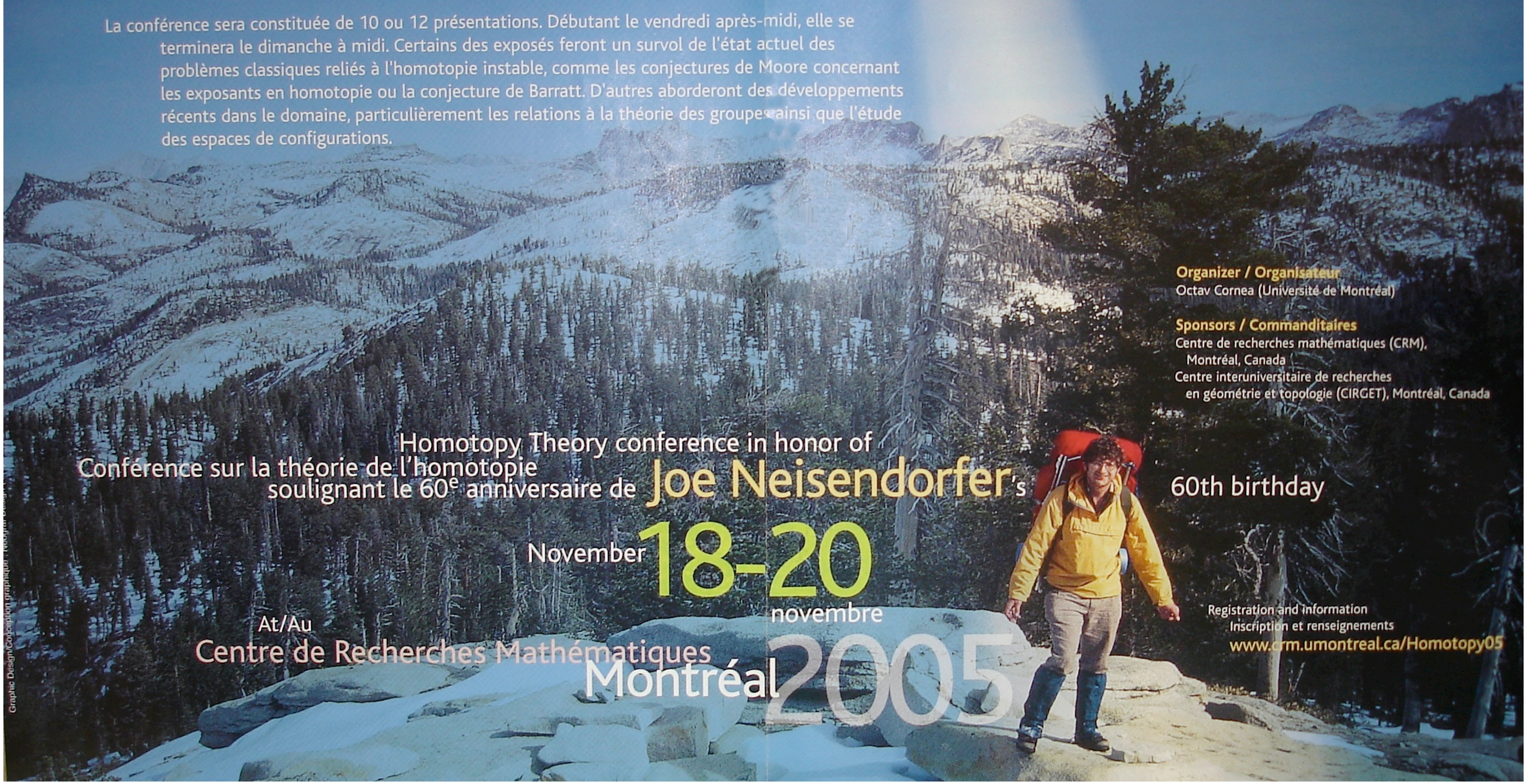

Even Manifolds

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1 A modest answer

There is a similar definition using $\mathbb{Z}/2\mathbb{Z}$ coefficients and in this case, Wu gave a very nice criterion in terms of the tangent bundle of M of this mod 2 intersection form to be even. Wu phrased his answer in terms of the stable tangent bundle, $\tau_M\colon M \to BO$, and what are now called the Wu classes $v_\ell \in H^\ell(BO;\mathbb{Z}/2\mathbb{Z})$:

Theorem 1.1 (Wu)**.** *The mod 2 intersection form of M^{4k} is even if and only if $\tau_M^*(v_{2k}) = 0$.*

Christan Bohr, Ronnie Lee and T. J. Li answered the **question** in terms of the evaluation homomorphism in the Universal Coefficients Theorem,

$$\mathrm{ev}\colon H^\ell(M;\mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(H_\ell(M;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})$$

as follows:

Theorem 1.2. *M^{4k} is even if and only if $\mathrm{ev}\big(\tau_M^*(v_{2k})\big) = 0$.*

There is an inclusion $\iota\colon \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2^\infty$ and an induced map on cohomology.

