\[ \varepsilon = \sqrt{\frac{R - R_L}{R_L}} \] gives eq. of curve \( u = \varepsilon A \)

- This is what happens in certain circumstances, even though these results are not from the N-S eqs.

**Problem**: we can conclude nothing about the stability of the bifurcation solution, since \( \partial \) was dropped from eq. It is that term which yields the necessary stability information.

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**Landau - Stuart - Watson Procedure**

\[- \phi_x + \nabla^2 \phi - R \phi_{xx} = 0\]

If \( \phi = A(t) \cos kx \sin z \), the eq. becomes,

\[-A \cos kx \sin z - (k^2 + 1)^3 A \cos kx \sin z + ARk^2 \cos kx ; \sin^2 z = 0\]

then, \( \dot{A} = \sum Rk^2 - (k^2 + 1)^3 \frac{3}{2} A \)

using \( (k^2 + 1)^3 = R_Lk^2 \),

then \( \dot{A} = k^2 (R - R_L) A \)

\[ \therefore \dot{A} = \nabla A ; \quad \nabla = k^2 (R - R_L) \]

So: growth rate \( \nabla \) depends on whether \( R \geq R_L \) and the same result is reached using a general \( A(t) \) time dependence.
viz. \( e^{\tau t} \) time dependence.

In 1944 Landau proposed the amplitude eq. of the form,

\[
\hat{A} = F(A)
\]

where \( F(A) = \tau A \) when \( A \) is small.

The first specific proposal for \( \hat{A} \) was

\[
\hat{A} = \tau A - a_{11} A^3 + \ldots \quad \text{-- Ricatti Eq.}
\]

Take a look at \( \hat{A} = \tau A - a_{11} A^3 \), use change of variables. \( y = \frac{1}{A^2} \)

Then, \( \frac{dy}{A^3} = -2 \frac{\hat{A}}{A^3} \)

and \( -\frac{A^2 y'}{2} = \tau A - a_{11} A^3 \)

\[
-\frac{y'}{2} = \frac{\tau}{A^2} - a_{11} = \tau y - a_{11}
\]

\[
\therefore \quad y' + 2\tau y = 2a_{11} \quad \text{so now it's linear.}
\]

Sol. \( y = \frac{a_{11}}{\tau} + Ce^{-2\tau t} \quad C = \text{const.} \)

\[
A = \frac{1}{\sqrt{\frac{a_{11}}{\tau} + Ce^{-2\tau t}}}
\]
Cases:
\[ A = F(A), \quad a_{11} = 0 \quad \Rightarrow \quad A = \frac{1}{\sqrt{e}} e^{\frac{t}{e}} \]
\[ \rightarrow \infty \quad a_{ii} \rightarrow \infty \quad \forall > 0 \]
\[ a_{ii} > 0 \quad \Rightarrow \quad A \rightarrow \sqrt{\frac{e}{d_{ii}}} \quad \text{as} t \rightarrow \infty \quad \forall > 0 \]

\( A \)
\[ \sqrt{\frac{e}{d_{ii}}} \]
\[ a_{11} \text{ absent} \]
\[ a_{11} \text{ present} \]

Now, take the model eq: \[ L \phi = D^2 \phi - K \phi_{xx} \]
\[ \phi = \phi_1 + \varepsilon \phi_2 + \varepsilon^2 \phi_3 + \ldots \]
\[ \phi_1 = 0 \quad \text{and} \quad \phi_1 = A \cos kx \sin z \quad \text{proven} \quad \hat{A} = \nabla A \]

Also, \[ L \phi_2 = \phi_1 \phi_2 = \frac{A^2}{4} \left( 1 + \cos 2kx \right) \sin 2z \]
and \[ \phi_2 = A^2 \left( \frac{1}{k} \sin 2z + \frac{1}{k \cos 2kx} \sin 2z \right) \]
with \( \hat{A} = \nabla A \) still.

Next eq:
\[ L \phi_3 \equiv -\frac{\partial \phi_3}{\partial t} + \nabla^2 \phi_3 - R \phi_{3xx} = \left( \phi_1 \phi_2 \right) \]
\[ = A^2 \left( \frac{1}{k} \sin 2z \cos kx \right) + \ldots \]
\[ \varphi_3 = A \sin z \cos kx + \ldots \]

So that
\[ \frac{dA}{dt} - k^2 (R - R_0) A = \left( \frac{1}{a_{11}} \right) A^3 \]

\[ \frac{dA}{dt} = -k^2 (R - R_0) A - a_{11} A^3 \]

Time-independent soln. : \[ A = 0 \]
\[ A^2 = \frac{k^2 (R - R_0)}{a_{11}} \]

*The A^2 soln is exactly the solution we get through bifurcation theory. So the bifurcation soln is the soln of the Landau eqn. So the Landau approach includes the bifurcation approach.*

Stability of solution, \[ A = \sqrt[3]{\frac{k^2 (R - R_0)}{a_{11}}} \]

To do stability analysis we consider perturbations.
What is stability of solution \( A = \sqrt{\frac{k^2 (R-R_c)}{a_{ii}}} \)

(we guessed from bifurcation theory that it was stable). 

