EXOTIC STRATIFICATIONS
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LAURENCE R. TAYLOR

Abstract. We discuss joint work with B. Hughes, S. Weinberger and B. Williams [7] on some examples of stratified spaces with two strata. We will describe the structure of neighborhoods of the lower stratum in the whole space in general and give some specific calculations when the lower stratum is a circle. Using some known K-theory calculations we produce examples where the neighborhood "improves" under crossing the stratified space with a high dimensional torus and other examples where it does not.

1. Stratified spaces

A manifold stratified space in the sense of Quinn with two strata is a pair (X, B) with B and $X - B$ manifolds together with some homotopy-theoretic gluing data. I want to describe a theory of neighborhoods of B in X and work out some examples (the exotic stratifications of the title but perhaps better referred to as non-exotic).

2. Neighborhood theories

Tubular neighborhood theorem for smooth embeddings is model.

(1) From a smooth embedding $B \to W$ we construct a vector bundle over B of dimension k, the codimension of the embedding.
(2) Vector bundles over B of dimension k are classified by maps of B into a classifying space.
(3) There is a smooth embedding of B into the total space of any vector bundle.
(4) There is an embedding of the total space of the vector bundle into W which is unique up to isotopy.
(5) All dimension k vector bundles over B occur as a normal bundle to some codimension k embedding.

Milnor [10] discovered microbundles and proved that the analogues of (2) and (5) hold for topological (and PL) embeddings. He also proved (1) and (4) hold "stably", that is for $B \to W \times \mathbb{R}^k$ for some k. Kister [8] proved micro-bundles are fibre bundles.

Hirsch, Browder and Rourke & Sanderson produced examples showing (1) and (4) need not hold without stabilization.

Rourke & Sanderson did produce a topological theory of neighborhoods, for locally-flat embeddings. There is a map from the classifying space for dimension k micro-bundles to the corresponding classifying space for neighborhoods and so the normal micro-bundle question is reduced to a lifting question.

In an apparently different direction Edwards proved that locally-flat embeddings have mapping-cylinder neighborhoods. In other words there is a map $f\colon N \to B$ such that the mapping-cylinder of f embeds in W as a neighborhood of B. Edwards showed that the map f is a manifold approximate fibration.

3. MAFS

A *manifold approximate fibration* or **MAF** is a map $p\colon M \to B$ where M and B are paracompact Hausdorff manifolds without boundary, p is a proper map, and p is an approximate fibration.

To define an approximate fibration, first define a *controlled map* from $q\colon Y \to B$ to $p\colon X \to B$: $F\colon q \to p$ is a level-preserving map $F\colon Y \times [0,1) \to X \times [0,1)$ such that the map $\hat{F}\colon Y \times [0,1] \to B \times [0,1]$ is continuous where $\hat{F}(y,t) = \big(p \times 1_{[0,1)}\big) \circ F(y,t)$ if $0 \leqslant t < 1$ and $\hat{F}(y,1) = \big(q(y),1\big)$.

An *approximate fibration* is a map $p\colon X \to B$ with the *controlled homotopy lifting property*: given any commutative diagram

$$\begin{array}{ccc} Z \times 0 & \xrightarrow{\alpha} & X \\ \cap & & \downarrow p \\ Z \times [0,1] & \xrightarrow{\beta} & B \end{array}$$

there exists a control map $F\colon \beta \to p$ such that $F(z,0,t) = \alpha(z)$ for $0 \leqslant t \leqslant 1$ and all $z \in Z$.

B, B & L classified **MAF**'s over B [4]. For ease of exposition assume B is connected. The *fibre germ* of a **MAF** $p\colon M \to B$ is the **MAF** $\mathfrak{p}\colon p^{-1}(U) \to U$ where $U \subset B$ is an open ball. The fibre germ can be proved to be well-defined up to controlled homeomorphism.

For a fixed fibre germ $\mathfrak{p}\colon V \to \mathbb{R}^i$, there is a classifying space $\mathbf{MAF}(\mathfrak{p})$ and a fibration $\mu\colon \mathbf{MAF}(\mathfrak{p}) \to \mathbf{BTOP}(i)$. The space of **MAF**'s over

B with fibre germ $\mathfrak{p}$ is homotopy equivalent to the space of lifts

$$\begin{tikzcd} & \mathbf{MAF}(\mathfrak{p}) \arrow[d, "\mu"] \\ B \arrow[r, "\tau_B"] \arrow[ur] & \mathbf{BTOP}(i) \end{tikzcd}$$

where τ_B classifies the tangent bundle to B. The fibre of μ is $\mathbf{BTOP}^c(V \to \mathbb{R}^i)$, the classifying space for the group of controlled-homeomorphisms [4]. We further showed that this space of is homotopy equivalent to the classifying space of bounded homeomorphisms of V, [5].

