Even Manifolds

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1 A modest answer

There is a similar definition using $\mathbb{Z}/2\mathbb{Z}$ coefficients and in this case, Wu gave a very nice criterion in terms of the tangent bundle of $M$ of this mod 2 intersection form to be even. Wu phrased his answer in terms of the stable tangent bundle, $\tau_M: M \to BO$, and what are now called the Wu classes $v_\ell \in H^\ell(BO; \mathbb{Z}/2\mathbb{Z})$

**Theorem 1.1 (Wu).** The mod 2 intersection form of $M^{4k}$ is even if and only if $\tau^*_M(v_{2k}) = 0$.

Christian Bohr, Ronnie Lee and T. J. Li answered the question in terms of the evaluation homomorphism in the Universal Coefficients Theorem,

$$\text{ev}: H^\ell(M; \mathbb{Z}/2\mathbb{Z}) \to \text{Hom}(H_\ell(M; \mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$$

as follows:

**Theorem 1.2.** $M^{4k}$ is even if and only if $\text{ev}(\tau^*_M(v_{2k})) = 0$.

There is an inclusion $\iota: \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2^\infty$ and an induced map on cohomology.

**Theorem 1.3.** $M^{4k}$ is even if and only if $\iota_*(\tau^*_M(v_{2k})) = 0$. 
There is an inclusion \( \iota: \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2^{\infty} \) and an induced map on cohomology.

**Theorem 1.3.** \( M^{4k} \) is even if and only if \( \iota_*(\tau^*_M(v_{2k})) = 0 \).

**Proof.**

\[
\begin{array}{cccccc}
& 0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \\
\text{Ext}(H_{2k-1}(M; \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) & \rightarrow & \text{Ext}(H_{2k-1}(M; \mathbb{Z}), \mathbb{Z}/2^{\infty}) \\
\downarrow & & \downarrow & & \\
H^{2k}(M; \mathbb{Z}/2\mathbb{Z}) & \rightarrow & H^{2k}(M; \mathbb{Z}/2^{\infty}) \\
\downarrow_{\text{ev}} & & \downarrow_{\text{ev}} & & \\
\text{Hom}(H_{2k}(M; \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) & \rightarrow & \text{Hom}(H_{2k}(M; \mathbb{Z}), \mathbb{Z}/2^{\infty}) \\
\downarrow & & \downarrow & & \\
0 & \rightarrow & 0 \\
\end{array}
\]

\( I_* \) is injective.

\[
\text{Ext}(H_{2k-1}(M; \mathbb{Z}), \mathbb{Z}/2^{\infty}) = 0
\]

\( \square \)
Let $v_{\ell}(2^\infty) = \iota_*(v_{\ell}) \in H^{\ell}(BSO; \mathbb{Z}/2^\infty)$.

**Theorem 1.4.** $M^{4k}$ is even if and only if $\tau^*_M(v_{2k}(2^\infty)) = 0$.

**Remark 1.5.** This characterizes evenness as the vanishing of a universal characteristic class and suggests the following shift of viewpoint, going back at least to Lashof.

Let $BSO\langle v_{\ell}(2^\infty) \rangle$ denote the homotopy fibre of the map $BSO \xrightarrow{v_{\ell}(2^\infty)} K(\mathbb{Z}/2^\infty; \ell)$ and let $p_2: BSO\langle v_{\ell}(2^\infty) \rangle \to BSO$ be the inclusion made into a fibration. Then

**Definition 1.6.** A $v_{2k}(2^\infty)$-structure on a bundle $\xi: X \to BO$ is a lift of $\xi$ to $BSO\langle v_{2k}(2^\infty) \rangle$.

**Remark 1.7.** The fibration is principal so the set of lifts is an $H^{2k-1}(X; \mathbb{Z}/2^\infty)$-torsor.
2 Related structures

One can also kill $v_{2k}$ or $\delta v_{2k}$, where $\delta$ is the integral Bockstein, to get principal fibrations

$$BSO\langle v_{2k} \rangle \xrightarrow{p_1} BSO \xrightarrow{v_{2k}} K(\mathbb{Z}/2\mathbb{Z}, 2k)$$

$$BSO\langle \delta v_{2k} \rangle \xrightarrow{p_3} BSO \xrightarrow{\delta v_{2k}} K(\mathbb{Z}, 2k + 1)$$

There are also $v_{2k}$-structures and $\delta v_{2k}$-structures on a bundle, defined as lifts. And the set of lifts are torsors.

Any $v_{2k}$-structure induces a canonical $v_{2k}(2^\infty)$-structure. Since

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

$$0 \rightarrow \mathbb{Z} \xrightarrow{\delta} \mathbb{Z}[\frac{1}{2}] \rightarrow \mathbb{Z}/2^\infty \rightarrow 0$$

commutes, any $v_{2k}(2^\infty)$-structure induces a canonical $\delta v_{2k}$-structure.

