The Exact Sequence of a Principal Fibration

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Principal fibrations and homotopy classes of maps

Examples

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Spaces are based as are maps so there is a base point $* = *_E \in E$ with $p(*) = *_B \in B$ and $\theta(p(*)) = *_C \in C$ the corresponding base points.

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  $[M^4, S^2]$ *comes up in studying broken Lefschetz fibrations on the 4-manifold* M^4*.*

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The proof is worth recalling.

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One can always use this model for the total space of a principal fibration: $b \in B$, $\lambda \in C^{[0,1]}$, $\lambda(0) = p(b)$.

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The action comes from an action of spaces $\Omega C \times E \to E$. A point in E is a point $b \in B$ and a path $\lambda \in C$ starting at $*_C$, the base point of C. Given a loop, ℓ, at $*_C$ and (b, λ) construct a new point in E, $(b, \ell + \lambda)$. □

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  - If f is null homotopic $\mathrm{Lift}_f(X, Y)$ is isomorphic to $[X, \Omega Y]$.

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- *The group* $[X, \Omega C]$ *acts on the set* $[X, E]$ *on the right.*
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1. $[X, \Omega C]$ still only depends on X and C.
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3. The map may depend on everything.

Maps of a 4-complex to a 2-sphere

The “Yes” answer to “Is this of any real use?” is best supplied by example.

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The sequence becomes

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Next the group $[X, \Omega\mathbf{HP}^\infty] = [X, S^3]$ fits in an exact sequence

$$H^2(X; \mathbb{Z}) \xrightarrow{Sq^2} H^4(X; \mathbb{Z}/2\mathbb{Z}) \to [X, S^3] \to H^3(X; \mathbb{Z}) \to 0$$

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For dimensional reasons, S^3 can be replaced by a 2-stage Postnikov system which is an infinite loop space so for the complexes considered here $[X,S^3]$ is an abelian group.

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A result of Larmore and Thomas [2] does. In this case it says

Theorem

Let X be a finite complex of dimension $\leqslant 4$. Fix $\gamma \in H^3(X;\mathbb{Z})$ and suppose there is a $k \geqslant 1$ such that $2^k\gamma = 0$. Pick $\gamma' \in H^2(X;\mathbb{Z}/2^k\mathbb{Z})$ with $\delta_k(\gamma') = \gamma$. Reduce γ' mod 2 and compute $Sq^2(\gamma') \in H^4(X;\mathbb{Z}/2\mathbb{Z})/Sq^2(H^2(X;\mathbb{Z})) \subset [X,S^3]$. For any $\bar\gamma \in [X,S^3]$ which maps to γ, $2^k\bar\gamma = Sq^2(\gamma')$.

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This theorem explains how to decide if a $\mathbb{Z}/2^k\mathbb{Z}$ summand of $H^3(X;\mathbb{Z})$ is a summand of $[X,S^3]$ or is hit by a $\mathbb{Z}/2^{k+1}\mathbb{Z}$ summand.

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Example

Suppose X is a complex of dimension $\leqslant 4$ and suppose that $Sq^2\colon H^2(X;\mathbb{Z}) \to H^4(X;\mathbb{Z}/2\mathbb{Z})$ and $Sq^2\colon H^2(X;\mathbb{Z}/2\mathbb{Z}) \to H^4(X;\mathbb{Z}/2\mathbb{Z})$ have the same image. Then $[X, S^3] = \mathrm{coker}(Sq^2) \oplus H^3(X;\mathbb{Z})$.

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If X is Habegger's manifold [1] or an Enrique's surface, then $Sq^2\colon H^2(X;\mathbb{Z}) \to H^4(X;\mathbb{Z}/2\mathbb{Z})$ is zero but $Sq^2\colon H^2(X;\mathbb{Z}/2\mathbb{Z}) \to H^4(X;\mathbb{Z}/2\mathbb{Z})$ is onto. Since $H^3(X;\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ it follows that $[X,S^3] \cong \mathbb{Z}/4\mathbb{Z}$.

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It remains to work out the homomorphism

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If the map $\pounds\mathbf{CP}^\infty \to \pounds\mathbf{HP}^\infty$ were an H-map, the calculation would be easy, but alas it is not.

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The H-space structure gives decompositions

$$\Omega\mathbf{CP}^\infty \times \mathbf{CP}^\infty \xrightarrow{\cong} \pounds\mathbf{CP}^\infty \to \pounds\mathbf{HP}^\infty \xrightarrow{\cong} \Omega\mathbf{HP}^\infty \times \mathbf{HP}^\infty$$

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An old formula of Rutter's [7] can be explained in terms of Zabrodsky's deviation from being an H-map [9]. Zabrodsky's deviation is a map

$$\Omega\mathbf{CP}^\infty \wedge \mathbf{CP}^\infty \to \Omega\mathbf{HP}^\infty$$

In the case here, it is only necessary to determine the deviation on the $S^3 \subset \Omega\mathbf{CP}^\infty \wedge \mathbf{CP}^\infty$ and this is determined by the induced map $H_3(£\mathbf{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z} \to H_3(£\mathbf{HP}^\infty;\mathbb{Z}) \cong \mathbb{Z}$ which is multiplication by ± 2. (This is not obvious.)
It follows that the map $\psi_f \colon H^1(X;\mathbb{Z}) \to [X, S^3]$ is determined as follows.

- For $x \in H^1(X;\mathbb{Z})$ calculate $x \cup p(f) \in H^3(X;\mathbb{Z})$.
  - Recall the surjection $[X, S^3] \to H^3(X;\mathbb{Z})$
  - Lift $x \cup p(f)$ to $[X, S^3]$ and then multiply by 2.
- Since $H^4(X;\mathbb{Z}/2\mathbb{Z}) \to [X, S^3] \to H^3(X;\mathbb{Z})$ is exact, the lift may not unique but twice it is.

Example (Pontryagin)

Let $X = S^2 \times S^1$. Then $H^2(X;\mathbb{Z}) \cong \mathbb{Z}$: let γ be a generator. If $\beta = c\gamma$ then there are maps $f\colon X \to S^2$ such that $\beta = p(f)$ and there is a bijection between $p_{\#}^{-1}(\beta)$ and $\mathbb{Z}$ if $c = 0$ and $\mathbb{Z}/2c\mathbb{Z}$ otherwise.

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Example

Let $X = S^2 \times S^1 \times S^1$. Let $\{\mathfrak{a}_1, \mathfrak{a}_2\} \subset H^1(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ be a basis and let $\{\mathfrak{a} = \mathfrak{a}_1 \cup \mathfrak{a}_2, \mathfrak{b}\} \subset H^2(X;\mathbb{Z})$ be a basis. It follows that $\{\mathfrak{b} \cup \mathfrak{a}_1, \mathfrak{b} \cup \mathfrak{a}_2\}$ is a basis for $H^3(X;\mathbb{Z})$. Then $\beta = a\mathfrak{a} + b\mathfrak{b}$ has square 0 if and only if $a \cdot b = 0$. If $b = 0$, then $\operatorname{coker}(\psi_f) = H^3(X;\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$. If $a = 0$ and $b \neq 0$, then the image of ψ_f is spanned by $(2b)\,\mathfrak{b} \cup \mathfrak{a}_1$ and $(2b)\,\mathfrak{b} \cup \mathfrak{a}_2$ and so $\operatorname{coker}(\psi_f) \cong \mathbb{Z}/2b\mathbb{Z} \oplus \mathbb{Z}/2b\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

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