An Alternative Sensitivity Function for the Campbell-Cochrane Model

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Abstract

In today's economic world of stocks, bonds, and mattresses for investing or hiding one's hard-earned money, an investor is faced with the difficult decisions of where and in what amount to entrust his assets. In order to better understand how investors make such decisions, many economists focus their research on discerning the appropriate utility function, which encapsulates the driving forces for decision-making. Campbell and Cochrane [**?**] proposed a consumption-based model of stock price motion that employs external habit as the motivator in investor decision-making. The current versions of and analysis based on the Campbell and Cochrane [**?**] external habit model assume the risk-free interest rate to be constant or linear. Neither of these assumptions is consistent with the statistical data. In addition, a linear risk-free rate eventually leads to a negative interest rate which would lead to unlimited borrowing. If this model is to be used to describe the economy, then a new risk-free rate must be incorporated successfully. A logistic risk-free rate is introduced which leads to a non-negative and bounded interest rate. Consequently, the logistic interest rate model better describes the behavior of the economy. The mathematical analysis is based on the method used in Chen, Cosimano, and Himonas [**?**] but this time an appropriate logistic-form equation for the risk-free rate is employed. The other properties of the Campbell and Cochrane model are not affected in a significant way. In particular, the equity premium and Sharpe ratio have the same properties.

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1 An Introduction

1.1 Pricing Assets

In today's economic world of stocks, bonds, and mattresses for investing or hiding one's hard-earned money, an investor is faced with the difficult decisions of where and in what amount to entrust his assets. In order to make these decisions, the investor must find a way to calculate the value of the earnings that a particular investment would offer. By making this calculation, the investor can then determine the price he is willing to pay for that particular investment. In short, the price of an asset p is what an investor *expects* the payoff in the future x to be, discounted slightly to account for the time the money could have been available to the investor. (We humans are naturally impatient creatures.) Written mathematically, this is

$$p = E[mx], \tag{1.1}$$

where E is the expectation operator, a mathematical function to deal with an investor's *expectations* about the future. m is the stochastic discount factor and is a multiplier that includes an accounting for the time discount we mentioned above. Let's consider the simple case of an asset over two periods. If we are at time t, then the payoff for holding a stock at time $t+1$ is the price of that stock along with any dividends payed out:

$$x_{t+1} = p_{t+1} + d_{t+1}.$$

Now in order to make sense of this and any payoff, we need to determine what its "worth" is to an investor. To do this, we define a utility function, a function that captures the "happiness" level of an investor related to the level of consumption he has. Economists insist that the utility function be increasing and concave. These demands reflect first the investor's ever-present desire for more happiness and second the notion of decreasing marginal utility – that the next bite adds less to one's utility, or happiness, than the last one did. A simple utility function is defined as a function of present and future levels of consumption: how an investor feels is related both to his immediate happiness from consumption and the time-discounted happiness from consumption that he expects to have in the future. In mathematical language, this becomes

$$U(c_t, c_{t+1}) = u(c_t) + \beta E_t[u(c_{t+1})].$$

Here U is the total utility of the investor of this period and the next, whereas u is the immediate utility of the investor as a function of consumption c_t at any time t. Also, β is taken from $[0, 1)$, so that in the aggregate utility, utility to be gained tomorrow means less to an investor than utility gained today. We again make use of the expectation operator E_t. However, note the subscript t: this is to be more specific about when our expectations are made. If we consider a quantity determined at time $t-1$ or t, say f_{t-1} or f_t, then our expectation at time t of that value is really just the value itself.

$$E_t[f_{t-1}] = f_{t-1}, E_t[f_t] = f_t.$$

Thus, with the subscript t we account for when the investor makes his expectation. For any time $s \leq t$, the quantity q_s, whatever it measures, is already known. Therefore we have

$$E_t[q_s] = q_s.$$

In order to price an asset, we give an investor the opportunity to buy as much as he wants. How does he decide how much to buy? An investor seeks to maximize his utility (wouldn't you?), so he must choose the amount A that he will buy to achieve this maximization:

$$\max_A \Big(u(c_t) + \beta E_t[u(c_{t+1})] \Big).$$

Moreover, he knows something about his consumption now and in the future in light of his purchasing decision:

$$c_t = C_t - A \cdot p_t \tag{1.2}$$

$$c_{t+1} = C_{t+1} + x_{t+1} \cdot A, \tag{1.3}$$

where we notate present and future consumption levels indicative of buying none of the asset as C_t and C_{t+1}. Consider these our "original state" consumption levels. In other words, today the investor gives up $A \cdot p_t$ units of consumption in exchange for $x_{t+1} \cdot A$ units of consumption in the future. Substituting these equations into our maximization equation, we have

$$\max_A \Big(u(C_t - A \cdot p_t) + \beta E_t[u(C_{t+1} + x_{t+1} \cdot A)] \Big).$$

Since we are dealing with an optimization problem, we take the derivative of the expression $u(C_t - A \cdot p_t) + \beta E_t[u(C_{t+1} + x_{t+1} \cdot A)]$ with respect to A and set it equal to zero.

$$-p_t u'(C_t - A \cdot p_t) + \beta E_t[x_{t+1} u'(C_{t+1} + x_{t+1} \cdot A)] \equiv 0$$

$$p_t u'(c_t) = \beta E_t[x_{t+1} u'(c_{t+1})]. \tag{1.4}$$

We can solve this equation for p_t by dividing by $u'(c_t)$, and since our expectations are taken at time t, we can treat $u'(c_t)$ as a constant.

$$p_t = \beta E_t \Big[\frac{u'(c_{t+1})}{u'(c_t)} x_{t+1} \Big]. \tag{1.5}$$

We can conclude that an investor will buy A so that the condition (1.4) holds. In fact, (1.4) makes plenty of sense: on the left side, we have the amount the investor gives up for the very next bit of the payoff, while the right side is the time-discounted expectation of the gain he will have from the purchase of the very next bit of payoff. In equilibrium, these amounts will be equal, for if they aren't, then the investor would buy more or less to make them so. From (1.5), we see that our stochastic discount factor m in our original pricing equation (1.1) becomes

$$m = \beta \frac{u'(c_{t+1})}{u'(c_t)}. \tag{1.6}$$

Above we analyzed a two-period asset pricing situation. But what about more realistic situations in which we care about more than just the next period? We can suppose that an investor is considering purchasing not just a next-term payoff x_{t+1} but a payoff stream $\{x_{t+j}\}_{j=1}^{\infty}$. If we were to follow the same method as above, we arrive at the pricing equation

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} x_{t+j} = E_t \sum_{j=1}^{\infty} m_{t,t+j} x_{t+j}. \tag{1.7}$$

Though this pricing equation and the equivalent two-period model above are both in discrete time, continuous-time models can be derived as well.

1.2 Mathematical Foundation

From a mathematical standpoint, there are many indispensible points of interest, especially concerning the existence and uniqueness of any solution we might be trying to find from a model. Below we will address this abstract mathematical character of the problem facing investors. This discussion will be deterministic, but stochastic factors can be added at the expense of complicating the analysis. The analysis we give will very closely follow the work by Daron Acemoglu in his book *Introduction to Modern Economic Growth* [**?**]. (This book is critical in understanding the theory behind the economic problem we are considering.)

At the beginning of the economic story is a maximization problem, not unlike our treatment of the pricing equation above. Let us first define the functions that play key roles in this story.

$$x(t) \in X \subset \mathbb{R}^{K_x}$$

is the state variable as a function of time. $x(t)$ gives K_x descriptors of the current economic situation facing our investor at time t.

$$y(t) \in Y \subset \mathbb{R}^{K_y}$$

is the control variable as a function of time. Like $x(t)$, $y(t)$ gives the K_y decisions that our investor makes based on his knowledge of $x(t)$. We take $K_x, K_y \geq 1$. In fact, we require that

$$y(t) \in G(t, x(t)),$$

where

$$G : \mathbb{Z}_+ \times X \to Y.$$

By this we mean to capture the notion that $y(t)$ can be chosen from a set of possible choices as constrained by the current economic status $x(t)$. Moreover, we write

$$x(t+1) = f(t, x(t), y(t))$$

to describe the (discrete) motion of the state x as a function of the previous state and the decisions made by the investor in that previous state. Finally, we define a payoff function

$$U : \mathbb{Z}_+ \times X \times Y \to \mathbb{R}.$$

Given a time t, a state $x(t)$, and a decision in response to that state $y(t)$, we have a numerical value $U(t, x(t), y(t))$ describing our reward. Finally, we take $x(0)$, our initial state, as a given.

With these functions, we state our investor's goal:

$$\sup_{\{x(t),y(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(t, x(t), y(t)). \tag{1.8}$$

This equation requires a good explanation, and we will try to provide one here. The sum $\sum_{t=0}^{\infty} \beta^t U(t, x(t), y(t))$ is called the objective function and represents the aggregate reward that an investor accrues from time $t = 0$ to infinity, the end of his life. The factors β^t are to account for the investor's impatience. We take $\beta \in [0, 1)$ so that for every term where $t \neq 0$, the reward from that period is discounted. For the investor, later rewards count less toward his total reward than does today's reward. The investor's goal is to find the sequence of control decisions $y(t)$ and the related states $x(t)$, $\{x(t), y(t)\}_{t=0}^{\infty}$, so that the aggregate reward is maximal. Economists often use a *change of variables* that makes $x(t)$ the state variable and $x(t+1)$ the control variable at time t. Moreover, the reward functions we will discuss in this paper are all *stationary*, in that they do not explicitly depend on time t, and are thus time-independent. For that reason, we will state the abstract problem facing an investor in stationary terms.

Problem 1. ***Investor's Problem.***

$$V^*(x(0)) = \sup_{\{x(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x(t), x(t+1)), \tag{1.9}$$

where we have

$$x(t+1) \in G(x(t)) \text{ for all } t \geq 0$$

and with given $x(0)$.

β, G, and U are as defined above, but without the time variable t. Moreover, we introduce the value function $V^* : \mathbb{R} \to \mathbb{R}$. If an investor starts with initial state $x(0)$, $V^*(x(0))$ gives him the supremum (or maximum, if it can be attained) of his lifetime aggregate reward. To solve this problem for an investor, we have two goals: to find the maximizing plan $\{x(t)\}_{t=0}^{\infty} \in X^{\infty}$ and to describe $V^*(x(0))$. As one might imagine, finding an infinite maximizing sequence could be a seemingly impossible task. However, if we were able to convert our problem from the search for a sequence to the search for a functional equation such that for an input of a state $x(t)$ it gives an output of a control $x(t+1)$, then we might be able to utilize many powerful tools of functional analysis to find such a function. Assuming that we can indeed make this conversion, and that the solutions to both the original problem and the conversion are equivalent and in one-to-one correspondence, the converted problem would be as below.

Problem 2. ***Conversion of Investor's Problem - The Bellman Equation.***

$$V(x) = \sup_{y \in G(x)} (U(x, y) + \beta V(y)) \tag{1.10}$$

for all $x \in X$.

(This formulation of the investor's problem is the most ciritical to this thesis. Although we do not present the preceding work here, we state the stochastic (non-deterministic) version of this formulation.

Problem 3. ***Stochastic Version of the Bellman Equation.***

$$V(x,z) = \sup_{y \in G(x,z)} \{U(x,y,z) + \mathbb{E}[V(y,z')|z]\} \tag{1.11}$$

for all $x \in X$ *and* $z \in Z$

Here z is a stochastic variable.)

Note that V^* from (1.9) is distinct from V of (1.10). V^* is an optimization given $x(0)$ over the possible sequences $\{x(t)\}_{t=0}^{\infty}$ of the sum of all discounted rewards, summed discretely. In contrast, V is an optimization over feasible controls of the sum of immediate reward $U(x)$ and discounted future value $\beta V(y)$. In this new statement of the problem, which is called the Bellman equation, the goal is to find a time invariant "policy" function so that, given any state $x \in X$ we know the optimal control $y \in G(x)$. An important economic characteristic to this statement of the problem is the recursive nature of the argument $U(x,y) + \beta V(y)$. Again, that $U(x,y)$ is the reward garnered for the present state x and the investor's response to it y, and $\beta V(y)$ is the future rewards for the decision made in view of the present state, which in turn has U imbedded in it.

In the statement of both versions of the investor's problem, we need to deal with the existence and nature of solutions. What is more, above we mentioned the hope that the two versions of the investor's problem are mathematically equivalent, and this too needs mathematical proof. In order to prove anything about these versions, we will need to make several assumptions about the functions and spaces involved. We first define the notion of *feasible plans*. If an investor has a state vector $x(t)$, we are concerned with the set of plans that remain possible for him as a result of those decisons. Thus, we define

$$\Phi(x(t)) = \{\{x(s)\}_{s=t}^{\infty} | x(s+1) \in G(x(s)) \text{ for } s \geq t\}. \tag{1.12}$$

Assumption 1. *$G(x)$ is nonempty for each $x \in X$, and for any initial state $x(0) \in X$ and any feasible plan $x \in \Phi(x(0))$, $lim_{n\to\infty} \sum_{t=0}^{n} \beta^t U(x(t), x(t+1))$ exists and is finite.*

In other words, given a state we assume that the set of possible controls is never empty and that, given a feasible plan and an initial state, the limit of the partial sums of the rewards both exists and is finite.

Assumption 2. *$X \subset \mathbb{R}^{K_x}$ is compact, G is nonempty-valued, compact-valued, and continuous. Also, $U : X_G \to \mathbb{R}$ is continuous, where $X_G = \{(x,y) \in X \times Y | y \in G(x)\} \subset X \times Y$.*

These assumptions are very natural in light of the conversion to a functional analysis problem in (1.10). Moreover, since X is compact and $G(x)$ is continuous and compact-valued, X_G is also compact. Finally, we know that the image of a compact space under a continuous function is bounded. Thus, $U(X_G)$ is bounded as well.

Assumption 3. *U is concave, and G is convex.*

In other words, if we have an $\alpha \in (0,1)$ and two points $(x_1, y_1), (x_2, y_2) \in X_G$, then we have

$$U(\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2) \geq \alpha U(x_1, y_1) + (1-\alpha)U(x_2, y_2),$$

where equality is reached only if $(x_1, y_1) = (x_2, y_2)$. Moreover, given $\alpha \in [0,1]$ and x_1, x_2 and $y_1 \in G(x_1), y_2 \in G(x_2)$, then

$$\alpha y_1 + (1-\alpha)y_2 \in G(\alpha x_1 + (1-\alpha)x_2).$$

Assumption 4. *Given $y \in X$, $U(\cdot, y)$ is strictly increasing in x and G is monotonic: if $x \leq x'$, then $G(x) \subset G(x')$.*

Here we require that U be increasing in the state variables, or that an investor prefers more to less in his initial situation. Also, given a better state, the control options available are also broader than otherwise. Both of these assumptions seem to fit our common sense. Finally, we make a strong assumption about the nature of U.