4. Tear-drops

Let $p\colon M \to B \times \mathbb{R}$ be a map. The *tear-drop* of p is the set $T(p) = M \amalg B$ with the tear-drop topology. The tear-drop topology is the minimal topology such that $M \subset T(p)$ is an open embedding and the function $c\colon T(p) \to B \times (-\infty, \infty]$ is continuous where $c(x) = p(x)$ for all $x \in M$ and $c(b) = (b, \infty)$ for all $b \in B$.

Hughes proved

Theorem 4.1. *The tear-drop $T(p)$ is a manifold stratified space with two strata (in the sense of Quinn) if and only if p is a* **MAF**.

A more difficult result from B, B, S & L is the following, [6].

Theorem 4.2. *If (X, B) is a manifold stratified space with two strata (in the sense of Quinn) then there is a* **MAF** *$p\colon M \to B \times \mathbb{R}$ and an embedding $T(p) \subset X$ which is the identity on B and whose image contains a neighborhood of B.*

Actually with more work, Hughes proved 4.1 and 4.2 without the two-strata hypothesis.

5. Tear-drops to mapping cylinder neighborhoods

One example of a two-strata manifold stratified space is a locally-flat embedding of one manifold in another, $B \subset W$. Edwards's theorem says that there is a mapping cylinder neighborhood and one can ask in general whether a particular tear-drop is actually a mapping cylinder.

Given a map $f\colon M \to B$, extend to $\check{f}\colon M \times [0, \infty) \to B \times \mathbb{R}$ by $\check{f}(x, t) = \big(f(x), t\big)$. For us the mapping cylinder of f is the tear-drop of $\check{f}$. If f is proper this is the usual mapping cylinder topology but for the approximate fibration $(0, 1) \subset [0, 1]$ it definitely is not.

If the mapping cylinder of f is a tear-drop neighborhood in a manifold stratified space, then $f\colon M \to B$ must be a **MAF** and if the fibre-germ of f is $\mathfrak{p}\colon V \to U$ then the fibre germ of $M\times(0,\infty) \xrightarrow{\check{f}|} B\times(0,\infty)$ is $\mathfrak{p}\times 1_{\mathbb{R}}\colon V\times\mathbb{R}\to U\times\mathbb{R}$. To solve the problem of whether a particular tear-drop is a mapping cylinder, we need to study the map ι in the commutative square

$$\begin{array}{ccc} \mathbf{MAF}(\mathfrak{p}) & \xrightarrow{\iota} & \mathbf{MAF}(\mathfrak{p}\times 1_{\mathbb{R}}) \\ \downarrow & & \downarrow \\ \mathbf{BTOP}(k) & \longrightarrow & \mathbf{BTOP}(k+1) \end{array} \tag{5.1}$$

The tangent bundle of B is classified by a map $\tau_B\colon B\to\mathbf{BTOP}(k)$. Let $E(\mathfrak{p}\times 1_{\mathbb{R}})\to\mathbf{BTOP}(k)$ denote the pull-back of $\mathbf{MAF}(\mathfrak{p}\times 1_{\mathbb{R}})\to\mathbf{BTOP}(k+1)$ to $\mathbf{BTOP}(k)$ and extend (5.1) to

$$\begin{array}{ccccc} \mathbf{MAF}(\mathfrak{p}) & \to & E(\mathfrak{p}\times 1_{\mathbb{R}}) & \longrightarrow & \mathbf{MAF}(\mathfrak{p}\times 1_{\mathbb{R}}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{BTOP}(k) & = & \mathbf{BTOP}(k) & \longrightarrow & \mathbf{BTOP}(k+1) \end{array} \tag{5.2}$$

Then a tear-drop $B\to\mathbf{MAF}(\mathfrak{p}\times 1_{\mathbb{R}})$ is equivalent to a lift of τ_B to $E(\mathfrak{p}\times 1_{\mathbb{R}})$.

In other theories of neighborhoods, when the total space and the subspace are crossed with the same space, usually the new neighborhood has properties similar to the old. For example, if $B\subset W$ has no normal micro-bundle, then neither does $B\times\mathbb{R}\subset W\times\mathbb{R}$. For tear-drops however, the fibre-germ does change and so sometimes the neighborhood can change.

