Model Problem

\[- \phi_e + \nabla^2 \phi - R \phi_{xx} = \phi \phi_e\]

\[\phi = \phi_{ee} = \phi_{xx} = 0 \quad m = 0, \pi\]

\(\phi\) bounded as \(x \to \pm \infty\)

- According to linear theory:

\[\phi = \sum \beta_{np} e^{T_{np} t} \sin p \varphi \cos n \kappa x\]

\(\text{sol:}\)

\[\phi = e^{\frac{n\kappa}{2}} \sin \varphi \cos n \kappa x \quad (n = 1, p = 1)\]

\[T = k^2(R - \lambda)\]

\[\lambda = \frac{(1 + k^2)^3}{k^2}, \quad R_{\min} = \frac{27}{4n \pi^2} - 1\]

\[n = 1, p = 1\]

\[s_0 = \phi \phi_e = - \phi_e + \nabla^2 \phi - R \phi_{xx}\]

- Try straight out expansion:
  - First iterate: \(s_0 = 0\)

  \[s_1 = \phi \phi_{1e} = e^{2 \frac{T}{k^2}} \sin \varphi \cos \frac{n \kappa}{2} \cos \kappa x\]

  \[= \frac{1}{4} e^{2 \frac{T}{k^2}} \sin \frac{2 \varphi}{2} (1 + \cos 2k x)\]

  \[s_2 = \phi \phi_{2e} + \phi \phi_{1e}\]

  \[\phi_2 = e^{2 \frac{T}{k^2}} \sum k_0 + k_2 \cos 2k x \cdot \sin \varphi\]

  - it has double growth rate.
\[ I \phi_2 = \frac{\partial}{\partial x} \phi_2 = e^{\frac{3}{32}} \frac{\partial}{\partial x} \left\{ \sin x \cos kx (k_0 + k_1 \cos 2kx) \right\} \]

\[ = e^{\frac{3}{32}} \left\{ \sin x \cos kx + \ldots \right\} \]

- Replication of first order problem.
- Unbounded sol.
- This is bad. In increased growth rate.

**Bifurcation Approach:**

- Since eigenfunction at criticality is steady (i.e., time dependence, \( \nabla_t = 0 \)), try to find a solution to nonlinear problem near criticality.
- Object is to solve: \( \nabla^2 \phi - R \phi_{xx} = \phi \phi_2 \) (same problem, but \( \phi_2 \)).
- Let \( \varepsilon \) be a small parameter,

\[ \phi = \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \varepsilon^3 \phi_3 + \ldots \]

\[ R = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \ldots \]

\[ \Delta \phi = \nabla^2 \phi - \left( R_0 \phi_{xx} \right) = (R - R_0) \phi_{xx} + \phi \phi_2 \]

Then, \( \frac{\partial}{\partial x} \left( \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \varepsilon^3 \phi_3 \right) = (\varepsilon R_1 + \varepsilon^2 R_2) (\varepsilon \phi_1 + \varepsilon^2 \phi_2 + \varepsilon^3 \phi_3) + (\varepsilon \phi_1 + \varepsilon^2 \phi_2 + \varepsilon^3 \phi_3) (\varepsilon \phi_2 + \varepsilon^2 \phi_2 + \varepsilon^3 \phi_3 + \ldots) \)
Equate coeffs. of like powers of \( \varepsilon \):

\[
\begin{align*}
2 \phi_1 &= 0 & : \Theta(\varepsilon^1) \\
2 \phi_2 &= R_1 \phi_{1,xx} + \phi_1 \phi_{1,xx} & : \Theta(\varepsilon^3)
\end{align*}
\]

\[
2 \phi_3 = R_1 \phi_{2,xx} + R_4 \phi_{1,xx} + \phi_{1,xx} \phi_{1,xx} + \phi_1 \phi_{1,xx} : \Theta(\varepsilon^3)
\]

1st eq. has eigenvalue of \((\text{this has been done many times})\): 

\[
\phi_1 = \cos k x \sin \varepsilon
\]

\[
\nabla^2 \phi_1 = -(k^2+1) \cos k x \sin \varepsilon
\]

\[
\nabla^6 \phi_1 = -(k^2+1)^3 \cos k x \sin \varepsilon = -R_L k^2 \cos k x \sin \varepsilon
\]

where \( R_L = \frac{(k^2+1)^3}{k^2} \)

2nd eq.

\[
2 \phi_2 = -k^2 R_1 \cos k x \sin \varepsilon + \cos^2 k x \sin \varepsilon \cos \varepsilon
\]

\[
= \frac{1}{4} k^2 \sin 2 \varepsilon + \frac{1}{4} \cos 2 k x \sin 2 \varepsilon
\]

from here on, \( R_1 = 0 \) otherwise we cannot have sol. (i.e., will not remain bounded as variable grows)

\[
\nabla^6 \phi_2 - R_L \phi_{2,xx} = \frac{1}{4} \sin 2 \varepsilon + \frac{1}{4} \cos 2 k x \sin 2 \varepsilon
\]

\[
\text{solv: } \phi_2 = A \sin 2 \varepsilon + B \cos 2 k x \sin 2 \varepsilon
\]

then: \( \text{subs and equating like terms:} \)

\[-64 A \sin 2 \varepsilon = \frac{1}{4} \sin 2 \varepsilon \implies A = -\frac{1}{256} \]

\[
R L \phi_3 = (4 k^2 + 1)^3 + 4 k^2 R_L \phi_3 = \frac{1}{4}
\]

\[
D \rule[5pt]{1mm}{2mm} \phi_3 = 64 (k^2+1)^3 + 4 k^2 R_L \phi_3 = \frac{1}{4}
\]
\[ R_2 = \frac{k^2}{k^2} \left( \frac{1}{512} + \frac{1}{960k^2R_L} \right) \] for nonsecular \( \phi_3 \).

This procedure could be carried out further, but we have what we need.

There exists a solution, time independent,
of the form

\[ \phi = \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \cdots \]
\[ R = R_L + \varepsilon^2 R_2 + \cdots \]

- Note that \( R_2 > 0 \). So that

\[ R - R_L = \varepsilon^2 R_2 > 0 \]

- Therefore, the solution exists only for \( R > R_L \).

The earlier solution of this disturbance eq. was \( u = 1 \). So we have two solutions,

1. \( u = 1 \)
2. \( u = 1 + \varepsilon \phi_1 + \cdots \)

- (a) Time dependent and no spatial dependence; exists for all \( R \).

- (b) Exists for \( R > R_L \), spatial dependence in \( x + \varepsilon \), time-independent.