Assumption 5. *U is continuously differentiable on the interior of X_G.*

The above assumption states that in the interior of the set where U makes sense, which is X_G, we require it to have as many derivatives as we wish.

As a result of the several assumptions we have made above, we can prove a few very important statements about solutions to (1.9), (1.10), and the relationship between them.

Theorem 1. *(**Equivalence of Values**) Given Assumption 1, we have that for any $x \in X$, a solution $V^*(x)$ to (1.9) is also a solution to (1.10). Conversely, we have that a solution $V(x)$ to (1.10) is also a solution to (1.9). Combining these results, we conclude that $V^*(x) = V(x)$ for each $x \in X$.*

This theorem thus uses Assumption 1 to show that the optimal values reached in (1.9) and (1.10) are indeed the same.

Theorem 2. *(**Principle of Optimality**) Assume Assumption 1 holds. Then let $x^* \in \Phi(x(0))$ be a feasible plan that is optimal in that it achieves $V^*(x(0))$ of (1.10). We then have*

$$V^*(x(t)) = U(x^*(t), x^*(t+1)) + \beta V^*(x^*(t+1)) \tag{1.13}$$

for $t \geq 0$ and $x^*(0) = x(0)$. Also, if any other $y^* \in \Phi(x(0))$ satisfies (1.13), then it optimizes (1.9) as well.

The **Principle of Optimality** is the great result of our efforts. It states that we can take an optimizing *sequence* x^* and write the rewards from it in the form of a sum of today's reward $U(x^*(t), u^*(t+1))$ and the discounted future reward $\beta V^*(x^*(t+1))$. From the **Equivalence of Values** we know that $V^* = V$, so (1.13) implies (1.10). Morevoer, the latter part of the theorem guarantees that we won't miss any solutions in converting from the sequence-based problem to the policy function problem, and vice versa.

Theorem 3. *(**Existence of Solutions**) If we take Assumptions 1 and 2 to be true, then there exists a unique, continuous, and bounded $V : X \to \mathbb{R}$ satisfying (1.10). Moreover, for a any $x(0) \in X$, there exists a feasible plan $x^* \in \Phi(x(0))$ that optimizes V.*

Note that, as yet, the **Existence of Solutions** will give us a unique value function V and the existence, but not *uniqueness*, of optimizing plans.

Theorem 4. *(**Concavity of the Value Function**) If Assumptions 1, 2, and 3 hold, then V from the **Existence of Solutions** is strictly concave.*

As a consequence of this strict concavity, we have the following corollary.

Corollary 1. *If we take Assumptions 1, 2, and 3 to be true, then there exists a unique optimal plan $x^* \in \Phi(x(0))$ for any $x(0) \in X$. Moreover, there exists a continuous policy function $\pi : X \to X$ so that $\pi(x^*(t)) = x^*(t+1)$.*

Theorem 5. *(**Monotonicity of the Value Function**) If Assumptions 1, 2, and 4 hold, and $V : X \to \mathbb{R}$ is the unique solution to the Bellman equation (1.10), then V is strictly increasing.*

The next theorem will give us the tools to use V in obtaining optimization results.

Theorem 6. *(**Differentiability of the Value Function**) Let Assumptions 1, 2, 3, and 5 hold, and take $\pi : X \to X$ be the continuous policy function from the Corollary above. Consider $x \in IntX$ and $\pi(x) \in G(x)$. Then V is differentiable at x and*

$$DV(x) = D_x U(x, \pi(x)). \tag{1.14}$$

Notice that without this theorem, the inherent problem with considering derivatives of V is the internally recursive nature of (1.10): since the expression for V contains V as well, differentiating is not a straightforward endeavor.

The above theorems have given a firm mathematical foundation to the fundamental formulation of a discrete-time, non-stochastic economic optimization problem . With the right assumptions, we know that given initial data $x(0)$, any solution to (1.9) is also a solution to the Bellman equation (1.10) and vice versa. Working with the Bellman equation allows us to use many powerful tools of functional analysis. Next, we know that sequential solutions to (1.9) also satisfy the Bellman equation. Importantly, we have a unique value function satisfying the Bellman equation (1.10) and for initial state $x(0)$ we are guaranteed to have an optimizing plan x. Because of the concavity of the value function, we also have that this optimizing plan is unique to the initial state. With these results in hand, we know that we can find unique optimizing plans for each given initial state. This tells us that the skeleton of the economic optimization problem, which must be formulated as in (1.9) and makes only the assumptions that the reward function U be concave and continuously differentiable on a domain, does make sense mathematically and has unique solutions. However, we need two more equations to fully characterize solutions to (1.9) and (1.10); we know solutions exist

and are unique, but if we have a given plan x, we as yet cannot be certain that it is indeed a solution.

If we begin with the Bellman equation

$$V(x) = \sup_{y \in G(x)} \Big(U(x, y) + \beta V(y) \Big),$$

we know that given Assumptions 1 - 5, V is strictly concave and continuously differentiable (see Theorems 4 and 6). Thus, we can discuss the problem using basic calculus tools: the global maximum of a strictly concave function is located where the derivative is zero. In fact, it turns out this description is both necessary and sufficient. Writing this mathematically, we have

$$D_y U(x, y^*) + \beta DV(y^*) = 0, \tag{1.15}$$

where y^* is the optimizing control and D is the gradient operator. (1.15) is called the Euler equation. Now although this condition is necessary and sufficient for an optimizing decision, we again run across the complication from the recursive nature of V. Since V is determined recursively as we discussed with the Bellman equation (1.10), we are as yet unable to solve for the optimal control y^* using (1.15).

From Theorem 6, we have that

$$DV(x) = D_x U(x, y^*).$$

We recall that by Corollary 1 we can write y^* as $\pi(x)$, where π is the continuous policy function. Using this notation and the above equation, we can rewrite the Euler equation as

$$D_y U(x, \pi(x)) + \beta D_x U(\pi(x), \pi(\pi(x))) = 0. \tag{1.16}$$

If we consider the one-dimensional case, we can explain using economic intuition. When x and y are real numbers, we have

$$\frac{\partial U(x, y^*)}{\partial y} + \beta V'(y^*) = 0,$$

or

$$\frac{\partial U(x, y^*)}{\partial y} = -\beta V'(y^*).$$

This means that the marginal utility from increasing the investor's utility today must be in balance with the loss to discounted marginal future value. But this explanation sits well with a basic understanding of economics. An investor will consume today up to the point where his marginal change in utility today exactly balances what he expects he is losing in a discounted sense in the future.

Our work above led us to (1.16), which is completely in terms of the policy function π and characterizes this as-yet unknown function. However, we still need one more condition to characterize fully any solutions, especially in infinite-dimensional problems. The Transversality Condition is this final condition, and it states that the (inner) product of the marginal

return from the state variable and the state variable itself not grow faster than or equal to $\frac{1}{\beta}$. Mathematically, we write

$$\lim_{t\to\infty} \beta^t D_x U(x^*(t), x^*(t+1)) \cdot x^*(t) = 0. \tag{1.17}$$

Another way to describe the transversality condition is to say that at any time t an investor only has a finite utility, regardless of the state x. Thus, the Transversality Condition means to avoid any situation where minute beneficial adjustments to the (infinite) controls would lead to undefined utility levels. If we are considering a case where the control variable is finite-dimensional, then the Euler equation is enough to guarantee finite marginal gains to utility.

The next theorem finally gives us a complete characterization of an optimal plan x^* to our original problem (1.9).

Theorem 7. *(**Euler Equation and the Transversality Condition**) We assume that Assumptions 1 - 5 all hold. Morevoer, we assume that $X \subset \mathbb{R}^K_+$. Then an interior sequence $\{x^*(t)\}_{t=0}^{\infty}$ is optimizing for (1.9) if and only if it satisfies both the Euler equation and the Transversality Condition.*

Again, we emphasize that the stochastic analog of this theory can be found in Acemoglu's text [**?**]. It is this non-deterministic analysis that truly provides the basis for our research in this paper.

1.3 Equity Premium Puzzle

Now that we have shown the fundamental economic problem (1.9) to be mathematically sensical and have fully characterized any abstract solutions to it, we can begin with modeling efforts. Since the theorems above only relied on characteristics of, among other things, the reward function U but not their functional form, the real efforts of economists to model the market is done in their choice for the utility function. The only limits for this choice are that the function be increasing but at a decreasing rate; this is to account for diminishing marginal returns, or the notion that although each next house, or car, or cookie, makes me "happier" than I was before, the next house I get will add less to my happiness than the one before it did. With the help of very basic mathematical tools, we have constructed an understanding of how assets are priced. But of what interest are these types of conclusions? In fact, these conclusions and related data are at the very heart of the most interesting finance research and modeling: the Equity Premium Puzzle. Before describing this puzzle in detail, we first introduce the notion of return and discuss the Sharpe ratio, an important indicator for stocks. The *gross return* on an asset is defined as the ratio of the next period's payout to this period's price:

$$R_{t+1} = \frac{x_{t+1}}{p_t}.$$

The gross return on an asset indicates to an investor the percent of his investment he gets back. If $R_{t+1} < 1$, then the investor is losing money; if $R_{t+1} = 1$, the investor is getting just

his investment back; and if $R_{t+1} > 1$, the investor is making money. With return defined, we can now discuss an extremely important measurement: the Sharpe ratio.

$$S = \frac{E(R^i) - R^f}{\sigma(R^i - R^f)}. \tag{1.18}$$

In (1.18), R^f is the risk-free rate of return, or more specifically the return on an investment that has zero inherent risk (think here of the rate on a U.S. Treasury bill); R^i is the return on an asset; $E(R^i) - R^f$ is called the excess return, or the expected return of an asset over and above the risk-free rate of return; and $\sigma(R^i - R^f)$ is the standard deviation of the excess return of an asset. The idea behind the Sharpe ratio is to scale expected excess returns by the variability of that expectation. This is clearer through example. If we consider a stock with a high return, say 10, yet comparably high (or higher) standard deviation on that return, say 13, then the Sharpe ratio will be low: $S = 10/13 = .77$. What's more, if a stock has a low expected return, say 5, and high standard deviation on its return, say 13 again, then the Sharpe ratio will be much lower: $S = 5/13 = .38$. Finally, if a stock has a high expected return, say 10, and a low standard deviation on the return, say 5, then the Sharpe ratio is quite high: $S = 10/5 = 2$. These examples give us a certain feeling about stocks and their Sharpe ratios. A high Sharpe ratio indicates a desirable stock since the excess return is high relative to the standard deviation on it: an investor can expect a solid return with not much (relative) variation on it. In contrast, a low Sharpe ratio indicates a low excess return relative to the standard deviation on it, indicating a less desirable stock: the investor expects a smaller return with more (relative) variability of that return.

In modeling stock motion, economists have calculated the average Sharpe ratio for stocks over the last 50 or so years. Average annual stock returns have been 9% with a standard deviation of 16%, whereas returns on Treasury bills have been 1% annually. These data lead to a Sharpe ratio of .5 annually, and this number has been economically reliable over the post-World War II period.

In trying to model the economy, economists have encountered problems. To understand these problems, we must briefly examine previous modeling efforts. The root of any model is the choice of utility function. As we discussed above, the only conditions on a utility function are that it be increasing ($u' > 0$) and concave ($u'' < 0$). The simplest functional form that achieves these conditions is

$$u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma}, \tag{1.19}$$

where $\gamma > 1$ and we add the factor $\frac{1}{1-\gamma}$ to simplify the first derivative. In this utility form, γ is actually a measure of risk-aversion. To understand the problem using this model, we go back to our simplest pricing equation (1.1). Dividing both sides by p, we have

$$1 = E\Big[m\frac{x}{p}\Big],$$

or in other words

$$1 = E_t\Big[m_{t+1}\frac{x_{t+1}}{p_t}\Big] = E_t[m_{t+1}R_t\Big]$$

for any payout x. Now let's consider a riskless bond with price $p = 1$. Pulling out the rate of return R_t^f, we see

$$\frac{1}{R_t^f} = E_t[m_{t+1}].$$

From (1.6) we see that

$$m_{t+1} = \beta\Big(\frac{c_{t+1}}{c_t}\Big)^{-\gamma},$$

so we have

$$\frac{1}{R_t^f} = E_t\Big[\beta\Big(\frac{c_{t+1}}{c_t}\Big)^{-\gamma}\Big].$$

In a lecture titled "Asset Pricing Theory and the Equity Premium" [**?**], John Cochrane provides a great explanation why this model fails. He first provides two important formulas. Using the above utility function, we have

$$E(R_{t+1} - R^f) \approx \gamma cov(\Delta c_{t+1}, R_{t+1}) \tag{1.20}$$

$$R^f \approx 1 + \delta + \gamma E(\Delta c_{t+1}) - \frac{1}{2}\gamma(\gamma - 1)\sigma^2(\Delta c_{t+1}). \tag{1.21}$$

Note that our first equation is for excess return, the numerator of the Sharpe ratio. Δc_{t+1} is called consumption growth and is a random variable. Cochrane also provides data for these quantities.

Annual data 1948 - 2002 in percents

$E(\Delta c)$	$\sigma(\Delta c)$	$E(R - R^f)$	$\sigma(R)$	$E(R^f)$	$corr(\Delta c, R)$	$cov(\Delta c, R)$
1.31	1.93	7.21	18.0	0.39	0.39	0.135

Using this data, we can solve for a value of γ, the level of risk-aversion.

$$\begin{aligned} E(R_{t+1} - R_f) &\approx \gamma cov(\Delta c_{t+1}, R_{t+1}) \\ 7.21 &= \gamma \times 0.135 \\ \gamma &= 53. \end{aligned} \tag{1.22}$$

This γ is *way* too big to actually describe our economy. To see why this is, let's consider a series of 50/50 bets. We focus on an investor with annual consumption at \$30K. Using the equation

$$\frac{\text{amount willing to pay to avoid bet}}{\text{size of bet}} = \gamma \frac{\text{size of bet}}{\text{consumption}}, \tag{1.23}$$

Cochrane computes the following table, which catalogues the amount this investor is willing to pay to avoid taking the bet in question.