Theorem 5.3. *If $p\colon M\to B\times\mathbb{R}$ is a* **MAF***, the tear-drop $T(p\times 1_{\mathbb{R}})$ has a mapping cylinder neighborhood.*

Corollary 5.4. *If (X,B) is a two-stratum manifold stratified space with tear-drop neighborhood $T(p)$ then $(X\times\mathbb{R},B\times\mathbb{R})$ is a two-stratum manifold stratified space with a mapping cylinder neighborhood.*

The fibre of $\mathbf{MAF}(\mathfrak{p})\to E(\mathfrak{p}\times 1_{\mathbb{R}})$ is the fibre of the map $\mathbf{BTOP}^c(V\to U)\to\mathbf{BTOP}^c(V\times\mathbb{R}\to U\times\mathbb{R}^1)$. To make calculations we restrict attention to a particular form of fibre germ.

We say the fibre germ is *trivial* when it is of the form $\mathfrak{p}\colon V\times\mathbb{R}^i\to\mathbb{R}^i$ where V is some compact manifold without boundary. When the fibre germ is trivial, Anderson & Hsiang [1] show that the fibre is the space of bounded concordances, $C^b(V\times\mathbb{R}^i\to\mathbb{R}^i)$, and they compute the homotopy groups of this space.

Theorem 5.5. *If $i+\dim V \geq 6$, then there exists a group isomorphism*

$$\alpha_k\colon \pi_k\big(C^b(V\times\mathbb{R}^i\to\mathbb{R}^i)\big)\longrightarrow\begin{cases}\mathrm{Wh}_1(\mathbb{Z}\pi_1 F) & \text{if } k=i-1\\ \widetilde{K}_0(\mathbb{Z}\pi_1 F) & \text{if } k=i-2\\ K_{2+k-i}(\mathbb{Z}\pi_1 F) & \text{if } 0\leqslant k<i-2.\end{cases}$$

Corollary 5.6. *(Edwards) If $B\subset W$ is locally-flat and dimension $B\geqslant 5$ then the embedding has a mapping cylinder neighborhood.*

6. Mapping cylinder neighborhoods to fibre bundles

A second question is when can a mapping cylinder neighborhood be improved to a fibre bundle neighborhood. Continue to assume that the fibre-germ is trivial, $\mathfrak{p}\colon V\times\mathbb{R}^i\to\mathbb{R}^i$. There is a map $\mathbf{BTOP}(V)\xrightarrow{\times 1_{\mathbb{R}^i}}\mathbf{BTOP}^c(V\times\mathbb{R}^i)$ and we showed in [4] that there is a commutative diagram

$$\begin{array}{ccccc}\mathbf{BTOP}(V) & \longrightarrow & \mathbf{BTOP}(V)\times\mathbf{BTOP}(i) & \longrightarrow & \mathbf{BTOP}(i)\\ \downarrow{\scriptstyle\times 1_{\mathbb{R}^i}} & & \downarrow{\scriptstyle\Psi} & & \|\\ \mathbf{BTOP}^c(V\times\mathbb{R}^i) & \longrightarrow & \mathbf{MAF}(\mathfrak{p}) & \longrightarrow & \mathbf{BTOP}(i)\end{array}$$

and a **MAF** with fibre-germ $\mathfrak{p}$ comes from a fibre bundle with fibre V if and only if the classifying lift for the **MAF**, $B\to\mathbf{MAF}(\mathfrak{p})$ has a lift to $\mathbf{BTOP}(V)\times\mathbf{BTOP}(i)$.

The main examples we consider are the following. Let $B=S^1$ and let (X,B) be a manifold stratified space with two strata. Suppose the dimension of X is at least 6. We assume that the fibre-germ for the teardrop neighborhood is a mapping cylinder of the **MAF** $\mathfrak{p}\colon V\times\mathbb{R}\to\mathbb{R}$. Any such stratified space is locally cone-like in the sense of Siebenmann.

Theorem 6.1. *For each integer $m\geqslant 5$, there exists a closed compact manifold V and a* **MAF** *over S^1 with fibre-germ $\mathfrak{p}\colon V\times\mathbb{R}\to\mathbb{R}$ such that the* **MAF** *over S^1 with fibre-germ $V\times\mathbb{R}^i\times\mathbb{R}\to\mathbb{R}^i\times\mathbb{R}$ is not controlled homeomorphic to a fibre bundle for any integer $i\geqslant 0$.*

7. The passage to K-theory

Since S^1 has trivial tangent bundle, lifts can be replaced by maps into fibres and since the fibres are classifying spaces, we are calculating components of the groups being classified. Our problem is to find elements $h\in\pi_0\big(\mathbf{TOP}^b(V\times\mathbb{R})\big)$ which are not in the image of the map

$$\pi_0\big(\mathbf{TOP}(V)\big)\to\pi_0\big(\mathbf{TOP}^b(V\times\mathbb{R}^1)\big)$$

These will be become mapping cylinders which are not bundles with fibre V. We also want that h pushed into $\pi_0\big(\mathbf{TOP}^b(V\times\mathbb{R}^{i+1})\big)$ is still not in the image of $\pi_0\big(\mathbf{TOP}(V)\big)$.