Theorem 1.3. *M^{4k} is even if and only if $\iota_*\big(\tau_M^*(v_{2k})\big) = 0$.*

There is an inclusion $\iota\colon \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2^\infty$ and an induced map on cohomology.

Theorem 1.3. *M^{4k} is even if and only if $\iota_*\big(\tau_M^*(v_{2k})\big) = 0$.*

Proof.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\operatorname{Ext}(H_{2k-1}(M;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \operatorname{Ext}(H_{2k-1}(M;\mathbb{Z}),\mathbb{Z}/2^\infty) \\
\downarrow & & \downarrow \\
H^{2k}(M;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\quad\iota_*\quad} & H^{2k}(M;\mathbb{Z}/2^\infty) \\
\text{ev}\downarrow & & \text{ev}\downarrow \\
\operatorname{Hom}(H_{2k}(M;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{I_*} & \operatorname{Hom}(H_{2k}(M;\mathbb{Z}),\mathbb{Z}/2^\infty) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

I_* is injective.

$$\operatorname{Ext}(H_{2k-1}(M;\mathbb{Z}),\mathbb{Z}/2^\infty) = 0$$

□

Let $v_\ell(2^\infty) = \iota_*(v_\ell) \in H^\ell(BSO; \mathbb{Z}/2^\infty)$.

Theorem 1.4. *M^{4k} is even if and only if $\tau_M^*\big(v_{2k}(2^\infty)\big) = 0$.*

Remark 1.5. This characterizes evenness as the vanishing of a universal characteristic class and suggests the following shift of viewpoint, going back at least to Lashof.

Let $BSO\langle v_\ell(2^\infty)\rangle$ denote the homotopy fibre of the map $BSO \xrightarrow{v_\ell(2^\infty)} K(\mathbb{Z}/2^\infty; \ell)$ and let $\mathfrak{p}_2 \colon BSO\langle v_\ell(2^\infty)\rangle \to BSO$ be the inclusion made into a fibration. Then

Definition 1.6. A $v_{2k}(2^\infty)$-structure on a bundle $\xi \colon X \to BO$ is a lift of ξ to $BSO\langle v_{2k}(2^\infty)\rangle$.

Remark 1.7. The fibration is principal so the set of lifts is an $H^{2k-1}(X; \mathbb{Z}/2^\infty)$-torsor.

2 Related structures

One can also kill v_{2k} or δv_{2k}, where δ is the integral Bockstein, to get principal fibrations

$$BSO\langle v_{2k}\rangle \xrightarrow{\mathfrak{p}_1} BSO \xrightarrow{v_{2k}} K(\mathbb{Z}/2\mathbb{Z}, 2k)$$

$$BSO\langle \delta v_{2k}\rangle \xrightarrow{\mathfrak{p}_3} BSO \xrightarrow{\delta v_{2k}} K(\mathbb{Z}, 2k+1)$$

There are also v_{2k}-structures and δv_{2k}-structures on a bundle, defined as lifts. And the set of lifts are torsors.

Any v_{2k}-structure induces a canonical $v_{2k}\,(2^{\infty})$-structure. Since

$$\begin{array}{ccccccccc} 0 \to \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z} & \to 0 \\ \| & & \downarrow & & \iota\downarrow & \\ 0 \to \mathbb{Z} & \longrightarrow & \mathbb{Z}[\frac{1}{2}] & \to & \mathbb{Z}/2^{\infty} & \to 0 \end{array}$$

commutes, any $v_{2k}\,(2^{\infty})$-structure induces a canonical δv_{2k}-structure.

Let δ_{∞} denote the Bockstein associated to the bottom exact sequence: δ denotes the Bockstein associated to the top exact sequence.