	Risk aversion γ			
Bet	2	10	50	100
\$10	\$0.00	\$0.01	\$0.05	\$0.10
\$100	\$0.20	\$1.00	\$4.99	\$9.94
\$1,000	\$20	\$99	\$435	\$665
\$10,000	\$2,000	\$6,921	\$9,430	\$9,718

Note that for $\gamma = 50$, the investor is willing to pay \$9,430 to avoid a \$10,000 bet. That makes no sense! Our reasons for dismissing such high γ are in fact two-fold. Even if we allow these results concerning the risk premia to stand for our high γ, this γ causes wild results in the interest rate. Going back to the second equation we listed above, we have

$$\begin{aligned} R^f &\approx 1 + \delta + \gamma E(\Delta c_{t+1}) - \frac{1}{2}\gamma(\gamma - 1)\sigma^2(\Delta c_{t+1}) \\ &\approx 1 + \delta + 533 \times 0.0131 - \frac{1}{2}53 \times 52 \times 0.0193^2 \\ &= 1 + \delta + 0.18. \end{aligned} \tag{1.24}$$

We can interpret these results in either of two ways, both of which are absurd. First, since $E(R^f) = 0.0039$, we can conclude that $\delta = -18\%$. This would mean that an investor prefers the future by a factor of 18% to the present and that people are amazingly patient, even by Neville Chamberlain's standards. This conclusion is completely contrary to our most basic notions of economics. Second, if we take δ to be a more realistic number (positive, for one thing), then this model would be predicting interest rates of $18 + \delta +$ inflation $\approx 25\%$. This is also ridiculous, especially in light of the data we have concerning the mean risk-free rate over the last 50 years.

So we can conclude that the utility function (1.19) cannot be used to describe accurately the motion of stock prices. In light of this, economists have tried to design other utility functions. Some economists have chosen to use not only the control variable c_t but also the state variable x_t in their formulations. These efforts have led to interesting and rather successful models. We focus on one of these efforts in the next section.

1.4 Campbell and Cochrane's Attempt

In their attempt to model empirical data, Campbell and Cochrane (CC) in "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior" [**?**] truly engage in economic revolution. Although the idea of habit as a major factor in individual utility is not unique to CC, in this paper CC claim to be successful in implementing habit into their model so that important economical data points are achieved.

CC's revolutionary contribution is the surplus consumption ratio

$$S_t^a \equiv \frac{C_t^a - X_t}{C_t^a}, \tag{1.25}$$

where C_t^a is the average consumption of individuals in the economy and X_t is the habitual level of consumption. CC argue that, rather than absolute levels of consumption, consumption relative to recent experiences is a truer indicator of how an individual feels. In their paper, CC point out that this provides a plausible explanation for the general fear of a recession: even if the loss to absolute consumption is small, the general response is very negative because the new consumption level is low relative to the recently incurred habitual level. In the surplus consumption ratio, habit X_t is externally determined, meaning that a representative individual has no effect on its determination. Moreover, in the case of equilibrium,

each individual will choose the same level of consumption, so in (1.25) we drop the "average" and have

$$S_t \equiv \frac{C_t - X_t}{C_t}. \tag{1.26}$$

CC also make the assumption that $C_t > X_t$. So we have that $S_t > 0$, and consumption very close to habit is a worst-case scenario. To ensure this, they define the motion of S_t in terms of its logarithm, $s_t \equiv lnS_t$:

$$s_{t+1} = (1-\phi)\bar{s} + \phi s_t + \lambda(s_t)(c_{t+1} - c_t - g), \tag{1.27}$$

where $c_t \equiv lnC_t$ and $g, \phi, \bar{s}$ are all parameters. Furthermore, we have that

$$\Delta c_{t+1} = c_{t+1} - c_t = g + \nu_{t+1}, \tag{1.28}$$

with ν_{t+1} a normally distributed random variable with mean 0 and variance σ^2. The research in this paper will focus on the functional form of $\lambda(s_t)$ in (1.27), called the sensitivity function. $\lambda(s_t)$ is defined in order to scale shocks to surplus consumption. Since $c_{t+1} - c_t - g = \nu_{t+1}$ is a random variable, the sensitivity function is meant to scale these random shocks in the following way: when times are good and there is high surplus consumption, we want the effect of random shocks to be low. In good times, we care less about random economical shocks. In contrast, when times are bad and surplus consumption is low, random shocks ought to factor heavily into our future surplus consumption. In bad times, any economical shock, be it positive or negative, will have a large impact on our utility.

The surplus consumption ratio is meant to provide a better gauge for how an individual feels than pure consumption alone. CC claim that a representative investor feels happier when his consumption is high relative to "habit," or the typical consumption level. In this model, identical individuals seek to maximize the function

$$E\sum_{t=0}^{\infty} \delta^t \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}, \tag{1.29}$$

where δ is the temporal discount factor and γ is a parameter. The utility function for this model is

$$u(C_t, X_t) = \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}. \tag{1.30}$$

In fact, an open question that will not be addressed in this paper is whether any other utility functions utilizing the surplus consumption ratio will yield similar or better modeling results. Though CC employed the revolutionary notion of habit influencing the perception about consumption in a particular way in their utility function, it stands to reason that there may be other ways to incorporate this idea mathematically. Setting this question aside and focusing on CC's utility function, we have marginal utility

$$u_C(C_t, X_t) = (C_t - X_t)^{-\gamma} = S_t^{-\gamma} C_t^{-\gamma} = (S_t C_t)^{-\gamma}. \tag{1.31}$$

Our stochastic discount factor is thus

$$M_{t+1} \equiv \delta \frac{u_C(C_{t+1}, X_{t+1})}{u_C(C_t, X_t)} = \delta \left(\frac{S_{t+1}}{S_t} \frac{C_{t+1}}{C_t} \right)^{-\gamma}. \tag{1.32}$$

CC began with the presumption that the risk-free rate is constant. They also assumed that habitual consumption X_t is predetermined at and near the steady-state level. From these assumptions they were able to solve for a functional form for λ. We find relation between $\lambda(s_t)$ and $r_{f,t+1}$ through the stochastic Euler equation $E_t[M_{t+1}R_{t+1}] = 1$. This leads us to

$$\begin{aligned} r_{f,t+1} &= ln(1/E_t[M_{t+1}]) \\ &= -ln\delta + \gamma g + \gamma(1-\phi)(\bar{s} - s_t) - \frac{\gamma^2\sigma^2}{2}(1+\lambda(s_t))^2. \end{aligned} \tag{1.33}$$

Later, Jessica Wachter [**?**] made the generalization that the risk-free rate ought to be linear rather than constant. The sensitivity function that resulted from this model is

$$\lambda(s - \bar{s}) = \begin{cases} \frac{\sqrt{1-2(s-\bar{s})}}{\bar{S}} - 1 & \text{if } s < \bar{s} + \frac{1-\bar{S}^2}{2} \\ 0 & \text{if } s \geq \bar{s} + \frac{1-\bar{S}^2}{2}, \end{cases} \tag{1.34}$$

where $ln\bar{S} = \bar{s}$ is the steady-state log surplus consumption ratio. The results using this slightly updated model had several unwanted properties. First, their resulting risk-free rate was (of course) linear with a (very small) positive slope. Thus, at some point the risk-free rate would become negative, creating an arbitrage situation. Second, their sensitivity function went to infinity as $s_t \to -\infty$ and also became negative for some $\tilde{s}_t$. In the case of negative $\lambda(s_t)$, CC redefined $\lambda(s_t)$ to be 0 for all s_t larger than this $\tilde{s}_t$. Thus, the sensitivity function went off to infinity and has a corner, and so is not smooth.

1.5 The CCH Attempt

Chen, Cosimano, and Himonas (CCH) have spent much time addressing the Campbell-Cochrane model from an analytic point-of-view. In [**?**] CCH used the sensitivity function of CC in their analysis. (In fact, this thesis will model very closely [**?**] in the analysis of proposed changes to the sensitivity function and the risk-free rate.) However, in later discussions they considered an alternative equation in an attempt at resolving these several difficulties in constructing his sensitivity function. CCH decided the form for the sensitivity function and solved back to the risk-free rate, as opposed to CC's method of choosing the risk-free rate and then solving for λ. In communication, CCH listed several necessary properties for the sensitivity function that come from the assumptions made by CC [**?**].

1. $\lambda(0) = \frac{1}{\bar{S}} - 1 > 0$,
2. $\lambda'(0) = \frac{-1}{\bar{S}} < 0$,

3. $\lambda''(0) = \frac{-1}{\bar{S}} < 0$, and

4. $\lim_{s\to\infty} \lambda(s) = 0$.

The constant $\bar{S}$ comes from CC's work, and since their two expressions for it are intrinsically linked to either a constant or a linear risk-free rate, we must rederive an expression for this constant. In CC [**?**] we find the following relationship:

$$S \equiv \frac{E(R^i) - R^f}{\sigma(R^i - R^f)} \approx \gamma\sigma(1 + \lambda(s_t)). \tag{1.35}$$

Later in this paper, we will be working in short-term interest structure, so we use the monthly Sharpe ratio $\left(S = .5 \times \frac{\sqrt{12}}{12} = 0.144\right)$ instead of the annual Sharpe ratio ($S = 0.5$). So we then have from above

$$0.144 = \gamma\sigma(1 + \lambda(s)) = \gamma\sigma\frac{1}{\bar{S}},$$

where the second equality follows from the first condition listed by CCH. Solving for $\bar{S}$, we have

$$\bar{S} = \frac{\gamma\sigma}{0.144}. \tag{1.36}$$

Moreover, CCH chose the logistic functional form for the sensitivity function:

$$\lambda(s) = A - \frac{B}{1 + Ce^{-Ds}},$$

where $A, B, C, D > 0$. Using the characteristics above, CCH derived the following sensitivity function.

$$\lambda(s) = \frac{(1 - \bar{S})(3 - \bar{S})}{\bar{S}\left(2 - \bar{S} + e^{\frac{3-\bar{S}}{1-\bar{S}}s}\right)}. \tag{1.37}$$

CCH's sensitivity function did solve the problem of infinite values that CC's sensitivity function had, yet the risk-free rate that resulted from their work had even more problems than before. This new risk-free rate runs off to infinity when $s \to -\infty$ and negative infinity when $s \to \infty$. In the graphs below, we compare the work of CC and CCH with both the risk-free rate and the sensitivity function.

Figure 1 displays the CC monthly risk-free rate as a bold line and the CCH monthly risk-free rate as a dashed line, both centered on 0. The parameter values are $\gamma = 2.001$, $\phi = 0.9896$, $b = 0$, $\bar{r} = 0.00783$, $\sigma_r = 0.005514$, $\bar{x} = 0.00157$, $\sigma = 0.00323$, $p_0 = 219.60$, $p_1 = 111.173$, and $\bar{S} = 0.04483$. The x-axis gives the log surplus consumption ratio on the support $[-150\sigma, 150\sigma] = [-0.8271, 0.8271]$. The y-axis records the CC monthly risk-free rate as a bold line and the CCH monthly risk-free rate as a dashed line, both centered on 0.

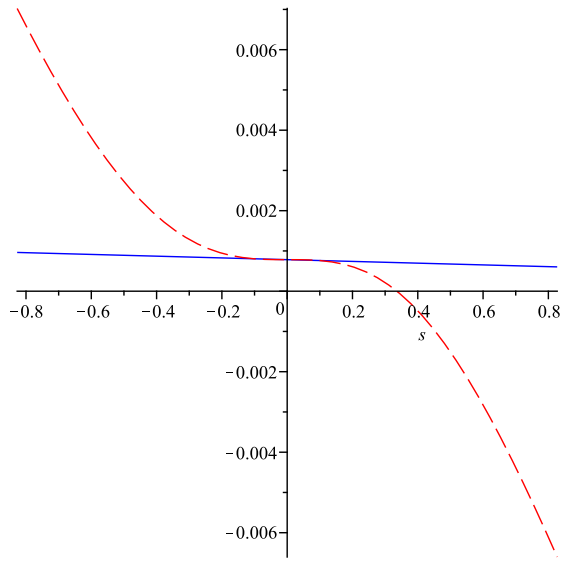


Figure 1

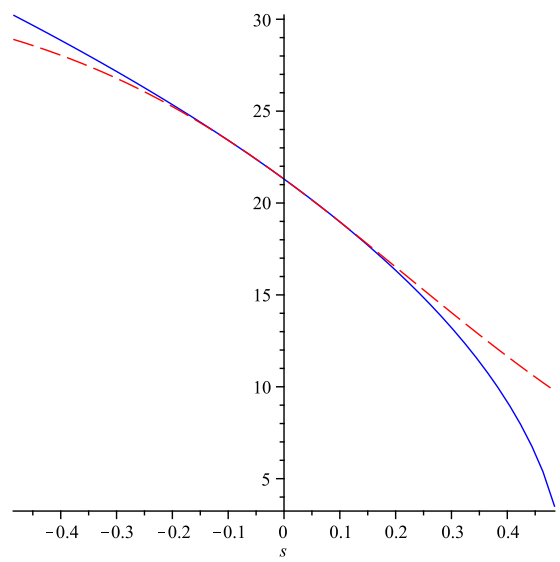


Figure 2

Figure 2 displays the CC sensitivity function as a bold line and the CCH sensitivity function as a dashed line, both centered on 0. The parameter values are $\gamma = 2.001$, $\phi = 0.9896$, $b = 0$, $\bar{r} = 0.00783$, $\sigma_r = 0.005514$, $\bar{x} = 0.00157$, $\sigma = 0.00323$, $p_0 = 219.60$, $p_1 = 111.173$, and $\bar{S} = 0.04483$. The x-axis gives the log surplus consumption ratio on the support $[-150\sigma, 150\sigma] = [-0.8271, 0.8271]$. The y-axis records the CCC sensitivity function as a bold line and the CCH sensitivity function as a dashed line, both centered on 0.

