Exotic stratifications

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Joint with:
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If \((X, A)\) and \((Y, B)\) are two pairs, then a map \(f : (X, A) \to (Y, B)\) is said to be strict, or stratum-preserving, if \(f(X \setminus A) \subseteq Y \setminus B\) and \(f(A) \subseteq B\). The subspace \(A\) of \(X\) is said to be forward tame if there exists a neighborhood \(N\) of \(A\) in \(X\) and a strict map \(H : (N \times I, A \times I \cup N \times \{0\}) \to (X, A)\) such that \(H(x, t) = x\) for all \((x, t) \in A \times I\) and \(H(x, 1) = x\) for all \(x \in N\).

Let \(\text{Maps}_s((X, A), (Y, B))\) denote the space of strict maps with the compact-open topology. The homotopy link of \(A\) in \(X\) is

\[
\text{holink}(X, A) = \text{Maps}_s\left(([0, 1], \{0\}), (X, A)\right).
\]

Evaluation at 0 defines a map \(q : \text{holink}(X, A) \to A\) which should be thought of as a model for a normal fibration of \(A\) in \(X\).

The pair \((X, A)\) is said to be a homotopically stratified pair if \(A\) is forward tame in \(X\) and if \(q : \text{holink}(X, A) \to A\) is a fibration. If in addition, the fiber of \(q : \text{holink}(X, A) \to A\) is finitely dominated, then \((X, A)\) is said to be homotopically stratified with finitely dominated local holinks. If the strata \(A\) and \(X \setminus A\) are manifolds (without boundary), \(X\) is a locally compact separable metric space, and \((X, A)\) is homotopically stratified with finitely dominated local holinks, then \((X, A)\) is a manifold stratified pair.
(1) From a smooth embedding $B \to W$ we construct a vector bundle over $B$ of dimension $k$, the codimension of the embedding.

(2) Vector bundles over $B$ of dimension $k$ are classified by maps of $B$ into a classifying space.

(3) There is a smooth embedding of $B$ into the total space of any vector bundle.

(4) There is an embedding of the total space of the vector bundle into $W$ which is unique up to isotopy.

(5) All dimension $k$ vector bundles over $B$ occur as a normal bundle to some codimension $k$ embedding.
Define a \emph{controlled map} from $q : Y \to B$ to $p : X \to B$: $F : q \to p$ to be a level-preserving map
\[ F : Y \times [0, 1) \to X \times [0, 1) \]
such that the map
\[ \hat{F} : Y \times [0, 1] \to B \times [0, 1] \]
is continuous where
\[ \hat{F}(y, t) = (p \times 1_{[0,1)}) \circ F(y, t) \]
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An approximate fibration is a map \( p: X \to B \) with the controlled homotopy lifting property. A manifold approximate fibration or MAF is a map \( p: M \to B \) where \( M \) and \( B \) are paracompact Hausdorff manifolds without boundary, \( p \) is a proper map, and \( p \) is an approximate fibration.
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The *fibre germ* of a **MAF** $p : M \to B$ is the **MAF** given by restriction

$$p : p^{-1}(U) \to U$$

where $U \subset B$ is an open ball.
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For a fixed fibre germ \( p: V \to \mathbb{R}^i \), there is a classifying space \( \text{MAF}(p) \) and a fibration
\[
\mu: \text{MAF}(p) \to \text{BTOP}(i)
\]
The fibre of \( \mu \) is
\[
\text{BTOP}^c(V \to \mathbb{R}^i)
\]
The space of MAF’s over $B$ with fibre germ $\mathfrak{p}$ is homotopy equivalent to the space of lifts

$$\begin{array}{ccc}
MAF(\mathfrak{p}) & \xrightarrow{\mu} & BTOP(i) \\
\downarrow \quad & & \downarrow \\
B & \xrightarrow{\tau_B} & BTOP(i)
\end{array}$$

where $\tau_B$ classifies the tangent bundle to $B$ provided $\dim V \geq 6$. 
Let $p: M \to B \times \mathbb{R}$ be a map. The \textit{tear-drop} of $p$ is the set $T(p) = M \sqcup B$ with the tear-drop topology. The tear-drop topology is the minimal topology such that $M \subset T(p)$ is an open embedding and the function $c: T(p) \to B \times (-\infty, \infty]$ is continuous where $c(x) = p(x)$ for all $x \in M$ and $c(b) = (b, \infty)$ for all $b \in B$. 

Theorem 4.1. The tear-drop $T(p)$ is a manifold stratified space with two strata if and only if $p$ is a MAF.

Theorem 4.2. If $(X, B)$ is a manifold stratified space with two strata with $\dim X \geq 6$, then there is a MAF $p: M \to B \times \mathbb{R}$ and an embedding $T(p) \subset X$ which is the identity on $B$ and whose image contains a neighborhood of $B$.

Actually with more work, Hughes proved 4.1 and 4.2 without the two-strata hypothesis.
\[
\begin{align*}
\text{MAF}(p) & \xrightarrow{\ell} \text{MAF}(p \times 1_{\mathbb{R}}) \\
\downarrow & \downarrow \\
\text{BTOP}(k) & \rightarrow \text{BTOP}(k + 1)
\end{align*}
\]
Theorem 5.5. If \( p: M \to B \times \mathbb{R} \) is a MAF, the tear-drop \( T(p \times 1_{\mathbb{R}}) \) has a mapping cylinder neighborhood.