3 Algebraic Topology

To amplify the last remark, note there are lifts

$$\begin{array}{ccccc} BSO\langle v_{2k}\rangle & \xrightarrow{\mathfrak{l}_{1\to 2}} & BSO\langle v_{2k}(2^\infty)\rangle & \xrightarrow{\mathfrak{l}_{2\to 3}} & BSO\langle \delta v_{2k}\rangle \\ \mathfrak{p}_1 \downarrow & & \mathfrak{p}_2 \downarrow & & \mathfrak{p}_3 \downarrow \\ BSO & = & BSO & = & BSO \end{array}$$

From the Serre spectral sequence, there exists classes $V_{2k} \in H^{2k}(BSO\langle \delta v_{2k}\rangle; \mathbb{Z})$ and $\psi_{2k} \in H^{2k-1}(BSO\langle v_{2k}(2^\infty)\rangle; \mathbb{Z}/2^\infty)$.

Lemma 3.1. $\delta_\infty\big(\psi_{2k}\big) = \mathfrak{l}^*_{2\to 3}(V_{2k})$; $\mathfrak{l}^*_{1\to 2}\big(\psi_{2k}\big) = 0$; $\delta_2\big(\psi_{2k}\big)$ *is the Wu class* $\mathfrak{p}_2^*\big(v_{2k}\big) \in H^{2k}(BSO\langle v_{2k}(2^\infty)\rangle; \mathbb{Z}/2\mathbb{Z})$. *The following diagram commutes*

$$\begin{array}{ccc} H_{2k}(BSO\langle v_{2k}(2^\infty)\rangle; \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{v_{2k}} & \mathbb{Z}/2\mathbb{Z} \\ \delta \downarrow & & \iota \downarrow \\ H_{2k-1}(BSO\langle v_{2k}(2^\infty)\rangle; \mathbb{Z}) & \xrightarrow{\psi_{2k}} & \mathbb{Z}/2^\infty \end{array}$$

Another way to think about even structures is that a bundle $\xi\colon X \to BSO$ has a $v_{2k}(2^\infty)$-structure provided there is a homomorphism h making

$$\begin{array}{ccc} H_{2k}(X;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{v_{2k}} & \mathbb{Z}/2\mathbb{Z} \\ \delta\downarrow & & \iota\downarrow \\ H_{2k-1}(X;\mathbb{Z}) & \xrightarrow{h} & \mathbb{Z}/2^\infty \end{array}$$

commute. If there is such an h, there are even structures such that $h = \psi_{2k}$. Even structures are a $H^{2k-1}(X;\mathbb{Z}/2^\infty)$-torsor: even structures with a fixed h are a ${}_2H^{2k-1}(X;\mathbb{Z}/2^\infty)$-torsor. These remarks follow from the action of the fibre of the total space of the principal fibration.

Silly Remark 3.2. A bundle ξ has $v_{2k}(\xi) = 0$ if and only if h can be taken to be trivial if and only if h restricted to ${}_2H_{2k-1}(X;\mathbb{Z})$ is trivial.

4 4-dimensional manifolds

In dimension four, $v_2 = w_2$, so $BSO\langle v_2\rangle = BSpin$ and $BSO\langle \delta v_2\rangle = BSpin^c$. The map $\psi_2 \colon \pi_1\big(BSO\langle v_2(2^\infty)\rangle\big) \to \mathbb{Z}/2^\infty$ is an isomorphism:

$$BSpin \to BSO\langle v_2(2^\infty)\rangle \xrightarrow{\psi_2} B\mathbb{Z}/2^\infty$$

displays the universal cover.

It follows from Silly Remark 3.2 that

Theorem 4.1 (Bohr and Lee & Li)**.** *Every even, compact* 4 *manifold* M *has a cyclic cover which is Spin: in particular, the cover corresponding to the kernel of* $\psi_2 \colon \pi_1(M) \to \mathbb{Z}/2^\infty$ *is Spin.*

and that

Theorem 4.2. *If* M *is an even* 4 *manifold, the cover corresponding to a subgroup* $\Gamma \subset \pi_1(M)$ *is Spin if and only if the composition*

$$_2H_1(\Gamma;\mathbb{Z}) \to {}_2H_1(\pi_1(M);\mathbb{Z}) \longrightarrow \mathbb{Z}/2^\infty$$

is trivial.

Less silly but still true

Theorem 4.3. *Let π be any finitely present group and let $h\colon \pi \longrightarrow \mathbb{Z}/2^\infty$ be any homomorphism. Then there exist even, compact 4 manifolds with $\pi_1(M) = \pi$ and with ψ_2 for that even structure being h.*

Since the universal cover of an even 4 manifold is Spin, Hopf shows that v_2 comes from $H^2(\pi; \mathbb{Z}/2\mathbb{Z})$. Take $v \in H^2(\pi; \mathbb{Z}/2\mathbb{Z})$ to be the composition

$$H_2(\pi; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\delta} H_1(\pi; \mathbb{Z}) \xrightarrow{h} \mathbb{Z}/2^\infty$$

and results in Teichner's thesis construct an M with the desired properties.