1.6 A New Attempt

With the above background concerning the CC model and previous work on the sensitivity function, the goal of this thesis is to refine the functional form of $\lambda(s_t)$. Instead of defining the risk-free rate to be constant or linear, we will use desired characteristics of the risk-free rate equation to derive a logistic-form equation – a functional form that we can force to remain positive and finite for all s_t. With this form for the risk-free rate, we can solve for a new functional form for $\lambda(s)$ by equating this risk-free rate equation with (1.33). We will then use this form for λ in the differential equation for the price-dividend function, determining an analytic solution with a certain radius of convergence. We use the well-known Cauchy-Kovalevksi Theorem in one dimension for linear second-order differential equations with analytic coefficients, proved in the Appendix, to guarantee the existence of such an analytic solution.

Theorem 8. *(**Cauchy-Kovalevksi Theorem for 1D**) The initial value problem*

$$y''(x) + a(x)y'(x) + b(x)y(x) = g(x), y(x_0) = y_0, y'(x_0) = y_1$$

with $a(x), b(x), g(x)$ analytic near x_0, has a unique analytic solution $y(x)$ near x_0 with radius of convergence equal to at least the smallest radius of convergence of $a(x), b(x), g(x)$.

(The motivation for this analytic-biased approach comes from CCH's work in [**?**], [**?**], and [**?**].) Although not in the current version of this paper, we will in the future use the following theorem, also proved in the Appendix, to determine the radius of convergence for this power series.

Theorem 9. *For $z \in \mathbb{C}$, the radius of convergence of a power series for $f(z)$ at z_0 is equal to the distance from z_0 to the boundary of the domain of holomorphicity.*

2 The New Sensitivity Function

2.1 Parameters

To generate our model, we need to pick certain parameters. For all the original (non-derived) parameters, we turn to CC [**?**]. (We get the standard deviation on the steady-state risk-free rate from Wachter [**?**] since it is not given in [**?**].)

We discussed above the constant $\bar{S} = e^{\bar{s}}$, the steady-state surplus consumption.

$$\bar{S} = \frac{\gamma\sigma}{.144}. \tag{2.1}$$

Moreover, we have the following expression for $\bar{S}$ derived in [**?**]:

$$\bar{S} = \sigma\sqrt{\frac{\gamma\phi}{1 - \phi - \frac{b}{\gamma}}}. \tag{2.2}$$

Table 1: Original Parameters

Parameters	Annually	Monthly
Mean consumption growth $\bar{x}$	0.0188	0.00157
Standard error on mean consumption growth σ	0.0112	0.00323
Risk-free interest rate $\bar{r}$	0.0094	0.000783
Standard deviation on risk-free rate σ_r	0.0191	0.00551
Utility curvature γ	2.001	
Habit persistence ϕ	0.882	0.9896

(This equation does come from the presumption that the risk-free rate is linear. This is in fact a generalization, for when $b = 0$, the risk-free rate is constant - the case CC were concerned with. We operate under the assumption that the risk-free rate is constant locally around the steady state, so we take $b = 0$.) We use the following approximation for $\bar{S}$:

$$\bar{S} = \sigma\sqrt{\frac{\gamma}{1-\phi}}.$$

2.1.1 Derivation of λ and $\bar{S}$ in CC Model

There are four assumptions in the Campbell-Cochrane model:

(**a**) the domain of the pricing kernel M_t is contained in the set $\mathbb{R}^+$ of all positive real numbers;

(**b**) the risk-free interest rate R_t is constant;

(**c**) the habitual consumption is predetermined at the steady-state level;

(**d**) the habitual consumption is predetermined near the steady-state level.

As above, we have the following expression for the risk-free rate:

$$\ln R_{t+1} = -\ln\beta + \gamma g + \gamma(\phi - 1)u - \frac{\sigma^2\gamma^2}{2}(1+\lambda(u))^2\,. \tag{2.3}$$

We immediately employ the second assumption – that the risk-free rate is constant. We set $A = -\ln R_{t+1} - \ln\beta + \gamma g$ to be a constant by using assumption (**b**) again. Equation (2.3) becomes

$$A + \gamma(\phi - 1)u - \frac{\sigma^2\gamma^2}{2}(1+\lambda(u))^2 = 0\,. \tag{2.4}$$

Solving equation (2.4) gives

$$\lambda(u) = \left[\frac{2}{\gamma^2\sigma^2}(A - \gamma(1-\phi)u)\right]^{1/2} - 1\,, \tag{2.5}$$

where $u = s_t - \bar{s}$. By substituting $B = \frac{2A}{\gamma 2\sigma 2}$, we obtain

$$\lambda(u) = \Big[B - \frac{2}{\gamma\sigma^2}(1-\phi)u\Big]^{1/2} - 1\,. \tag{2.6}$$

The goal is to solve for B. We have the two evolution equations

$$c_{t+1} = c_t + g + \nu_{t+1} \tag{2.7}$$
$$s_{t+1} = (1-\phi)\bar{s} + \phi s_t + \lambda(u)\nu_{t+1} \tag{2.8}$$

Plugging in, we have

$$s_{t+1} = (1-\phi)\bar{s} + \phi s_t + \lambda(u)(c_{t+1} - c_t - g). \tag{2.9}$$

From the surplus consumption ratio $S_t = \frac{C_t - X_t}{C_t}$ we have

$$s_{t+1} = \ln\Big(\frac{C_{t+1} - X_{t+1}}{C_t}\Big). \tag{2.10}$$

Since S is a function of C and X, the total derivative of s_{t+1} is

$$\frac{\partial s_{t+1}}{\partial \ln X_{t+1}} d(\ln X_{t+1}) + \frac{\partial s_{t+1}}{\partial \ln C_{t+1}} d(\ln C_{t+1}) = 0 + \lambda(u) d(\ln C_{t+1}). \tag{2.11}$$

Computing the partial derivatives, we arrive at

$$\begin{aligned}
\frac{\partial s_{t+1}}{\partial \ln X_{t+1}} &= \frac{1}{C_{t+1} - X_{t+1}} \frac{\partial}{\partial \ln X_{t+1}} (C_{t+1} - X_{t+1}) \\
&= -\frac{1}{C_{t+1} - X_{t+1}} \frac{\partial}{\partial \ln X_{t+1}} (e^{\ln X_{t+1}}) \\
&= -\frac{X_{t+1}}{C_{t+1} - X_{t+1}} \\
&= -\frac{X_{t+1} + C_{t+1} - C_{t+1}}{C_{t+1} - X_{t+1}} \\
&= -\Big(\frac{C_{t+1}}{C_{t+1} - X_{t+1}} - 1\Big) \\
&= -\Big(\frac{1}{S_{t+1}} - 1\Big).
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
\frac{\partial s_{t+1}}{\partial C_{t+1}} &= \frac{1}{C_{t+1} - X_{t+1}} \frac{\partial}{\partial \ln C_{t+1}} (C_{t+1} - X_{t+1}) - 1 \\
&= \frac{C_{t+1}}{C_{t+1} - X_{t+1}} - 1 \\
&= \frac{1}{S_{t+1}} - 1.
\end{aligned} \tag{2.13}$$

If we substitute these into the total derivative above, we have

$$-\Big(\frac{1}{S_{t+1}}-1\Big)d(\ln X_{t+1})+\Big(\frac{1}{S_{t+1}}-1\Big)d(\ln C_{t+1})=\lambda(u)d(\ln C_{t+1}). \tag{2.14}$$

Dividing by $d(\ln C_{t+1})$ and solving for $\frac{d(\ln X_{t+1})}{d(\ln C_{t+1})}$, we have

$$-\Big(\frac{1}{S_{t+1}}-1\Big)\frac{d(\ln X_{t+1})}{d(\ln C_{t+1})}=\lambda(u)-\Big(\frac{1}{S_{t+1}}-1\Big) \tag{2.15}$$

$$\frac{d(\ln X_{t+1})}{d(\ln C_{t+1})}=1-\frac{\lambda(u)}{\frac{1}{S_{t+1}}-1}=1-\frac{\lambda(u)}{e^{-s_{t+1}}-1}. \tag{2.16}$$

From the third assumption, we want that $\frac{d(\ln X_{t+1})}{d(\ln C_{t+1})}(\bar{s})\equiv 0$. Using this in the equation above, we find that

$$\lambda(0)=e^{-\bar{s}}-1. \tag{2.17}$$

We can use this result to find the constant B:

$$\Big[B-\frac{2}{\gamma\sigma^2}(1-\phi)\cdot 0\Big]^{1/2}-1=e^{-\bar{s}}-1$$

$$B=e^{-2\bar{s}}. \tag{2.18}$$

We can now employ the fourth assumption to derive an expression for $\lambda'(0)$.

$$\begin{aligned}\frac{d}{ds_t}\frac{d(\ln X_{t+1})}{d(\ln C_{t+1})}&=\frac{d}{ds_t}\Big(1-\frac{\lambda(u)}{\frac{1}{S_{t+1}}-1}\Big)\\&=-\Big[\frac{(e^{-s_{t+1}}-1)\lambda'(u)\frac{du}{ds_t}+\lambda(u)(-\frac{ds_{t+1}}{ds_t}e^{-s_{t+1}})}{(e^{-s_{t+1}}-1)^2}\Big]\\&=\frac{-\lambda'(u)\frac{du}{ds_t}}{e^{-s_{t+1}}-1}-\frac{\frac{ds_{t+1}}{ds_t}\lambda(u)e^{-s_{t+1}}}{(e^{-s_{t+1}}-1)^2}.\end{aligned} \tag{2.19}$$

We compute the derivatives within the expression:

$$\frac{du}{ds_t}=\frac{d(s_t-\bar{s})}{ds_t}=1 \tag{2.20}$$

$$\frac{ds_{t+1}}{ds_t}=\frac{d\Big((1-\phi)\bar{s}+\phi s_t+\lambda(u)\nu_{t+1}\Big)}{ds_t}=-\phi+\lambda'(u)\nu_{t+1}. \tag{2.21}$$

Plugging in, we have

$$\begin{aligned}\frac{d}{ds_t}\frac{d(\ln X_{t+1})}{d(\ln C_{t+1})}&=\frac{-\lambda'(u)}{e^{-s_{t+1}}-1}-\frac{(-\phi+\lambda'(u)\nu_{t+1})\lambda(u)e^{-s_{t+1}}}{(e^{-s_{t+1}}-1)^2}\\&=-\frac{\lambda'(u)}{\frac{1}{S_{t+1}}-1}-\frac{\phi\lambda(u)}{\Big(\frac{1}{S_{t+1}}-1\Big)^2}\frac{1}{S_{t+1}}-\frac{\lambda'(u)\lambda(u)\nu_{t+1}}{\Big(\frac{1}{S_{t+1}}-1\Big)^2}\frac{1}{S_{t+1}}.\end{aligned} \tag{2.22}$$

Evaluating at $u = 0$ we have

$$-\frac{\lambda'(0)}{\frac{1}{\bar{S}}-1} - \frac{\phi\lambda(0)}{\left(\frac{1}{\bar{S}}-1\right)^2}\frac{1}{\bar{S}} - \frac{\lambda'(0)\lambda(0)\nu_{t+1}}{\left(\frac{1}{\bar{S}}-1\right)^2}\frac{1}{\bar{S}} \equiv 0. \tag{2.23}$$

Moreover, taking expectations we can eliminate the term with ν_{t+1}.

$$\begin{aligned}
-\frac{\lambda'(0)}{\frac{1}{\bar{S}}-1} - \frac{\phi\lambda(0)}{\left(\frac{1}{\bar{S}}-1\right)^2} &\equiv 0 \\
-\lambda'(0)(1-\bar{S}) - \phi\lambda(0) &\equiv 0 \\
\lambda'(0) &= \frac{\phi\lambda(0)}{\bar{S}-1} = \frac{\phi(e^{-\bar{s}}-1)}{e^{\bar{s}}-1} \\
&= \phi e^{-\bar{s}}
\end{aligned} \tag{2.24}$$

Now if we compute λ' and evaluate at $u = 0$, we can estimate $\bar{S}$.

$$\lambda(u) = \left(e^{-2\bar{s}} - \frac{2}{\gamma\sigma^2}(1-\phi)u\right)^{\frac{1}{2}} - 1$$

$$\lambda'(u) = \frac{1}{2\left(e^{-2\bar{s}} - \frac{2}{\gamma\sigma^2}(1-\phi)u\right)^{\frac{1}{2}}} \cdot \frac{-2(1-\phi)}{\gamma\sigma^2} \tag{2.25}$$

$$\lambda'(0) = \frac{1}{(e^{-2\bar{s}}-0)^{\frac{1}{2}}}\frac{-(1-\phi)}{\gamma\sigma^2} \equiv -\phi e^{-\bar{s}} \tag{2.26}$$

$$\begin{aligned}
\frac{1}{e^{-\bar{s}}}\frac{-(1-\phi)}{\gamma\sigma^2} &= -\phi e^{-\bar{s}} \\
e^{-\bar{s}} &= \frac{1}{\sigma}\sqrt{\frac{1-\phi}{\phi\gamma}} \\
\bar{S} = e^{\bar{s}} &= \sigma\sqrt{\frac{\phi\gamma}{1-\phi}}.
\end{aligned} \tag{2.27}$$

Since ϕ is empirically close to 1, we approximate $\bar{S}$ with the following expression:

$$\bar{S} = \sigma\sqrt{\frac{\gamma}{1-\phi}}.$$

This makes our final expression for the CC's sensitivity function (or rather, an approximation for it)

$$\lambda(u) = \frac{1}{\bar{S}}\sqrt{1-2u} - 1. \tag{2.28}$$

2.2 Deriving the Theoretical Risk-Free Rate Equation

From Cochrane's *Asset Pricing* [**?**] we have the theoretical equation

$$r_t^f(s_t)dt = -E_t\left[\frac{d\Lambda}{\Lambda}\right], \tag{2.29}$$

where $\Lambda_t \equiv e^{-\beta t}u'(c_t)$ is the continuous-time discount factor. This is the continuous time equivalent of $R_t^f = \frac{1}{E_t[M_{t+1}]}$. From CC's choice of utility function, we have $\Lambda_t = e^{-\beta t}[S_t C_t]^{-\gamma}$. We now derive the differential of Λ, $d\Lambda_t$. For this, we require the assistance of Ito's Lemma from Shreve's *Stochastic Calculus for Finance II: Continuous-Time Models* [**?**].