Let $\delta_{\infty}$ denote the Bockstein associated to the bottom exact sequence: $\delta$ denotes the Bockstein associated to the top exact sequence.
3 Algebraic Topology

To amplify the last remark, note there are lifts

\[
\begin{array}{cccc}
BSO\langle v_{2k} \rangle & \xrightarrow{\ell_{1\to 2}} & BSO\langle v_{2k}(2^\infty) \rangle & \xrightarrow{\ell_{2\to 3}} & BSO\langle \delta v_{2k} \rangle \\
p_1 \downarrow & & p_2 \downarrow & & p_3 \downarrow \\
BSO & = & BSO & = & BSO
\end{array}
\]

From the Serre spectral sequence, there exists classes \( V_{2k} \in H^{2k}(BSO\langle \delta v_{2k} \rangle; \mathbb{Z}) \) and \( \psi_{2k} \in H^{2k-1}(BSO\langle v_{2k}(2^\infty) \rangle; \mathbb{Z}/2^\infty) \).

**Lemma 3.1.** \( \delta_\infty(\psi_{2k}) = \ell_{2\to 3}(V_{2k}); \ell_{1\to 2}(\psi_{2k}) = 0; \) \( \delta_2(\psi_{2k}) \) is the Wu class \( p_2^*(v_{2k}) \in H^{2k}(BSO\langle v_{2k}(2^\infty) \rangle; \mathbb{Z}/2\mathbb{Z}) \). The following diagram commutes

\[
\begin{array}{cccc}
H_{2k}(BSO\langle v_{2k}(2^\infty) \rangle; \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{v_{2k}} & \mathbb{Z}/2\mathbb{Z} \\
\delta \downarrow & & \ell \downarrow \\
H_{2k-1}(BSO\langle v_{2k}(2^\infty) \rangle; \mathbb{Z}) & \xrightarrow{\psi_{2k}} & \mathbb{Z}/2^\infty
\end{array}
\]
Another way to think about even structures is that a bundle $\xi: X \to BSO$ has a $v_2k(2^\infty)$-structure provided there is a homomorphism $h$ making

$$
\begin{array}{ccc}
H_{2k}(X; \mathbb{Z}/2\mathbb{Z}) & \overset{v_2k}{\longrightarrow} & \mathbb{Z}/2\mathbb{Z} \\
\delta \downarrow & & \downarrow \iota \\
H_{2k-1}(X; \mathbb{Z}) & \overset{h}{\longrightarrow} & \mathbb{Z}/2^\infty
\end{array}
$$

commute. If there is such an $h$, there are even structures such that $h = \psi_{2k}$. Even structures are a $H^{2k-1}(X; \mathbb{Z}/2^\infty)$-torsor: even structures with a fixed $h$ are a $2H^{2k-1}(X; \mathbb{Z}/2^\infty)$-torsor. These remarks follow from the action of the fibre of the total space of the principal fibration.

**Silly Remark 3.2.** A bundle $\xi$ has $v_2k(\xi) = 0$ if and only if $h$ can be taken to be trivial if and only if $h$ restricted to $2H_{2k-1}(X; \mathbb{Z})$ is trivial.
4 4-dimensional manifolds

In dimension four, \( v_2 = w_2 \), so \( BSO\langle v_2 \rangle = BSpin \) and \( BSO\langle \delta v_2 \rangle = BSpin^c \).

The map \( \psi_2: \pi_1( BSO\langle v_2(2^\infty) \rangle ) \rightarrow \mathbb{Z}/2^\infty \) is an isomorphism:

\[
BSpin \rightarrow BSO\langle v_2(2^\infty) \rangle \xrightarrow{\psi_2} B\mathbb{Z}/2^\infty
\]
displays the universal cover.

It follows from Silly Remark 3.2 that

**Theorem 4.1** (Bohr and Lee & Li). *Every even, compact 4 manifold \( M \) has a cyclic cover which is Spin: in particular, the cover corresponding to the kernel of \( \psi_2: \pi_1(M) \rightarrow \mathbb{Z}/2^\infty \) is Spin.*

and that

**Theorem 4.2.** *If \( M \) is an even 4 manifold, the cover corresponding to a subgroup \( \Gamma \subset \pi_1(M) \) is Spin if and only if the composition*

\[
_2H_1(\Gamma; \mathbb{Z}) \rightarrow _2H_1(\pi_1(M); \mathbb{Z}) \rightarrow \mathbb{Z}/2^\infty
\]

*is trivial.*

Less silly but still true
**Theorem 4.3.** Let $\pi$ be any finitely present group and let $h: \pi \to \mathbb{Z}/2\infty$ be any homomorphism. Then there exist even, compact 4 manifolds with $\pi_1(M) = \pi$ and with $\psi_2$ for that even structure being $h$.