Corollary 5.6. If \((X, B)\) is a two-stratum manifold stratified space with tear-drop neighborhood \( T(p) \) then \((X \times \mathbb{R}, B \times \mathbb{R})\) is a two-stratum manifold stratified space with a mapping cylinder neighborhood.
We say the fibre germ is *trivial* when it is of the form $p : V \times \mathbb{R}^i \to \mathbb{R}^i$ where $V$ is some compact manifold without boundary.
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When the fibre germ is trivial, Anderson & Hsiang show that the fibre is the space of bounded concordances, $C^b(V \times \mathbb{R}^i \to \mathbb{R}^i)$ is the fibre of the stabilization map

$$\text{MAF}(p) \to E(p \times 1_{\mathbb{R}})$$
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**Theorem 5.7.** If $i + \dim V \geq 6$, then there exists a group isomorphism

$$\alpha_k: \pi_k(C^b(V \times \mathbb{R}^i \to \mathbb{R}^i)) \to \begin{cases} \text{Wh}_1(\mathbb{Z}\pi_1 F) & \text{if } k = i - 1 \\ \tilde{K}_0(\mathbb{Z}\pi_1 F) & \text{if } k = i - 2 \\ K_{2+k-i}(\mathbb{Z}\pi_1 F) & \text{if } 0 \leq k < i - 2. \end{cases}$$

**Corollary 5.8.** (Edwards) If $B \subset W$ is locally-flat and dimension $B \geq 5$ then the embedding has a mapping cylinder neighborhood.
\[
\begin{align*}
\text{BTOP}(V) & \rightarrow \text{BTOP}(V) \times \text{BTOP}(i) \\
\downarrow \times 1_{\mathbb{R}^i} & \quad \quad \downarrow \Psi \\
\text{BTOP}^c(V \times \mathbb{R}^i) & \rightarrow \quad \text{MAF}(p)
\end{align*}
\]
\[ \text{BTOP}(V) \rightarrow \text{BTOP}(V) \times \text{BTOP}(i) \]
\[ \downarrow \times 1_{\mathbb{R}^i} \]
\[ \text{BTOP}^c(V \times \mathbb{R}^i) \rightarrow \text{MAF}(p) \]

**Theorem 6.9.** For each integer \( m \geq 5 \), there exists a closed compact \( m \)-manifold \( V \) and a MAF over \( p: W \rightarrow S^1 \) with fibre-germ \( p: V \times \mathbb{R} \rightarrow \mathbb{R} \) such that the MAF over \( S^1 \) with fibre-germ \( V \times \mathbb{R}^i \times \mathbb{R} \rightarrow \mathbb{R}^i \times \mathbb{R} \) is not controlled homeomorphic to a fibre bundle for any integer \( i \geq 0 \).
A MAF $p: M \to S^1$ with trivial fibre-germ, is determined by an element $h: \pi_0(\text{TOP}^b(V \times \mathbb{R}))$.

There exists a crossed homomorphism

$$
\beta: \pi_0(\text{TOP}^b(V \times \mathbb{R})) \to \text{Wh}(\mathbb{Z}\pi_1 V)
$$

defined by using the bounded homeomorphism to construct an inertial $h$-cobordism and then taking the torsion.
Theorem 7.10. Let $h \in \pi_0(TOP^b(V \times \mathbb{R}))$ and let $p: M \to S^1$ be the associated MAF with $\dim M \geq 6$.

(1) The following are equivalent.
   (a) $p$ is controlled homeomorphic to a fibre bundle projection with fibre $V$.
   (b) $\beta(h) = 0 \in Wh(\mathbb{Z}\pi_1V)$.

(2) The following are equivalent.
   (a) $p \times 1_\mathbb{R}$ is controlled homeomorphic to a fibre bundle projection with fibre $V$.
   (b) $\beta(h) \in \text{Im } N \subset Wh(\mathbb{Z}\pi_1V)$.

(3) There exists a subgroup $G$ of $\tilde{K}_0(\mathbb{Z}\pi_1V)$ and a function

$$N_0: G \to Wh(\mathbb{Z}\pi_1V)/\text{Im } N$$

such that the following are equivalent.
   (a) $p \times 1_\mathbb{R^2}$ is controlled homeomorphic to a fibre bundle projection with fibre $V$.
   (b) $\beta(h) \in Wh(\mathbb{Z}\pi_1V)/\text{Im } N$ is in $N_0(G)$. 

Theorem 8.11. Let $h \in \pi_0(\text{TOP}^b(V \times \mathbb{R}))$ and let $p: M \to S^1$ be the associated MAF with $\dim M \geq 6$.

(1) The following are equivalent.
   (a) $p$ is controlled homeomorphic to a fibre bundle projection.
   (b) $\beta(h) \in \text{Im} (1 - h_*) \subset \text{Wh}(\mathbb{Z}\pi_1 V)$.

(2) The following are equivalent.
   (a) $p \times 1_{\mathbb{R}}$ is controlled homeomorphic to a fibre bundle projection.
   (b) $\beta(h) \in \text{Im} N + \text{Im} (1 - h_*) \subset \text{Wh}(\mathbb{Z}\pi_1 V)$.

(3) If $\tilde{K}_k(\mathbb{Z}\pi_1 V) = 0$ for $k \leq 0$ then the following are equivalent.
   (a) $p \times 1_{\mathbb{R}^i}$ is controlled homeomorphic to a fibre bundle projection.
   (b) $\beta(h) \in \text{Im} N + \text{Im} (1 - h_*) \subset \text{Wh}(\mathbb{Z}\pi_1 V)$.