Theorem 10. ***(Ito's Lemma)*** *Consider a function $F(t, X_1(t), X_2(t))$, where $X_1(t)$ and $X_2(t)$ are stochastic processes, and t is time. We can thus write these stochastic processes in the form*

$$dX_1(t) = f_1(t)dt + S_1(t)dW_1(t)$$

and

$$dX_2(t) = f_2(t)dt + S_2(t)dW_2(t),$$

with $W_1(t)$ and $W_2(t)$ Brownian motions. Then in differential form we have

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X_1}dX_1 + \frac{\partial F}{\partial X_2}dX_2 + \frac{1}{2}\left(\frac{\partial^2 F}{\partial X_1^2}(dX_1)^2 + 2\frac{\partial^2 F}{\partial X_1 \partial X_2}dX_1 dX_2 + \frac{\partial^2 F}{\partial X_2^2}(dX_2)^2\right). \tag{2.30}$$

Moreover, we have the multiplication rules for differentials

$$dtdt = 0, dtdW = 0, dW_1 dW_2 = \rho dt, \tag{2.31}$$

where ρ is the correlation between the Brownian motions W_1 and W_2.

Remember that $\Lambda = \Lambda(t, C_t, S_t)$. From their definitions we know S_t and C_t to be stochastic processes. Therefore, we arrive at the following expression:

$$d\Lambda = \frac{\partial \Lambda}{\partial t}dt + \frac{\partial \Lambda}{\partial S}dS + \frac{\partial \Lambda}{\partial C}dC + \frac{1}{2}\left(\frac{\partial^2 \Lambda}{\partial S^2}(dS)^2 + 2\frac{\partial^2 \Lambda}{\partial S \partial C}dSdC + \frac{\partial^2 \Lambda}{\partial C^2}(dC)^2\right). \tag{2.32}$$

We can quickly compute the partial derivatives. They are

$$\frac{\partial \Lambda}{\partial S} = -\gamma\frac{\Lambda}{S}, \ \frac{\partial \Lambda}{\partial C} = -\gamma\frac{\Lambda}{C}, \ \frac{\partial \Lambda}{\partial t} = -\beta\Lambda, \tag{2.33}$$

$$\frac{\partial^2 \Lambda}{\partial S^2} = \gamma(\gamma+1)\frac{\Lambda}{S^2}, \ \frac{\partial^2 \Lambda}{\partial C^2} = \gamma(\gamma+1)\frac{\Lambda}{C^2}, \ \text{ and } \frac{\partial^2 \Lambda}{\partial S \partial C} = \gamma^2\frac{\Lambda}{SC}.$$

We have that $C(x) = e^x$ and that $S(s) = e^s$, where x and s are stochastic processes. Ito's Lemma tells us that

$$dS = \frac{\partial S}{\partial s}ds + \frac{1}{2}\frac{\partial^2 S}{\partial s^2}(ds)^2 \tag{2.34}$$

$$dC = \frac{\partial C}{\partial x}dx + \frac{1}{2}\frac{\partial^2 C}{\partial x^2}(dx)^2. \tag{2.35}$$

This implies

$$\frac{dS}{S} = ds + \frac{1}{2}(ds)^2. \tag{2.36}$$

We then see that

$$\left(\frac{dS}{S}\right)^2 = (ds + \frac{1}{2}(ds)^2)^2 = (ds)^2 + (ds)^3 + \frac{1}{4}(ds)^4 = (ds)^2, \tag{2.37}$$

where the higher order terms of ds vanish by direct application of Ito's Lemma.

$$\frac{dC}{C} = dx + \frac{1}{2}(dx)^2. \tag{2.38}$$

We then see using the same reasoning as above that

$$\left(\frac{dC}{C}\right)^2 = (dc)^2. \tag{2.39}$$

We substitute in the partial derivatives of Λ (2.33) into (2.32) to find

$$d\Lambda = -\beta\Lambda dt - \gamma\frac{\Lambda}{S}dS - \gamma\frac{\Lambda}{C}dC + \frac{1}{2}\left(\gamma(\gamma+1)\frac{\Lambda}{S^2}(dS)^2 + 2\gamma^2\frac{\Lambda}{SC}dSdC + \gamma(\gamma+1)\frac{\Lambda}{C^2}(dC)^2\right). \tag{2.40}$$

Grouping common terms together and using the rules for the stochastic processes for S (2.36) and C (2.38) yields

$$\frac{d\Lambda}{\Lambda} = -\beta dt - \gamma\left(ds + \frac{1}{2}(ds)^2\right) - \gamma\left(dx + \frac{1}{2}(dx)^2\right) + \frac{1}{2}\Big(\gamma(\gamma+1)(ds)^2 + 2\gamma^2\,(dx)\,(ds) + \gamma(\gamma+1)(dx)^2\Big).$$

We once again combine common terms. Moreover, following diffusion model techniques in (Cochrane, p.491-492) [**?**] we derive the following continuous-time motion equations for $x = lnC$ and $s = lnS$ from their discrete time versions (1.28) and (1.27):

$$dx = \bar{x}dt + \sigma d\omega \tag{2.41}$$

$$ds = (\phi - 1)(s - \bar{s})dt + \lambda(s - \bar{s})\sigma d\omega. \tag{2.42}$$

We will later need $(ds)^2$, $(dx)^2$, and $dsdx$. Computing and using Ito's rules, we have

$$(ds)^2 = \Big((\phi - 1)(s - \bar{s})dt + \lambda(s - \bar{s})\sigma d\omega\Big)^2 = \lambda^2(s - \bar{s})\sigma^2 dt \tag{2.43}$$

$$(dx)^2 = (\bar{x}dt + \sigma d\omega)^2 = \sigma^2 dt \tag{2.44}$$

$$ds \cdot dx = \sigma^2\lambda(s - \bar{s})dt. \tag{2.45}$$

So we see that

$$\begin{aligned} dx + \frac{1}{2}(dx)^2 &= (\bar{x}dt + \sigma d\omega) + \frac{1}{2}(\sigma^2 dt) \\ &= (\bar{x} + \frac{1}{2}\sigma^2)dt + \sigma d\omega. \end{aligned} \tag{2.46}$$

$$\begin{aligned} ds + \frac{1}{2}(ds)^2 &= \Big((\phi - 1)(s - \bar{s})dt + \lambda(s - \bar{s})\sigma d\omega\Big) + \frac{1}{2}(\lambda^2(s - \bar{s})\sigma^2 dt) \\ &= \Big((\phi - 1)(s - \bar{s}) + \sigma^2\lambda^2(s - \bar{s})\Big)dt + \sigma\lambda(s - \bar{s})d\omega \end{aligned} \tag{2.47}$$

Now working with our original equation, we have

$$\begin{aligned} \frac{d\Lambda}{\Lambda} &= -\beta dt - \gamma\Big(ds + \frac{1}{2}(ds)^2\Big) - \gamma\Big((dx + \frac{1}{2}(dx)^2\Big) + \frac{1}{2}\Big(\gamma(\gamma+1)(ds)^2 + 2\gamma^2 dxds + \gamma(\gamma+1)(dx)^2\Big) \\ &= -\beta dt - \gamma ds - \frac{1}{2}\gamma(ds)^2 - \gamma dx - \frac{1}{2}(dx)^2 + \frac{1}{2}\gamma^2(ds)^2 + \frac{1}{2}\gamma(ds)^2 + \gamma^2 dxds + \frac{1}{2}\gamma^2(dx)^2 + \frac{1}{2}\gamma(dx)^2 \\ &= -\beta dt - \gamma\Big((\phi - 1)(s - \bar{s})dt + \lambda(s - \bar{s})\sigma d\omega\Big) - \gamma\Big(\bar{x}dt + \sigma d\omega\Big) \\ &+ \frac{1}{2}\gamma^2\sigma^2\lambda^2(s - \bar{s})dt + \gamma^2\sigma^2\lambda(s - \bar{s})dt + \frac{1}{2}\gamma^2\sigma^2 dt \\ &= \Big(- \beta - \gamma(\phi - 1)(s - \bar{s}) - \gamma\bar{x} + \frac{\gamma^2\sigma^2}{2}(1 + \lambda(s - \bar{s}))^2\Big) - \gamma\sigma(1 + \lambda(s - \bar{s}))d\omega. \end{aligned} \tag{2.48}$$

If we center around 0 instead of $\bar{s}$, we have the stochastic process for the stochastic discount factor in the CC model:

$$\frac{d\Lambda}{\Lambda} = \left(-\gamma(\phi - 1)s - \beta - \gamma\bar{x} + \frac{1}{2}\gamma^2\sigma^2\left(1 + \lambda(s)\right)^2\right)dt - \gamma\sigma\left(1 + \lambda(s)\right)d\omega. \tag{2.49}$$

Taking expectations, we then see

$$\begin{aligned} r_t^f(s_t)dt = &- E_t\Big[\left(-\gamma(\phi - 1)s - \beta - \gamma\bar{x} + \frac{1}{2}\gamma^2\sigma^2\left(1 + \lambda(s)\right)^2\right)dt - \gamma\sigma\left(1 + \lambda(s)\right)d\omega\Big] \\ = &\Big(\gamma(\phi - 1)s + \beta + \gamma\bar{x} - \frac{1}{2}\gamma^2\sigma^2(1 + \lambda(s))^2\Big)dt. \end{aligned} \tag{2.50}$$

Thus we have the theoretical log risk-free rate

$$\boxed{r_t^f(s_t) = \gamma(\phi - 1)s + \beta + \gamma\bar{x} - \frac{1}{2}\gamma^2\sigma^2(1 + \lambda(s))^2.} \tag{2.51}$$

Now a general logistic function - the functional form we want for our risk-free rate - has the form

$$L(t) = A + \frac{B}{1 + Ce^{Dt}} \tag{2.52}$$

for real constants A, B, C, and $D > 0$. We can solve for two important constants necessary for our model, β and D, in terms of A, B, C, and given parameters. From CC, we have the following two important relationships:

$$\lambda(0) = \frac{1}{\bar{S}} - 1 \tag{2.53}$$

$$\lambda'(0) = \frac{-1}{\bar{S}}. \tag{2.54}$$

We begin with the equivalence

$$\gamma(\phi - 1)s + \beta + \gamma\bar{x} - \frac{1}{2}\gamma^2\sigma^2(1 + \lambda(s))^2 \equiv A + \frac{B}{1 + Ce^{Ds}}.$$

Solving for our sensitivity function λ, we arrive at

$$\boxed{\lambda = \frac{\sqrt{2\beta - 2A + 2\gamma\bar{x} + 2\gamma(\phi - 1)s - \frac{2B}{1+Ce^{Ds}}}}{\gamma\sigma} - 1.} \tag{2.55}$$

Thus, we have a new sensitivity function equation that comes from the logistic nature we imposed on the risk-free rate. Using the first of our relationships above (2.53) and (2.51), we have

$$\beta + \gamma\bar{x} - \frac{\gamma^2\sigma^2}{2}\left(\frac{1}{\bar{S}}\right)^2 = \bar{r}. \tag{2.56}$$

From this and our expression for $\bar{S}$ [1] we can solve for the β that is necessary for the relationship to hold in terms of A, B, C, and other given parameters.

$$\boxed{\beta = \bar{r} - \gamma\bar{x} + \frac{\gamma(1 - \phi)}{2}.} \tag{2.57}$$

Differentiating our expression for λ, we have

$$\lambda'(s) = \frac{1}{2\gamma\sigma\sqrt{2\beta + 2\gamma\bar{x} - 2A + 2\gamma(\phi - 1)s - \frac{2B}{1+Ce^{Ds}}}}\left(2\gamma(\phi - 1) + \frac{2BCDe^{Ds}}{(1 + Ce^{Ds})^2}\right). \tag{2.58}$$

We can use the second of our relationships (2.54) to solve for D. We also employ the relationship (2.1).

$$\lambda'(0) = \frac{1}{2\gamma\sigma\sqrt{2\beta - 2A + 2\gamma\bar{x} - \frac{2B}{1+C}}}\left(2\gamma(\phi - 1) + \frac{2BCD}{(1 + C)^2}\right) \equiv \frac{-.144}{\gamma\sigma}. \tag{2.59}$$

$$\boxed{D = \frac{-(1 + C)^2}{BC}\left(.144\sqrt{2\beta - 2A + 2\gamma\bar{x} - \frac{2B}{1 + C}} + \gamma(\phi - 1)\right).} \tag{2.60}$$

[1] We actually use the approximation for $\bar{S}$: $\bar{S} \approx \sigma\sqrt{\frac{\gamma}{1-\phi}}$.

2.3 Deriving the Logistic Risk-Free Rate Equation

As stated above, a general logistic function comes in the form

$$L(t) = A + \frac{B}{1 + Ce^{Dt}} \tag{2.61}$$

for real constants A, B, C, D. For L to be decreasing, we require that $D > 0$. Using (2.60) and certain desired characteristics, we can solve for the constants A, B, and C.

We want our risk-free interest rate to have its inflection point at the steady state $\bar{s}$ having the value of the historic risk-free interest rate $\bar{r}$. We also design r^f to have supremum $\bar{r}e^{\sigma_r}$ and infimum $\bar{r}e^{-\sigma_r}$, where σ_r is the standard deviation of $r^f = lnR^f$. (We use e^{σ_r} rather than σ_r because of the form of the parameters specified in [**?**].) In Figure 2 of [**?**] makes clear that r_t^f is in negative correlation with s_t, so that the risk-free rate ought to be a decreasing function of $s_t = \ln S_t$.

Our desired form for the risk-free rate is

$$r_t^f(s_t) = A + \frac{B}{1 + Ce^{Ds_t}}, \tag{2.62}$$

where we center around the steady-state surplus consumption $\bar{s} \equiv 0$ to simplify algebra. To make r_t^f decreasing we take $D > 0$. We want

$$r_t^f(0) = A + \frac{B}{1 + C} \equiv \bar{r}. \tag{2.63}$$

Moreover, we want

$$r_t^f(\infty) = A + \frac{B}{1 + Ce^{\infty}} = A \equiv \bar{r}e^{-\sigma_r}. \tag{2.64}$$

We also want

$$r_t^f(-\infty) = A + \frac{B}{1 + Ce^{-\infty}} = A + B \equiv \bar{r}e^{\sigma_r}. \tag{2.65}$$

We then get

$$B = \bar{r}(e^{\sigma_r} - e^{-\sigma_r}). \tag{2.66}$$

Plugging these two constants into (2.63), we get that

$$C = \frac{e^{\sigma_r} - 1}{1 - e^{-\sigma_r}}. \tag{2.67}$$

Using (2.57) we compute that $\beta = 0.008064$, and plugging into (2.60) we see that $D = 22.562$.