Both Bohr and Lee & Li construct examples of even 4 manifolds for which the cover corresponding to the kernel of ψ_2 is the minimal cyclic cover which is Spin.

For completeness, note that the semi-dihedral group of order 16 has $H_1(SD_{16}; \mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and one can find examples for which ψ_2 is the projection onto $\mathbb{Z}/4\mathbb{Z}$. The evident 4-fold cover is certainly Spin, but so is the 2-fold sub-cover with group $\mathbb{Z}/8\mathbb{Z} \subset SD_{16}$. In fact, given any even 4 manifold with $\pi_1 \cong SD_{16}$, the double cover with fundamental group $\mathbb{Z}/8\mathbb{Z}$ is Spin.

What can one say about the converse to the Bohr, Lee & Li result?
If M^4 has a cyclic Spin cover, must M be even?

To begin more generally, suppose $\widetilde{M} \to M^4 \to B\pi$ is a cover and that $\widetilde{M}$ is Spin. Consider the Serre spectral sequence with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

$H^0\big(B\pi\ ;\ H^2(\widetilde{M}\ ;\ \mathbb{Z}/2\mathbb{Z})\big)$	$\cdot$	$\cdot$	$\cdot$
$H^0\big(B\pi\ ;\ H^1(\widetilde{M}\ ;\ \mathbb{Z}/2\mathbb{Z})\big)$	$H^1\big(B\pi\ ;\ H^1(\widetilde{M}\ ;\ \mathbb{Z}/2\mathbb{Z})\big)$	$\cdot$	$\cdot$
$H^0\big(B\pi\ ;\ \mathbb{Z}/2\mathbb{Z}\big)$	$H^1\big(B\pi\ ;\ \mathbb{Z}/2\mathbb{Z}\big)$	$H^2\big(B\pi\ ;\ \mathbb{Z}/2\mathbb{Z}\big)$	$H^3\big(B\pi\ ;\ \mathbb{Z}/2\mathbb{Z}\big)$

The total degree two line is in red.
Compare this spectral sequence to the one with $\mathbb{Z}/2^\infty$ coefficients.

Lemma 4.4. *If $H_2(B\pi;\mathbb{Z})$ is odd torsion, $H^2(B\pi;\mathbb{Z}/2^\infty) = 0$.*

EG 4.5. $H^2(B\pi;\mathbb{Z}/2^\infty) = 0$ for $\pi = \mathbb{Z}/2^r\mathbb{Z}$, $D_{2^{r+2}}$, $Q_{2^{r+2}}$ and $SD_{2^{r+3}}$.

If $H_1(\widetilde{M};\mathbb{Z})$ has no 2-torsion, then $H^1(\widetilde{M};\mathbb{Z}/2^\infty)$ is 2-divisible and hence $H^1\big(B\pi;H^1(\widetilde{M};\mathbb{Z}/2^\infty)\big) = 0$ if π is a finite 2-group.

Theorem 4.6. *If $\widetilde{M} \to M \to B\pi$ is a cover with $\widetilde{M}$ Spin, and if $H_1(\widetilde{M};\mathbb{Z})$ has no 2-torsion and if π is a finite 2-group with $H^2(B\pi;\mathbb{Z}/2^\infty) = 0$, then M is even.*

To construct examples for which M is not even, note

Theorem 4.7. *If $\widetilde{M} \to M \to B\pi$ is a cover with $\widetilde{M}$ Spin, if $H_1(\widetilde{M};\mathbb{Z}) = \bigoplus_r \mathbb{Z}/2\mathbb{Z}$ and if $v_2(M)$ is non-zero in $E_\infty^{1,1}$, then M is not even.*

This follows since $H_1(\widetilde{M};\mathbb{Z}) = \bigoplus_r \mathbb{Z}/2\mathbb{Z}$ implies $H^1(\widetilde{M};\mathbb{Z}/2\mathbb{Z}) \to H^1(\widetilde{M};\mathbb{Z}/2^\infty)$ is an isomorphism.

EG 4.8. Use results in Teichner's thesis to construct an M^4 with $\pi_1 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $v_2 = x \cup y$ where x, $y \in H^1(B\pi;\mathbb{Z}/2\mathbb{Z})$ are a basis. Then M is not even but it has a Spin double cover.

One can repackage these results as results on free actions of finite groups on Spin 4 manifolds.