Since the universal cover of an even 4 manifold is Spin, Hopf shows that $v_2$ comes from $H^2(\pi; \mathbb{Z}/2\mathbb{Z})$. Take $v \in H^2(\pi; \mathbb{Z}/2\mathbb{Z})$ to be the composition

$$
H_2(\pi; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\delta} H_1(\pi; \mathbb{Z}) \xrightarrow{h} \mathbb{Z}/2\infty
$$

and results in Teichner’s thesis construct an $M$ with the desired properties.

Both Bohr and Lee & Li construct examples of even 4 manifolds for which the cover corresponding to the kernel of $\psi_2$ is the minimal cyclic cover which is Spin.

For completeness, note that the semi-dihedral group of order 16 has $H_1(SD_{16}; \mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and one can find examples for which $\psi_2$ is the projection onto $\mathbb{Z}/4\mathbb{Z}$. The evident 4-fold cover is certainly Spin, but so is the 2-fold sub-cover with group $\mathbb{Z}/8\mathbb{Z} \subset SD_{16}$. In fact, given any even 4 manifold with $\pi_1 \cong SD_{16}$, the double cover with fundamental group $\mathbb{Z}/8\mathbb{Z}$ is Spin.

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What can one say about the converse to the Bohr, Lee & Li result?
If $M^4$ has a cyclic Spin cover, must $M$ be even?
To begin more generally, suppose $\widetilde{M} \to M^4 \to B\pi$ is a cover and that $\widetilde{M}$ is Spin. Consider the Serre spectral sequence with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

$$
\begin{array}{cccc}
H^0(B\pi; H^2(\widetilde{M}; \mathbb{Z}/2\mathbb{Z})) & \cdot & \cdot & \cdot \\
H^0(B\pi; H^1(\widetilde{M}; \mathbb{Z}/2\mathbb{Z})) & H^1(B\pi; H^1(\widetilde{M}; \mathbb{Z}/2\mathbb{Z})) & \cdot & \cdot \\
H^0(B\pi; \mathbb{Z}/2\mathbb{Z}) & H^1(B\pi; \mathbb{Z}/2\mathbb{Z}) & H^2(B\pi; \mathbb{Z}/2\mathbb{Z}) & H^3(B\pi; \mathbb{Z}/2\mathbb{Z})
\end{array}
$$

The total degree two line is in red.

Compare this spectral sequence to the one with $\mathbb{Z}/2\infty$ coefficients.

**Lemma 4.4.** If $H_2(B\pi; \mathbb{Z})$ is odd torsion, $H^2(B\pi; \mathbb{Z}/2\infty) = 0$.

**EG 4.5.** $H^2(B\pi; \mathbb{Z}/2\infty) = 0$ for $\pi = \mathbb{Z}/2^r\mathbb{Z}$, $D_{2^r+2}$, $Q_{2^r+2}$ and $SD_{2^r+3}$.

If $H_1(\widetilde{M}; \mathbb{Z})$ has no 2-torsion, then $H^1(\widetilde{M}; \mathbb{Z}/2\infty)$ is 2-divisible and hence $H^1(B\pi; H^1(\widetilde{M}; \mathbb{Z}/2\infty)) = 0$ if $\pi$ is a finite 2-group.

**Theorem 4.6.** If $\widetilde{M} \to M \to B\pi$ is a cover with $\widetilde{M}$ Spin, and if $H_1(\widetilde{M}; \mathbb{Z})$ has no 2-torsion and if $\pi$ is a finite 2-group with $H^2(B\pi; \mathbb{Z}/2\infty) = 0$, then $M$ is even.
To construct examples for which $M$ is not even, note

**Theorem 4.7.** If $\tilde{M} \rightarrow M \rightarrow B\pi$ is a cover with $\tilde{M}$ Spin, if $H_1(\tilde{M};\mathbb{Z}) = \bigoplus_r \mathbb{Z}/2\mathbb{Z}$ and if $v_2(M)$ is non-zero in $E_{1,1}^{1,1}$, then $M$ is not even.

This follows since $H_1(\tilde{M};\mathbb{Z}) = \bigoplus_r \mathbb{Z}/2\mathbb{Z}$ implies $H^1(\tilde{M};\mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(\tilde{M};\mathbb{Z}/2\mathbb{Z})$ is an isomorphism.

**EG 4.8.** Use results in Teichner’s thesis to construct an $M^4$ with $\pi_1 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $v_2 = x \cup y$ where $x, y \in H^1(B\pi;\mathbb{Z}/2\mathbb{Z})$ are a basis. Then $M$ is not even but it has a Spin double cover.