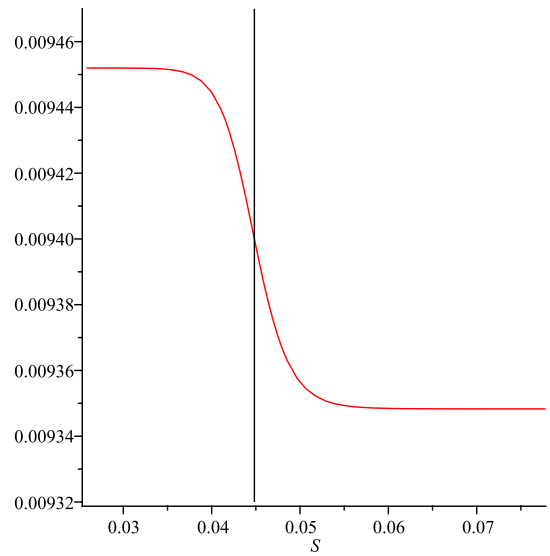


Figure 3

Figure 3 displays the new sensitivity function, centered on $\bar{S}$. The parameter values are $\gamma = 2.001$, $\phi = 0.9896$, $b = 0$, $\bar{r} = 0.00783$, $\sigma_r = 0.005514$, $\bar{x} = 0.00157$, $\sigma = 0.00323$, $p_0 = 219.60$, $p_1 = 111.173$, and $\bar{S} = 0.04483$. The x-axis gives the surplus consumption ratio on the support of $[\bar{S}e^{-100\sigma_r}, \bar{S}e^{100\sigma_r}] = [0.03245, 0.06195]$. The y-axis records the annual risk-free interest rate.

This gives us our *final* form for $\lambda(s)$ (when $b > 0$):

$$\boxed{\lambda(s) = \frac{\sqrt{C_0 + C_1 s + \frac{C_2}{1+C_3 e^{C_4 s}}}}{C_5} - 1,} \tag{2.68}$$

where we define our constants as

$$\begin{aligned} C_0 &= 2\beta + 2\gamma\bar{x} - 2A \\ C_1 &= 2\gamma(\phi - 1) \\ C_2 &= -2B \\ C_3 &= C \\ C_4 &= D \\ C_5 &= \gamma\sigma, \end{aligned} \tag{2.69}$$

with β and D determined from (2.57) and (2.60), respectively.

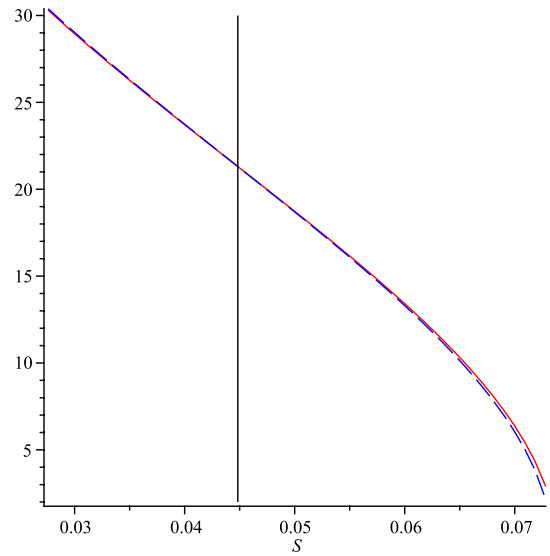


Figure 4

Figure 4 displays the new sensitivity function as a bold line and the CC sensitivity function as a dashed line, both centered on $\bar{S}$. The parameter values are $\gamma = 2.001$, $\phi = 0.9896$, $b = 0$, $\bar{r} = 0.00783$, $\sigma_r = 0.005514$, $\bar{x} = 0.00157$, $\sigma = 0.00323$, $p_0 = 219.60$, $p_1 = 111.173$, and $\bar{S} = 0.04483$. The x-axis gives the surplus consumption ratio on the support $[\bar{S}e^{-150\sigma}, \bar{S}e^{150\sigma}] = [0.02760, 0.07281]$. The y-axis records the new sensitivity function as a bold line and the CC sensitivity function as a dashed line.

3 Deriving the ODE

We have from Cochrane's *Asset Pricing* [**?**] the following continuous-time Euler condition:

$$0 = \Lambda_t D_t dt + E_t[d(\Lambda_t P_t)]. \tag{3.1}$$

Rewriting in terms of the price-dividend ratio $p = \frac{P}{D}$, we have

$$\begin{aligned} 0 &= D_t dt + E_t\Big[\frac{d(\Lambda_t P_t)}{\Lambda_t}\Big] \\ 0 &= \frac{1}{p} dt + E_t\Big[\frac{d[\Lambda p D]}{\Lambda p D}\Big]. \end{aligned} \tag{3.2}$$

We let $F(\Lambda, p, D) = \Lambda p D$ and using Ito's Lemma, we see

$$\begin{aligned} dF =& \frac{\partial F}{\partial \Lambda} d\Lambda + \frac{\partial F}{\partial p} dp + \frac{\partial F}{\partial D} dD + \frac{1}{2}\frac{\partial^2 F}{\partial \Lambda^2}(d\Lambda)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial p^2}(dp)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial D^2}(dD)^2 \\ &+ \frac{\partial^2 F}{\partial \Lambda \partial p}(d\Lambda)(dp) + \frac{\partial^2 F}{\partial D \partial p}(dD)(dp) + \frac{\partial^2 F}{\partial \Lambda \partial D}(d\Lambda)(dD). \end{aligned} \tag{3.3}$$

We see directly that

$$\frac{1}{2}\frac{\partial^2 F}{\partial \Lambda^2}(d\Lambda)^2 = \frac{1}{2}\frac{\partial^2(\Lambda p D)}{\partial \Lambda^2}(d\Lambda)^2 = 0.$$

The same occurs for $\frac{1}{2}\frac{\partial^2 F}{\partial p^2}(dp)^2$ and $\frac{1}{2}\frac{\partial^2 F}{\partial D^2}(dD)$. Computing our other derivatives and plugging in, we have

$$\frac{1}{p}dt + E_t\Big[\frac{d\Lambda}{\Lambda} + \frac{dp}{p} + \frac{dD}{D} + \frac{d\Lambda dp}{\Lambda p} + \frac{dDdp}{Dp} + \frac{d\Lambda dD}{\Lambda D}\Big] = 0. \tag{3.4}$$

We assume that the price dividend function $p = \frac{P}{D}$ is a C^2 function of $s = lnS$. Then using Ito's Lemma, we have

$$dp = p'(s)ds + \frac{1}{2}p''(s)(ds)^2. \tag{3.5}$$

Recall we have a description of the stochastic process ds. Plugging in the continuous motion equation from (2.42), $ds = (\phi - 1)(s - \bar{s})dt + \lambda(s - \bar{s})\sigma d\omega$, we see

$$dp = p'(s)\Big((\phi-1)(s-\bar{s})dt + \lambda(s-\bar{s})\sigma d\omega\Big) + p''(s)\Big((\phi-1)(s-\bar{s})dt + \lambda(s-\bar{s})\sigma d\omega\Big)^2. \tag{3.6}$$

We previously computed $(ds)^2$ in (2.43), so using this to simplify our expression, we see

$$dp = \Big(p'(s)(\phi-1)(s-\bar{s}) + \frac{1}{2}p''(s)\lambda^2(s-\bar{s})\sigma^2\Big)dt + p'(s)\lambda(s-\bar{s})\sigma d\omega. \tag{3.7}$$

Thus by solving for the price-dividend function p, we can describe the motion of the price-dividend ratio in continuous time.

We have one final assumption to make: we assume we are in equilibrium, and thus that $C = D$. By replacing D with C, we can use relation (2.38) and (2.47) in the pricing equation. We now calculate some of the factors from our Euler equation, again centering around 0 instead of $\bar{s}$.

$$\frac{dC}{C} = dx + \frac{1}{2}(dx)^2 = (\bar{x} + \frac{1}{2}\sigma^2)dt + \sigma d\omega \tag{3.8}$$

$$\begin{aligned} dp &= p'(s)ds + \frac{1}{2}p''(s)(ds)^2 = p'\Big((\phi-1)sdt + \sigma\lambda d\omega\Big) + \frac{1}{2}p''(\sigma^2\lambda^2 dt) \\ &= \Big(p'(\phi-1)s + \frac{1}{2}p''\sigma^2\lambda^2\Big)dt + p'\lambda\sigma d\omega \end{aligned} \tag{3.9}$$

$$\frac{d\Lambda}{\Lambda}\frac{dp}{p} = \frac{-\gamma\sigma^2\lambda(1+\lambda)p'dt}{p} \tag{3.10}$$

$$\frac{dC}{C}\frac{dp}{p} = \frac{p'\lambda\sigma^2 dt}{p} \tag{3.11}$$

$$\frac{d\Lambda}{\Lambda}\frac{dC}{C} = -\gamma\sigma^2(1+\lambda)dt. \tag{3.12}$$

When substituting into our pricing equation, we can use $E_t[d\omega] = 0$ and arrive at

$$0 = \frac{1}{p}dt + \left(-\gamma(\phi-1)s - \beta - \gamma\bar{x} + \frac{1}{2}\gamma^2\sigma^2\left(1+\lambda(s)\right)^2\right)dt + \frac{(p'(\phi-1)s + \frac{1}{2}p''\sigma^2\lambda^2)dt}{p}$$
$$+ (\bar{x} + \frac{1}{2}\sigma^2)dt + \frac{-\gamma\sigma^2\lambda(1+\lambda)p'dt}{p} + \frac{p'\lambda\sigma^2 dt}{p} + -\gamma\sigma^2(1+\lambda)dt. \tag{3.13}$$

Multiplying by $\frac{p}{dt}$, we get

$$0 = 1 + (-\gamma(\phi-1)s - \beta - \gamma\bar{x} + \frac{1}{2}\gamma^2\sigma^2(1+\lambda)^2)p + p'(\phi-1)s + \frac{1}{2}p''\lambda^2\sigma^2$$
$$+ (\bar{x} + \frac{1}{2}\sigma^2)p - \gamma\sigma^2\lambda(1+\lambda)p' + \lambda\sigma^2 p' - \gamma\sigma^2(1+\lambda)p. \tag{3.14}$$

$$\left(\frac{1}{2}\lambda^2\sigma^2\right)p'' = \left(\gamma\sigma^2\lambda(1+\lambda) - \lambda\sigma^2 - (\phi-1)s\right)p'$$
$$+ \left(\gamma(\phi-1)s + \beta + \gamma\bar{x} - \frac{1}{2}\gamma^2\sigma^2(1+\lambda)^2 - \bar{x} - \frac{1}{2}\sigma^2 + \gamma\sigma^2(1+\lambda)\right)p - 1. \tag{3.15}$$

So we can now write our pricing equation as a second order ODE in p:

$$\boxed{c_2(s)p''(s) = c_1(s)p'(s) + c_0(s)p(s) - 1,} \tag{3.16}$$

with coefficients c_2, c_1, c_0 defined as

$$c_2(s) = \frac{\lambda^2(s)\sigma^2}{2}, \tag{3.17}$$
$$c_1(s) = \gamma\sigma^2\lambda(s)(1+\lambda(s)) - \sigma^2\lambda(s) - (\phi-1)s, \tag{3.18}$$
$$c_0(s) = \gamma(\phi-1)s + \beta + (\gamma-1)\bar{x} - \frac{\gamma^2\sigma^2}{2}(1+\lambda(s))^2 + \gamma\sigma^2(1+\lambda(s)) - \frac{\sigma^2}{2}. \tag{3.19}$$

The normal form of our ODE is then

$$p''(s) = \frac{c_1(s)}{c_2(s)}p'(s) + \frac{c_0(s)}{c_2(s)}p(s) - \frac{1}{c_2(s)} \tag{3.20}$$

where we define c_i are as above.

The coefficients above are completely general and do not involve expanded forms of $\lambda(s)$. Thus, if we ever find need to alter our sensitivity function again, there is no need to rederive the above differential equation.

4 Solving the ODE

Our hope in solving the ODE is to employ the Cauchy-Kovalevski Theorem in the special case of a linear ODE with analytic coefficients.

Theorem 11. *Consider*

$$y''(x) + a(x)y'(x) + b(x)y(x) = g(x), y(0) = y_0, y'(0) = y_1,$$

where $a(x), b(x), g(x)$ *are analytic about* x_0*. This IVP ODE has a* unique *solution* $p(x)$ *near* x_0 *which is analytic with radius of convergence* r_0 *equal to at least the smallest radius of convergence of* a, b, *and* g.

In the Introduction to Complex Analysis in the Appendix, we discussed the largest possible radius for a Taylor series expansion of a holomorphic function. Our ODE is obviously linear. Since all the coefficients of our ODE are smooth, and since $C_0 + C_1 \cdot 0 + \frac{C_2}{1+C_3e^{C_4 \cdot 0}} = .0207 > 0$, we trust that there will be *some* radius of convergence for all the coefficients, and thus a unique analytic solution with *some* radius of convergence. We must at some point compute this radius of convergence. For now we will take for granted that there is such an analytic solution and radius of convergence and set out to determine the recursion formula for the coefficients of our solution for p. We focus on $\lambda(s)$ since $\lambda(s)$ shows up in one form or another in each of our coefficients.