There exists a crossed homomorphism

$$\beta\colon \pi_0\big(\mathbf{TOP}^b(V\times\mathbb{R})\big)\to \mathrm{Wh}(\mathbb{Z}\pi_1 V)$$

defined by using the bounded homeomorphism to construct an inertial h-cobordism and then taking the torsion.

Theorem 7.1. *Let $h\in\pi_0\big(\mathbf{TOP}^b(V\times\mathbb{R})\big)$ and let $p\colon M\to S^1$ be the associated* **MAF** *with* $\dim M\geqslant 6$*.*

(1) *The following are equivalent.*
  (a) *p is controlled homeomorphic to a fibre bundle projection with fibre V.*
  (b) *$\beta(h)=0\in\mathrm{Wh}(\mathbb{Z}\pi_1V)$.*

(2) *The following are equivalent.*
  (a) *$p\times 1_{\mathbb{R}}$ is controlled homeomorphic to a fibre bundle projection with fibre V.*
  (b) *$\beta(h)\in Im\, N\subset \mathrm{Wh}(\mathbb{Z}\pi_1V)$.*

(3) *There exists a subgroup G of $\widetilde{K}_0(\mathbb{Z}\pi_1V)$ and a function*

$$N_0\colon G\to \mathrm{Wh}(\mathbb{Z}\pi_1V)/Im\, N$$

such that the following are equivalent.
  (a) *$p\times 1_{\mathbb{R}^2}$ is controlled homeomorphic to a fibre bundle projection with fibre V.*
  (b) *$\beta(h)\in \mathrm{Wh}(\mathbb{Z}\pi_1V)/Im\, N$ is in $N_0(G)$.*

One then gets examples by taking $\pi_1(V)=\pi_1=\mathbb{Z}/5\mathbb{Z}$. Then

- By a result of Lawson [9], there exist V such that β is onto.
- $\mathrm{Wh}(\mathbb{Z}\pi_1)\cong\mathbb{Z}$, $\widetilde{K}_0(\mathbb{Z}\pi_1)=0$
- The involution on Wh is trivial.

Remark 7.2. If $p\times 1_M$ is controlled homeomorphic to a fibre bundle projection for some i-dimensional manifold M. then $p\times 1_{\mathbb{R}^i}$ is controlled homeomorphic to a fibre bundle projection

8. Going from not fibre bundle with V as fibre to no fibre bundle at all

The previous section shows that there are explicit calculations one can do to determine if some **MAF** with fibre-germ $\mathfrak{p}\colon V\times\mathbb{R}^i\to\mathbb{R}^i$ has a tear-drop which is and/or is not the mapping cylinder of a bundle with fibre V, but we wish to construct examples for which the teardrop is not the mapping cylinder of any bundle. Associated to the fibre-germ there is a controlled homeomorphism $h\colon V \times \mathbb{R}^i \to V \times \mathbb{R}^i$ and the induced map $h_*\colon \mathrm{Wh}(\mathbb{Z}\pi_1) \to \mathrm{Wh}(\mathbb{Z}\pi_1)$ only depends on the mapping cylinder.

Theorem 8.1. *Let $h \in \pi_0\big(\mathbf{TOP}^b(V \times \mathbb{R})\big)$ and let $p\colon M \to S^1$ be the associated* **MAF** *with $\dim M \geqslant 6$.*

(1) *The following are equivalent.*
  (a) *p is controlled homeomorphic to a fibre bundle projection.*
  (b) *$\beta(h) \in Im\,(1 - h_*) \subset \mathrm{Wh}(\mathbb{Z}\pi_1 V)$.*

(2) *The following are equivalent.*
  (a) *$p \times 1_{\mathbb{R}}$ is controlled homeomorphic to a fibre bundle projection.*
  (b) *$\beta(h) \in Im\,N + Im\,(1 - h_*) \subset \mathrm{Wh}(\mathbb{Z}\pi_1 V)$.*

(3) *If $\widetilde{K}_k(\mathbb{Z}\pi_1 V) = 0$ for $k \leqslant 0$ then the following are equivalent.*
  (a) *$p \times 1_{\mathbb{R}^i}$ is controlled homeomorphic to a fibre bundle projection.*
  (b) *$\beta(h) = 0 \in \mathrm{Wh}(\mathbb{Z}\pi_1 V)/\big(Im\,N + Im\,(1 - h_*)\big)$.*

Remark 8.2. If Theorem 8.1 says that there is no fibre bundle structure, then there is no fibre bundle structure even when non-manifold fibres are allowed.

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E-mail address: taylor.2@nd.edu

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46656