0$ and $x_3 \cup v = 0$ then $\langle x_1, u \cup v, x_3 \rangle = 0$. Symmetry implies

Theorem 6. $[x_0, x_1, x_2, x_3] = 0$ whenever it is defined and, with indices mod 4, $x_i = u \cup v$ and $x_{i-1} \cup u = 0 = x_{i+1} \cup v$.

Examples

Theorem 7. Let W be a Λ-oriented, compact bordism between two connected n-dimensional manifolds. Assume $H_1(W, \partial W; \mathbb{Z}) \cong \mathbb{Z}$. Let $\iota_\pm \colon \partial_\pm W \to W$ denote the inclusions. Then $H^n(W; \mathbb{Z}) \cong \mathbb{Z}$. The two boundary components can be oriented so that if $[w_0, w_1, w_2, w_3] \in H^n(W)$ then

$$\left[\iota_-^*(w_0), \iota_-^*(w_1), \iota_-^*(w_2), \iota_-^*(w_3)\right] = \left[\iota_+^*(w_0), \iota_+^*(w_1), \iota_+^*(w_2), \iota_+^*(w_3)\right]$$

It turns out to be relatively easy to understand Massey products of three classes of degree 1 since these only depend on the fundamental group. A good source of examples are 3-manifolds. Perhaps the most famous example is Massey's proof that the triple product can be used to show that the Borromean rings are linked.

A second famous example is the Heisenberg manifold, M: real upper 3×3 triangular matrices modulo the subgroup of integer ones. The integral cohomology is torsion-free and H^1 is generated by two classes x_1, x_2 which are dual to the loops $t \mapsto \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$.

Then $\langle x_1, x_1, x_2 \rangle$ is indivisible in H^2. By Theorem 4

$$[x_1, x_1, x_1, x_2] = 0 \quad \text{and by duality} \quad [x_2, x_1, x_1, x_2] = \pm 1$$

Symmetry forces

$$[x_1, x_1, x_2, x_2] = [x_2, x_1, x_1, x_2] \quad \text{and by duality} \quad [x_2, x_1, x_2, x_2] = 0$$

so with the correct choice of orientation,

$$\langle x_1, x_1, x_2 \rangle = x_2^* \qquad \text{and} \qquad \langle x_1, x_2, x_2 \rangle = x_1^* \ .$$

A theorem of T. Miller says that a closed, compact, $(k-1)$-connected manifold of dimension less than $4k-1$ is formal.

Many people produced examples at the boundary, $(k-1)$-connected manifold of dimension $4k-1$ which are not formal. M. Katz requested examples of such manifolds with all products from H^k being zero and all of H^{3k-1} spanned by Massey products. He also wanted certain cohomology groups to be torsion-free. Dranishnikov and Rudyak produced such examples in many dimensions. Here is a different construction which gives examples in all dimensions.

Start with $T^4 \times M$. Let $H^1(T^4)$ be generated by t_0, t_1, t_2 and t_3, the pull-backs of a generator of $H^1(S^1)$ under the four projections. Let $z_0 = t_0 \cup x_2$, $z_1 = t_1 \cup x_1$, $z_2 = t_2 \cup x_1$ and $z_3 = t_3 \cup x_2$. Note $[z_0, z_1, z_2, z_3] = 1$ and $[z_i, z_1, z_2, z_3] = 0$, $i \in \{1, 2, 3\}$.

Do surgery to kill π_1 and all of H^2 except for a $\mathbb{Z}^4$. If W is the trace of the surgery this can be done so there are classes $w_i \in H^2(W)$ so that $[w_i, w_1, w_2, w_3]$ is defined and the w_i map to the z_i. Let K^7 denote the other end of the trace of the surgery and let $\hat{z}_i \in H^2(K)$ denote the image of w_i. Check that the w_i span $H^2(W;\mathbb{Z})$ and $w_i \cup w_j = 0$ for all i, $j \in \{0, 1, 2, 3\}$. Furthermore the $[\hat{z}_i, \hat{z}_1, \hat{z}_2, \hat{z}_3]$ are defined and have the same values as the unhatted versions. (Theorem 7).

Apply symmetry to produce four Massey products which form the dual basis to the $\hat{z}_i$. The manifold K^7 satisfies all of Katz's requirements.

The above construction using $S^{k-1} \times S^{k-1} \times S^{k-1} \times S^{k-1}$ in place of T^4 produces examples in all dimensions.