4.1 Recursion to Find the Coefficients for the Polynomial Approximation for $\lambda(s)$

First, we define the term

$$\Lambda = \lambda + 1 = \frac{\sqrt{C_0 + C_1 s + \frac{C_2}{1+C_3e^{C_4 s}}}}{C_5}. \tag{4.1}$$

Now our goal is to write Λ as a power series. Formally, we have

$$\Lambda = \sum_{k=0}^{\infty} a_k s^k. \tag{4.2}$$

We must find the coefficients a_k. Before we do that, however, we will solve for coefficients b_k in the power series

$$\Lambda^2 = \sum_{k=0}^{\infty} b_k s^k. \tag{4.3}$$

We do this to sidestep the square root in our expression for Λ. We can later solve recursively for the a_k's, and we then immediately have the coefficients for the power series for λ.

$$\begin{aligned}
\Lambda^2 &= \frac{C_0 + C_1 s + \frac{C_2}{1+C_3e^{C_4 s}}}{C_5^2} \\
C_5^2\Lambda^2 &= C_0 + C_1 s + \frac{C_2}{1 + C_3e^{C_4 s}} \\
C_5^2\Lambda^2(1 + C_3e^{C_4 s}) &= C_0(1 + C_3e^{C_4 s}) + C_1 s(1 + C_3e^{C_4 s}) + C_2 \\
C_5^2\Lambda^2 + C_5^2C_3\Lambda^2e^{C_4 s} &= C_0 + C_0C_3e^{C_4 s} + C_1 s + C_1C_3e^{C_4 s}s + C_2. \qquad (4.4)
\end{aligned}$$

Using the formal power series for Λ^2 we compute:

$$C_5^2 \sum_{k=0}^{\infty} b_k s^k + C_5^2 C_3 \Big(\sum_{k=0}^{\infty} b_k s^k \Big) \Big(\sum_{k=0}^{\infty} \frac{C_4^k}{k!} s^k \Big) = C_0 + C_0 C_3 \sum_{k=0}^{\infty} \frac{C_4^k}{k!} s^k + C_1 s + C_1 C_3 \sum_{k=0}^{\infty} \frac{C_4^k}{k!} s^{k+1} + C_2$$

$$C_5^2 \sum_{k=0}^{\infty} b_k s^k + C_5^2 C_3 \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} \frac{b_{k-j} C_4^j}{j!} \Big) s^k = C_0 + C_0 C_3 \sum_{k=0}^{\infty} \frac{C_4^k}{k!} s^k + C_1 s + C_1 C_3 \sum_{k=0}^{\infty} \frac{C_4^k}{k!} s^{k+1} + C_2. \tag{4.5}$$

With the above calculation, we can begin recursively calculating the b_k's. From the zero-th power of s, we have

$$\begin{aligned} C_5^2 b_0 + C_3 C_5^2 b_0 &= C_0 + C_2 + C_0 C_3 \\ b_0 &= \frac{C_0 + C_2 + C_0 C_3}{C_5^2 (1 + C_3)}. \end{aligned} \tag{4.6}$$

From the first power s, we have

$$\begin{aligned} C_5^2 b_1 + C_3 C_5^2 \sum_{j=0}^{1} \frac{b_{1-j} C_4^j}{j!} &= C_1 + \frac{C_0 C_3 C_4}{1!} + \frac{C_1 C_3 C_4^0}{0!} \\ C_5^2 b_1 + C_3 C_5^2 b_1 + C_3 C_5^2 b_0 C_4 &= C_1 + C_0 C_3 C_4 + C_1 C_3 \\ C_5^2 b_1 + C_3 C_5^2 b_1 &= C_1 + C_0 C_3 C_4 + C_1 C_3 - C_3 C_5^2 C_4 b_0 \\ b_1 &= \frac{C_1 + C_0 C_3 C_4 + C_1 C_3 - C_3 C_4 C_5^2 b_0}{C_5^2 (1 + C_3)} \\ &= \frac{C_1 + C_0 C_3 C_4 + C_1 C_3 - C_3 C_4 C_5^2 \left(\frac{C_0 + C_2 + C_0 C_3}{C_5^2 (1 + C_3)} \right)}{C_5^2 (1 + C_3)} \\ &= \frac{C_1 (1 + C_3)^2 - C_2 C_3 C_4}{C_5^2 (1 + C_3)^2}. \end{aligned} \tag{4.7}$$

We do one more non-general computation: from the second power of s, we have

$$\begin{aligned} C_5^2 b_2 + C_3 C_5^2 \sum_{j=0}^{2} \frac{b_{2-j} C_4^j}{j!} &= \frac{C_0 C_3 C_4^2}{2!} + \frac{C_1 C_3 C_4}{1!} \\ C_5^2 b_2 + C_3 C_5^2 b_2 &= \frac{C_0 C_3 C_4^2}{2} + C_1 C_3 C_4 - C_3 C_4 C_5^2 b_1 - \frac{C_3 C_4^2 C_5^2 b_0}{2} \\ b_2 &= \frac{\frac{C_0 C_3 C_4^2}{2} + C_1 C_3 C_4 - C_3 C_4 C_5^2 b_1 - \frac{C_3 C_4^2 C_5^2 b_0}{2}}{C_5^2 (1 + C_3)}. \end{aligned} \tag{4.8}$$

Finally, from the general kth power of s, we have

$$C_5^2 b_k + C_3 C_5^2 \sum_{j=0}^{k} \frac{b_{k-j} C_4^j}{j!} = \frac{C_0 C_3 C_4^k}{k!} + \frac{C_1 C_3 C_4^{k-1}}{(k-1)!}$$

$$C_5^2 b_k + C_3 C_5^2 b_k = \frac{C_0 C_3 C_4^k}{k!} + \frac{C_1 C_3 C_4^{k-1}}{(k-1)!} - C_3 C_5^2 \sum_{j=1}^{k} \frac{b_{k-j} C_4^j}{j!}$$

$$b_k = \frac{1}{C_5^2(1+C_3)} \Big(\frac{C_0 C_3 C_4^k}{k!} + \frac{C_1 C_3 C_4^{k-1}}{(k-1)!} - C_3 C_5^2 \sum_{j=1}^{k} \frac{b_{k-j} C_4^j}{j!} \Big). \quad (4.9)$$

With these expressions for the b_k, we set to solving for the a_k, the coefficients for Λ. We have

$$\Lambda^2 = \sum_{k=0}^{\infty} b_k s^k = \Big(\sum_{k=0}^{\infty} a_k s^k \Big)^2 = \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} a_{k-j} a_j \Big) s^k. \quad (4.10)$$

But then we immediately see that

$$b_k = \sum_{j=0}^{k} a_{k-j} a_j. \quad (4.11)$$

From this relationship we see

$$b_0 = \sum_{j=0}^{0} a_0 a_0 = a_0^2 \Rightarrow a_0 = \sqrt{b_0} \quad (4.12)$$

$$b_1 = \sum_{j=0}^{1} a_{1-j} a_j = 2 a_1 a_0 \Rightarrow a_1 = \frac{1}{2a_0} b_1 \quad (4.13)$$

$$b_k = \sum_{j=0}^{k} a_{k-j} a_j = a_k a_0 + a_{k-1} a_1 + \cdots a_1 a_{k-1} + a_0 a_k$$

$$b_k - \sum_{j=1}^{k-1} a_{k-j} a_j = 2 a_k a_0 \Rightarrow a_k = \frac{1}{2a_0} \Big(b_k - \sum_{j=1}^{k-1} a_{k-j} a_j \Big). \quad (4.14)$$

So we now have all the coefficients for the power series representation $\sum_{k=0}^{\infty} a_k x^k = \Lambda$. From this power series we immediately derive the power series for λ:

$$\lambda = \sum_{k=0}^{\infty} \tilde{a_k} x^k, \quad (4.15)$$

where we have the coefficients

$$\tilde{a_0} = \sqrt{b_0} - 1$$
$$\tilde{a_1} = a_1 = \frac{1}{2a_0} b_1$$
$$\tilde{a_k} = a_k = \frac{1}{2a_0}\Big(b_k - \sum_{j=1}^{k-1} a_{k-j} a_j\Big). \tag{4.16}$$

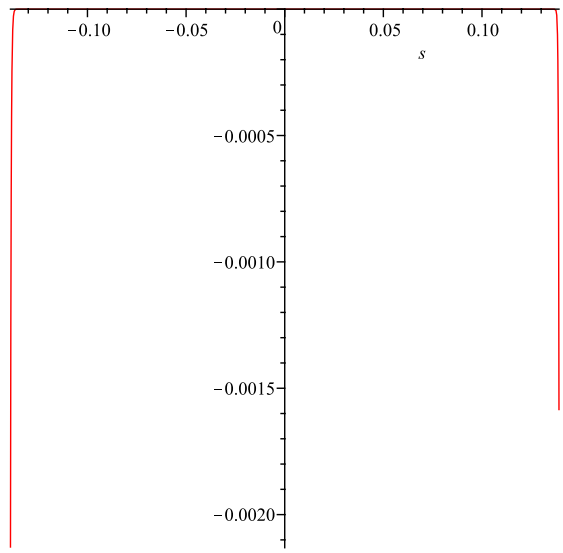


Figure 5

Figure 5 displays difference between the 475th and 400th approximations of λ centered on 0. The parameter values are $\gamma = 2.001$, $\phi = 0.9896$, $b = 0$, $\bar{r} = 0.00783$, $\sigma_r = 0.005514$, $\bar{x} = 0.00157$, $\sigma = 0.00323$, $p_0 = 219.60$, $p_1 = 111.173$, and $\bar{S} = 0.04483$. The x-axis gives the log surplus consumption ratio on the support of $[-42\sigma, 42\sigma] = [-0.1358, 0.1358]$. The y-axis records the difference between the 475th and 400th approximations of λ centered on 0. Note that the difference between the 475th and 400th approximations has shrunk to zero over nearly the entire interval. We will use the 475th approximation throughout the rest of the paper.

4.2 Recursion to Solve for the Price-Dividend Power Series

We have

$$p(s) = \sum_{k=0}^{\infty} p_k s^k, p'(s) = \sum_{k=0}^{\infty} (k+1) p_{k+1} s^k, p''(s) = \sum_{k=0}^{\infty} (k+1)(k+2) p_{k+2} s^k.$$

We are concerned with the differential equation

$$c_2(s) p''(s) = c_1(s) p'(s) + c_0(s) p(s) - 1, \tag{4.17}$$

with coefficients

$$c_2(s) = \frac{\lambda^2(s)\sigma^2}{2} = \sum_{n=0}^{\infty} c_{2,n} s^n, \tag{4.18}$$

$$c_1(s) = \gamma\sigma^2\lambda(s)(1+\lambda(s)) - \sigma^2\lambda(s) - (\phi-1)s = \sum_{n=0}^{\infty} c_{1,n} s^n, \tag{4.19}$$

$$c_0(s) = \gamma(\phi-1)s + \beta + (\gamma-1)\bar{x} - \frac{\gamma^2\sigma^2}{2}(1+\lambda(s))^2 + \gamma\sigma^2(1+\lambda(s)) - \frac{\sigma^2}{2} = \sum_{n=0}^{\infty} c_{0,n} s^n. \tag{4.20}$$

With the power-series expression for $\lambda(s)$ we derived above, we can indeed find $c_{i,k}$ for each $i = 0, 1, 2$ and for all k. We will do this below, but first we solve formally for the coefficients p_k in the power-series expression for $p(s)$.

We first multiply power series to find the terms in our ODE:

$$\begin{aligned} c_2(s)p''(s) &= \Big(\sum_{n=0}^{\infty} c_{2,n} s^n\Big)\Big(\sum_{n=0}^{\infty} (n+1)(n+2)p_{n+2} s^n\Big) \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} (k+1)(k+2)c_{2,n-k}p_{n+2}\Big)s^n \end{aligned} \tag{4.21}$$

$$c_1(s)p'(s) = \Big(\sum_{n=0}^{\infty} c_{1,n} s^n\Big)\Big(\sum_{n=0}^{\infty} (n+1)p_{n+1} s^n\Big) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} (k+1)c_{1,n-k}p_{k+1}\Big)s^n \tag{4.22}$$

$$c_0(s)p(s) = \Big(\sum_{n=0}^{\infty} c_{0,n} s^n\Big)\Big(\sum_{n=0}^{\infty} p_n s^n\Big) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} c_{0,n-k}p_k\Big)s^n \tag{4.23}$$

Inputing these expressions into the ODE, we have

$$\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} (k+1)(k+2)c_{2,n-k}p_{k+2}\Big)s^n = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} (k+1)c_{1,n-k}p_{k+1} + \sum_{k=0}^{n} c_{0,n-k}p_k\Big)s^n - 1 \tag{4.24}$$

We can now solve for relations that will lead to a recursive formula. From the constant terms we have

$$\begin{aligned} 1\cdot 2\cdot c_{2,0}p_2 &= 1\cdot c_{1,0}p_1 + c_{0,0}p_0 - 1 \\ p_2 &= \frac{c_{1,0}p_1 + c_{0,0}p_0 - 1}{2c_{2,0}}. \end{aligned} \tag{4.25}$$

Later we will derive p_0 and p_1, the initial conditions.

We now derive a closed recursive formula for p_k. From the kth order terms we have

$$\sum_{k=0}^{n} (k+1)(k+2)c_{2,n-k}p_{k+2} = \sum_{k=0}^{n} (k+1)c_{1,n-k}p_{k+1} + \sum_{k=0}^{n} c_{0,n-k}p_k. \tag{4.26}$$

We solve for our highest order term, p_{n+2}. Reindexing, we have

$$\sum_{k=2}^{n+2}(k-1)kc_{2,n-k+2}p_k = \sum_{k=1}^{n+1}kc_{1,n-k+1}p_k + \sum_{k=0}^{n}c_{0,n-k}p_k \tag{4.27}$$

$$(n+2-1)(n+2)c_{2,n-(n+2)+2}p_{n+2} + \sum_{k=2}^{n+1}(k-1)kc_{2,n-k+2}p_k = \sum_{k=1}^{n+1}kc_{1,n-k+1}p_k + \sum_{k=0}^{n}c_{0,n-k}p_k \tag{4.28}$$

$$(n+1)(n+2)c_{2,0}p_{n+2} = \sum_{k=1}^{n+1}kc_{1,n-k+1}p_k + \sum_{k=0}^{n}c_{0,n-k}p_k - \sum_{k=2}^{n+1}(k-1)kc_{2,n-k+2}p_k \tag{4.29}$$

$$= 1\cdot c_{1,n}p_1 + (n+1)c_{1,0}p_{n+1} + \sum_{k=2}^{n}kc_{1,n-k+1}p_k \tag{4.30}$$

$$+ c_{0,n}p_0 + c_{0,n-1}p_1 + \sum_{k=2}^{n}c_{0,n-k}p_k \tag{4.31}$$

$$- \Big(n(n+1)c_{2,1}p_{n+1} + \sum_{k=2}^{n}(k-1)kc_{2,n-k+2}p_k\Big) \tag{4.32}$$

$$\begin{aligned}(n+1)(n+2)c_{2,0}p_{n+2} &= c_{1,n}p_1 + (n+1)c_{1,0}p_{n+1} + c_{0,n}p_0 + c_{0,n-1}p_1 - n(n+1)c_{2,1}p_{n+1} \\ &+ \sum_{k=2}^{n}\Big(kc_{1,n-k+1} + c_{0,n-k} - (k-1)kc_{2,n-k+2}\Big)p_k.\end{aligned} \tag{4.33}$$

We have solved for p_{n+2}, giving us a second-order recursion. After deriving p_0 and p_1 we will have a closed formula for the coefficients of the power series expression for $p(s)$.