One can repackagethese results as results on free actions of finite groups on Spin 4 manifolds.
5 Group actions on Spin 4 manifolds

Throughout this section, let $M^4$ be a compact, closed, Spin 4 manifold and let $G$ be a finite group acting freely on $M$.

If $G$ has odd order, $M/G$ is Spin so $16 \cdot |G|$ divides $\sigma(M)$ by Rochlin’s Theorem.

**Theorem 5.1.** Let $\sigma(M)$ denote the signature of $M$. If $H_1(M; \mathbb{Z})$ has no 2-torsion and if $H_2(BG; \mathbb{Z}) = 0$, then $8 \cdot |G|$ divides $\sigma(M)$.

Some hypotheses were omitted in the lecture for the next three results.

**Theorem 5.2.** Let $\sigma(M)$ denote the signature of $M$. If the 2-Sylow subgroup of $G$ is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and if $H_1(M; \mathbb{Z})$ has no 2-torsion then $4 \cdot |G|$ divides $\sigma(M)$.

**Theorem 5.3.** Assume the hypotheses of 5.2. Further assume

$$\sigma(M) \equiv 4 \cdot |G| \mod 8 \cdot |G|$$

then $M/G$ is odd. If $v_2(M/G) \in H^2(BG; \mathbb{Z}/2\mathbb{Z})$ and if $\iota: \mathbb{Z}/2\mathbb{Z} \subset G$ is any subgroup of order 2, $\iota^*(v_2(M/G)) \neq 0$. 
EG 5.4. Let $K^4$ be a K3 surface, a simply-connected algebraic surface of signature 16. Habegger constructed free involutions on $K$ as did Enriques. The quotient $K/\mathbb{Z}/2\mathbb{Z}$ is an even manifold of signature 8 as required by Theorem 5.1.

Hitchin constructed a free action of $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ on $K$ so Theorem 5.2 is best possible. In order for Theorem 5.3 to hold, $v_2(K/G) \in H^2(G; \mathbb{Z}/2\mathbb{Z})$ is $x^2 + y^2 + xy$.

The conditions in Theorem 5.3 are hard to achieve. If $G = \bigoplus \mathbb{Z}/2\mathbb{Z}$, then for any $\alpha \in H^2(BG; \mathbb{Z}/2\mathbb{Z})$ there exists an $i: \mathbb{Z}/2\mathbb{Z} \subset G$ such that $i^*(\alpha) = 0$.

**Theorem 5.5.** If $H_1(M; \mathbb{Z})$ has no 2-torsion and if $\bigoplus \mathbb{Z}/2\mathbb{Z} \subset G$ is the 2-Sylow subgroup then $8 \cdot |G|$ divides $\sigma(M)$.
8 Even bordism

In dimension $4k$, even bordism consists of $4k$ manifolds with a $v_{2k}(2^\infty)$-structure modulo those which bound a $4k + 1$-manifold with a $v_{2k}(2^\infty)$-structure which restricts. Even bordism is easy to relate to $\delta v_{2k}$-bordism: there is a fibration

$$BSO\langle v_{2k}(2^\infty) \rangle \to BSO\langle \delta v_{2k} \rangle \to K(\mathbb{Z}[\frac{1}{2}], 2k)$$

and a spectral sequence

$$H_p(K(\mathbb{Z}[\frac{1}{2}], 2k) ; MSO_q\langle v_{2k}(2^\infty) \rangle) \Rightarrow MSO_{p+q}\langle \delta v_{2k} \rangle$$

By Serre mod-$\mathcal{C}$ theory $MSO_*\langle v_{2k}(2^\infty) \rangle \to MSO_*$ is a rational isomorphism with kernel and cokernel 2-torsion; similarly, $MSO_*\langle \delta v_{2k} \rangle \to MSO_*\langle K(\mathbb{Z}, 2k) \rangle$ is a rational isomorphism with kernel and cokernel finitely-generated 2-torsion.

It follows from the spectral sequence that

$$MSO_{4k}\langle v_{2k}(2^\infty) \rangle \to MSO_{4k}\langle \delta v_{2k} \rangle$$

is injective.

In dimension 4 the calculation can be done in many ways.
Theorem 8.1. $MSO_4\langle v_2(2^\infty) \rangle \cong \mathbb{Z}$ with the signature divided by 8 giving the isomorphism.

One can further check that $MSO_3\langle v_2(2^\infty) \rangle \cong \mathbb{Z}/2^\infty$ and $MSO_5\langle v_2(2^\infty) \rangle \cong \mathbb{Z}/2^\infty \oplus \mathbb{Z}/2^\infty$. 
References