Perturbation of \( A \): \( A = \bar{A} + \alpha \)

as before, \( \frac{dA}{dt} = k^2 (R-R_c)A - a_{ii} A^3 \)

\[ \frac{d\alpha}{dt} = k^2 (R-R_c) \bar{A} + k^2 (R-R_c) \alpha - a_{ii} (\bar{A} + \alpha)^3 \]

\[ = k^2 (R-R_c) \alpha - 3a_{ii} \bar{A}^2 \alpha \]

\[ = k^2 (R-R_c) \alpha - 3k^2 (R-R_c) \alpha \]

\[ \frac{d\alpha}{dt} = -2k^2 (R-R_c) \alpha \]

\[ \therefore \text{ Stable } \left( \frac{d\alpha}{dt} < 0 \right) \text{ if } R > R_c \]

Recall that this solution only existed if \( R > R_c \).

\[ \therefore \text{ guess from bifurcation theory was correct. } \]

\[ \therefore \text{ For the conditions of existence, the solution is stable } \]

[Diagram of stability region]
Now suppose \( a_{ii} < 0 \) \( \forall i \).

Then the solution eq becomes,

\[
k^2 (R - R_0) A + b_{ii} A^3 = 0
\]

solutions, \( A = 0 \)

\[
A^2 = \frac{-k^2 (R - R_0)}{b_{ii}}
\]

\( A \) must be real, which is only true if \( R < R_L \) \( (b_{ii} > 0) \)

This gives \textit{subcritical bifurcation},

\[
\begin{array}{c}
A \\
\downarrow \\
\searrow \\
R_L
\end{array}
\]

and,

\[
\frac{dA}{dt} = k^2 (R - R_0) + b_{ii} A^3 = -2k^2 (R - R_L) A
\]

\( \Rightarrow \text{unstable since } R < R_L \)

Thus, the sign of \( a_{ii} \) decides whether you have a \textit{supercritical} or \textit{subcritical bifurcation}.

\( a_{ii} \) is the \textit{Landau constant}.\]
For \( \frac{dA}{dE} = rA \pm a_{11} A^3 \)
\( a_{11} > 0 \) Taylor - Couette, (super)
\( a_{11} \geq 0 \) Bénard (depending on structure of the problem)
\( a_{11} < 0 \) Channel flow, (sub)

**Bifurcation Theorem**

\[
\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(R) x_j + F_i(x_1, \ldots, x_n)
\]

in vector form:

(1) \[
\frac{dx}{dt} = Ax + F, \quad \text{where } A = A(R) \text{ and } A \text{ is a real parameter.}
\]

\( F \) is a polynomial function of \( x_1, x_2, \ldots, x_n \) of degree 2 at least.

\( F(0) = 0 \Rightarrow x = 0 \) is a solution of (1)

Consider linearized problem,

\[
\frac{dx}{dt} = Ax
\]

Sol.
\[
x = x_0 e^{At}, \quad T = s = \omega A =
\]
Stability is decided by eigenvalues of $A$.

1. If all eigenvalues of $A$ have negative real parts, then $x = 0$ is a stable (asymptotically) solution of (1) for sufficiently small disturbances.

2. If $A$ has an eigenvalue with a positive real part, then $x = 0$ is an unstable solution of (1).

3. If as $R$ increases through some $R_c$, a single eigenvalue of $A$ changes from real negative to real positive, then a solution bifurcates from $x = 0$ at $R = R_c$.

4. Under conditions of (3), and if all eigenvalues have real parts when $R < R_c$, the bifurcation solution is stable if supercritical and unstable if subcritical.
How about if $\tau$ is complex?

\[
\frac{dx}{dt} = Ax, \quad x = e^{\tau t} ; \quad \tau = \sigma + i\omega
\]

Canonical form:

\[
\frac{dx_1}{dt} = \sigma x_1 + i\omega x_2
\]

\[
\frac{dx_2}{dt} = -i\omega x_1 + \sigma x_2
\]

In complex notation, $\tau = x_1 + ix_2$

\[
\frac{d\tau}{dt} = \tau \cdot \bar{\tau}
\]

At critical point, when $\tau = 0$,

\[
\frac{dx_1}{dt} = i\omega x_2, \quad \frac{dx_2}{dt} = -i\omega x_1,
\]

\[
\frac{d^2x_1}{dt^2} + \omega^2 x_1 = 0
\]

Sol: $x_1 = \begin{cases} \cos \omega t \\ \sin \omega t \end{cases}$

Thus, eigenfunction of linear problem at criticality is periodic, with period $\frac{2\pi}{\omega}$.

Eq. in plane Poiseuille flow: linear exponential growth given by $e^{i\omega(x-ct)} = \text{temporal growth}$.