5 Group actions on Spin 4 manifolds

Throughout this section, let M^4 be a compact, closed, Spin 4 manifold and let G be a finite group acting freely on M.

If G has odd order, M/G is Spin so $16 \cdot |G|$ divides $\sigma(M)$ by Rochlin's Theorem.

Theorem 5.1. *Let $\sigma(M)$ denote the signature of M. If $H_1(M;\mathbb{Z})$ has no 2-torsion and if $H_2(BG;\mathbb{Z}) = 0$, then $8 \cdot |G|$ divides $\sigma(M)$.*

Some hypotheses were omitted in the lecture for the next three results.

Theorem 5.2. *Let $\sigma(M)$ denote the signature of M. If the 2-Sylow subgroup of G is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and if $H_1(M;\mathbb{Z})$ has no 2-torsion then $4 \cdot |G|$ divides $\sigma(M)$.*

Theorem 5.3. *Assume the hypotheses of 5.2. Further assume*

$$\sigma(M) \equiv 4 \cdot |G| \mod 8 \cdot |G|$$

then M/G is odd. If $v_2(M/G) \in H^2(BG;\mathbb{Z}/2\mathbb{Z})$ and if $\iota\colon \mathbb{Z}/2\mathbb{Z} \subset G$ is any subgroup of order 2, $\iota^*\big(v_2(M/G)\big) \neq 0$.

EG 5.4. Let K^4 be a K3 surface, a simply-connected algebraic surface of signature 16. Habegger constructed free involutions on K as did Enriques. The quotient $K/_{\mathbb{Z}/2\mathbb{Z}}$ is an even manifold of signature 8 as required by Theorem 5.1.

Hitchin constructed a free action of $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ on K so Theorem 5.2 is best possible. In order for Theorem 5.3 to hold, $v_2(K/G) \in H^2(G; \mathbb{Z}/2\mathbb{Z})$ is $x^2 + y^2 + xy$.

The conditions in Theorem 5.3 are hard to achieve. If $G = \bigoplus_3 \mathbb{Z}/2\mathbb{Z}$, then for any $\alpha \in H^2(BG; \mathbb{Z}/2\mathbb{Z})$ there exists an $\iota \colon \mathbb{Z}/2\mathbb{Z} \subset G$ such that $\iota^*(\alpha) = 0$.

Theorem 5.5. *If $H_1(M; \mathbb{Z})$ has no 2-torsion and if $\bigoplus_3 \mathbb{Z}/2\mathbb{Z} \subset G$ is the 2-Sylow subgroup then $8 \cdot |G|$ divides $\sigma(M)$.*

8 Even bordism

In dimension $4k$, even bordism consists of $4k$ manifolds with a $v_{2k}(2^\infty)$-structure modulo those which bound a $4k+1$-manifold with a $v_{2k}(2^\infty)$-structure which restricts. Even bordism is easy to relate to δv_{2k}-bordism: there is a fibration

$$BSO\langle v_{2k}(2^\infty)\rangle \to BSO\langle \delta v_{2k}\rangle \to K\left(\mathbb{Z}\left[\tfrac{1}{2}\right], 2k\right)$$

and a spectral sequence

$$H_p\left(K\left(\mathbb{Z}\left[\tfrac{1}{2}\right], 2k\right); MSO_q\langle v_{2k}(2^\infty)\rangle\right) \Rightarrow MSO_{p+q}\langle \delta v_{2k}\rangle$$

By Serre mod-$\mathfrak{C}$ theory $MSO_*\langle v_{2k}(2^\infty)\rangle \to MSO_*$ is a rational isomorphism with kernel and cokernel 2-torsion; similarly, $MSO_*\langle \delta v_{2k}\rangle \to MSO_*\big(K(\mathbb{Z}, 2k)\big)$ is a rational isomorphism with kernel and cokernel finitely-generated 2-torsion.

It follows from the spectral sequence that

$$MSO_{4k}\langle v_{2k}(2^\infty)\rangle \to MSO_{4k}\langle \delta v_{2k}\rangle$$

is injective.

In dimension 4 the calculation can be done in many ways.

Theorem 8.1. *$MSO_4\langle v_2(2^\infty)\rangle \cong \mathbb{Z}$ with the signature divided by 8 giving the isomorphism.*

One can further check that $MSO_3\langle v_2(2^\infty)\rangle \cong \mathbb{Z}/2^\infty$ and $MSO_5\langle v_2(2^\infty)\rangle \cong \mathbb{Z}/2^\infty \oplus \mathbb{Z}/2^\infty$.

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