Using the power-series expression we derived for λ (4.15), we find $c_{i,k}$.

$$c_2(s) = \frac{\lambda^2(s)\sigma^2}{2} = \frac{\sigma^2}{2}\Big(\sum_{k=0}^{\infty}\tilde{a_k}s^k\Big)^2 \tag{4.34}$$

$$= \frac{\sigma^2}{2}\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k}\tilde{a_{k-j}}\tilde{a_j}\Big)s^k; \tag{4.35}$$

So

$$c_{2,i} = \frac{\sigma^2}{2}\sum_{j=0}^{i}\tilde{a_{i-j}}\tilde{a_j}. \tag{4.36}$$

$$c_1(s) = \sum_{k=0}^{\infty} c_{1,k} s^k = \gamma\sigma^2\lambda(1+\lambda) - \sigma^2\lambda - (\phi-1)s \tag{4.37}$$

$$= \gamma\sigma^2\Big(\sum_{k=0}^{\infty} \tilde{a_k} s^k\Big)\Big(1 + \sum_{k=0}^{\infty} \tilde{a_k} s^k\Big) - \sigma^2 \sum_{k=0}^{\infty} \tilde{a_k} s^k - (\phi-1)s \tag{4.38}$$

$$= \gamma\sigma^2 \sum_{k=0}^{\infty} \tilde{a_k} s^k + \gamma\sigma^2\Big(\sum_{k=0}^{\infty} \tilde{a_k} s^k\Big)^2 - \sigma^2 \sum_{k=0}^{\infty} \tilde{a_k} s^k - (\phi-1)s \tag{4.39}$$

$$= \sigma^2(\gamma-1)\sum_{k=0}^{\infty} \tilde{a_k} s^k + (1-\phi)s + \gamma\sigma^2 \sum_{k=0}^{\infty}\sum_{j=0}^{k} \Big(\tilde{a_{k-j}}\tilde{a_j}\Big) s^k. \tag{4.40}$$

So we see that

$$\begin{aligned} c_{1,0} &= \sigma^2(\gamma-1)\tilde{a_0} + \gamma\sigma^2 \tilde{a_0}^2 \\ c_{1,1} &= \sigma^2(\gamma-1)\tilde{a_1} + (1-\phi) + 2\gamma\sigma^2\tilde{a_1}\tilde{a_0} \\ c_{1,i} &= \sigma^2(\gamma-1)\tilde{a_i} + \gamma\sigma^2 \sum_{k=0}^{i} \tilde{a_{i-k}}\tilde{a_k} \text{ for all } i \geq 2. \end{aligned} \tag{4.41}$$

$$c_0(s) = \sum_{k=0}^{\infty} c_{0,k} s^k = \gamma(\phi-1)s + \beta + (\gamma-1)\bar{x} - \frac{\gamma^2\sigma^2}{2}(1+\lambda)^2 + \gamma\sigma^2(1+\lambda) - \frac{\sigma^2}{2} \tag{4.42}$$

$$= \gamma(\phi-1)s + \beta + (\gamma-1)\bar{x} - \frac{\sigma^2}{2} + \gamma(\phi-1)s - \frac{\gamma^2\sigma^2}{2}\Big(1 + \sum_{k=0}^{\infty} \tilde{a_k} s^k\Big)^2 + \gamma\sigma^2 + \gamma\sigma^2 \sum_{k=0}^{\infty} \tilde{a_k} s^k \tag{4.43}$$

$$= \gamma(\phi-1)s + \beta + (\gamma-1)\bar{x} + \gamma\sigma^2 - \frac{\gamma^2\sigma^2}{2}\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k} \bar{a_{k-j}}\bar{a_j}\Big) s^k + \gamma\sigma^2 \sum_{k=0}^{\infty} \tilde{a_k} s^k. \tag{4.44}$$

where

$$\bar{a_0} = \tilde{a_0} + 1 \tag{4.45}$$

$$\bar{a_i} = \tilde{a_i} \text{ for all } i \geq 1. \tag{4.46}$$

We conclude that

$$c_{0,0} = \beta + (\gamma-1)\bar{x} - \frac{\sigma^2}{2} + \gamma\sigma^2 - \frac{\gamma^2\sigma^2}{2}\bar{a_0}^2 + \gamma\sigma^2\tilde{a_0} \tag{4.47}$$

$$c_{0,1} = \gamma(\phi-1) - \gamma^2\sigma^2\bar{a_1}\bar{a_0} + \gamma\sigma^2\tilde{a_1} \tag{4.48}$$

$$c_{0,i} = -\frac{\gamma^2\sigma^2}{2}\sum_{j=0}^{k} \bar{a_{k-j}}\bar{a_j} + \gamma\sigma \text{ for all } i \geq 2. \tag{4.49}$$

4.3 Initial Conditions

We must now derive the initial conditions p_0 and p_1. p_0 is quite simple: we choose p_0 to match CC's price-dividend function at $s = 0$. We find the annual data in CC's paper [**?**], and converting to monthly terms we choose $p_0 = 18.3 \cdot 12 = 219.6$. Deriving p_1 takes a bit more work. We begin with instantaneous return on equity:

$$R^e(s)dt = \frac{dP}{P} + \frac{Ddt}{P}, \tag{4.50}$$

where P is the price of the asset and D is the dividend stream of the asset. We also have that $P = pD$, where p is of course our price-dividend ratio. Substituting this, we then have

$$\begin{aligned} R^e(s)dt &= \frac{d(pD)}{pD} + \frac{Ddt}{pD} \\ &= \frac{p(dD) + Ddp + (dp)(dD)}{pD} + \frac{Ddt}{pD} \\ &= \frac{dD}{D} + \frac{dp}{p} + \frac{dpdD}{pD} + \frac{dt}{p} \\ &= \frac{dC}{C} + \frac{dp}{p} + \frac{dpdC}{pC} + \frac{dt}{p}. \end{aligned} \tag{4.51}$$

The last equality follows from our assumption of equilibrium, in which we take $C = D$. Moreover, we have the following equation from [**?**] for the instantaneous return on bonds:

$$-R^b(s)dt = E_t\Big[\frac{d\Lambda}{\Lambda}\Big]. \tag{4.52}$$

From the derivation of our ODE, we also have (3.4):

$$\frac{1}{p}dt + E_t\Big[\frac{d\Lambda}{\Lambda} + \frac{dp}{p} + \frac{dD}{D} + \frac{d\Lambda dp}{\Lambda p} + \frac{dDdp}{Dp} + \frac{d\Lambda dD}{\Lambda D}\Big] = 0. \tag{4.53}$$

We use the above equations in working toward an expression for the expected instantaneous risk premium.

$$\begin{aligned} \Big(E_t\Big(R^e(s)\Big) - R^b(s)\Big)dt &= E_t\Big[\frac{dC}{C} + \frac{dp}{p} + \frac{dpdC}{pC} + \frac{dt}{p}\Big] + E_t\Big[\frac{d\Lambda}{\Lambda}\Big] \\ &= \frac{dt}{p} + E_t\Big[\frac{dC}{C} + \frac{dp}{p} + \frac{dpdC}{pC} + \frac{d\Lambda}{\Lambda}\Big] \\ &= -E_t\Big[\frac{d\Lambda dC}{\Lambda C} + \frac{d\Lambda dp}{\Lambda p}\Big]. \end{aligned} \tag{4.54}$$

Calculating the internal terms of the above expression (4.54), we can refer back to (2.49),

(2.46), and (3.7).

$$\frac{d\Lambda}{\Lambda} = \Big[\gamma(1-\phi)s - \beta - \gamma\bar{x} + \frac{\gamma^2\sigma^2}{2}(1+\lambda)^2\Big]dt - \gamma\sigma(1+\lambda)^2 d\omega \tag{4.55}$$

$$\frac{dC}{C} = (\bar{x} + \frac{1}{2}\sigma^2)dt + \sigma d\omega \tag{4.56}$$

$$dp = (p'(s)(\phi-1)s + \frac{1}{2}p''(s)\lambda(s)^2\sigma^2)dt + p'(s)\lambda(s)\sigma d\omega \tag{4.57}$$

$$\frac{d\Lambda}{\Lambda}\frac{dC}{C} = -\gamma\sigma^2(1+\lambda)dt \tag{4.58}$$

$$\frac{d\Lambda}{\Lambda}dp = -\gamma\sigma^2(1+\lambda)\lambda p' dt \tag{4.59}$$

$$\frac{d\Lambda}{\Lambda}\frac{dp}{p} = -\gamma\sigma^2(1+\lambda)\lambda\frac{p'}{p}dt. \tag{4.60}$$

From these calculations, (4.54) becomes

$$\begin{aligned} -E_t\Big[\frac{d\Lambda dC}{\Lambda C} + \frac{d\Lambda dp}{\Lambda p}\Big] &= E_t\Big[\gamma\sigma^2(1+\lambda)\Big(1+\lambda\frac{p'}{p}\Big)dt\Big] \\ &= \gamma\sigma^2(1+\lambda)\Big(1+\lambda\frac{p'}{p}\Big)dt, \end{aligned} \tag{4.61}$$

with the last equality coming since the term inside the expectation is not stochastic. We conclude that

$$\Big(E_t\Big(R^e(s)\Big) - R^b(s)\Big) = \gamma\sigma^2(1+\lambda)\Big(1+\lambda\frac{p'}{p}\Big), \tag{4.62}$$

and thus

$$p' = \frac{p}{\lambda}\Big[\frac{E_t(R^e(s)) - R^b(s)}{\gamma\sigma^2(1+\lambda)} - 1\Big]. \tag{4.63}$$

Note that $p_1 = p'(0)$. Evaluating, we have

$$p_1 = p'(0) = \frac{\Big(E_t(R^e(0)) - R^b(0) - \gamma\sigma^2(1+\lambda(0))\Big)p_0}{\gamma\sigma^2\Big(1+\lambda(0)\Big)\lambda(0)}. \tag{4.64}$$

We get $E_t(R^e(0) - R^b(0)) = \frac{.066}{12} = 0.0055$ from CC's paper [**?**] (converted from annual to monthly).

4.4 Radius of Convergence

We now have a power series solution to our differential equation, as was promised by Cauchy-Kovalevski. What remains is to find the radius of convergence for this power series by using Cauchy-Kovalevksi. Our work above focused on finding a power series for λ and a good power-series approximation since this is the function that drives the radius of convergence.

Since we are using the Cauchy-Kovalevski Theorem, we will have to find the radius of convergence for the power series of the price-dividend ratio by finding the smallest radius of convergence for the coefficients of the ODE. The form of the ODE we used to derive the recursion formula (4.17) is not in standard form yet. We rewrite it as

$$p''(s) - \frac{c_1(s)}{c_2(s)}p'(s) - \frac{c_0(s)}{c_2(s)}p(s) = -\frac{1}{c_2(s)}, \tag{4.65}$$

where the $c_i(s)$ are as defined above.

In analyzing our coefficients, our first hurdle is in dealing with the inversion of our power series: although we have computed power series for $c_2(s), c_1(s), c_0(s)$, we must find a way to write $\frac{1}{c_i(s)}$ as a power series. Beginning with $f(x) = \sum_{k=0}^{\infty} a_k x^k$, we write formally $\frac{1}{f(x)} = \sum_{k=o}^{\infty} b_k x^k$. Multiplying, we get

$$1 = \Big(\sum_{k=0}^{\infty} a_k x^k\Big)\Big(\sum_{j=o}^{\infty} b_j x^j\Big) = \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k} a_{k-j} b_j\Big).$$

By doing simple multiplication of series, we can get a feel for the b_k.

$$(a_0 + a_1 x + \cdots)(b_0 + b_1 x + \cdots) = a_0 b_0 + (a_1 b_0 + a_0 b_1)x + \cdots = 1.$$

Immediately we see that

$$b_0 = \frac{1}{a_0} \tag{4.66}$$

$$a_0 b_1 + a_1 b_0 = 0 \tag{4.67}$$

so that

$$b_1 = \frac{-a_1 b_0}{a_0}.$$

Following this strategy, we conclude that

$$\begin{aligned} b_0 &= \frac{1}{a_0} \\ b_n &= \frac{-\sum_{k=0}^{n-1} a_{n-k} b_k}{a_0}. \end{aligned} \tag{4.68}$$

Using this result, we can rewrite our coefficients:

$$\begin{aligned} \frac{c_1(s)}{c_2(s)} &= \frac{\sum_{k=0}^{\infty} c_{1,k} x^k}{\sum_{j=0}^{\infty} c_{2,j} x^j} = \Big(\sum_{k=0}^{\infty} c_{1,k} x^k\Big)\Big(\sum_{j=0}^{\infty} c_{2,j}^{-} x^j\Big) = \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k} c_{1,k-j} c_{2,j}^{-}\Big) x^k \\ \frac{c_0(s)}{c_2(s)} &= \frac{\sum_{k=0}^{\infty} c_{0,k} x^k}{\sum_{j=0}^{\infty} c_{2,j} x^j} = \Big(\sum_{k=0}^{\infty} c_{0,k} x^k\Big)\Big(\sum_{j=0}^{\infty} c_{2,j}^{-} x^j\Big) = \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k} c_{0,k-j} c_{2,j}^{-}\Big) x^k \\ \frac{1}{c_2(s)} &= \frac{1}{\sum_{k=0}^{\infty} c_{2,k} x^k} = \sum_{k=0}^{\infty} c_{2,k}^{-} x^k, \end{aligned} \tag{4.69}$$

where

$$\bar{c_{2,0}} = \frac{1}{c_{2,0}}$$
$$\bar{c_{2,n}} = -\sum_{k=0}^{n-1} \frac{c_{2,n-k}\bar{c_{2,k}}}{c_{2,0}} \text{ if } n \neq 0. \tag{4.70}$$

Using the 475th approximation of sensitivity function λ, each of the c_i's is a smooth function over the support $[-42\sigma, 42\sigma] = [-0.1358, 0.1358]$. It is only necessary to determine whether dividing by $c_2(s)$ preserves a non-zero radius of convergence. Figure 6 demonstrates that $\frac{1}{c_2(s)}$ converges on the support $[-43\sigma, 43\sigma] = [-0.1390, 0.1390]$ since $c_2(s)$ is never zero in this range. So by the Cauchy-Kovalevski Theorem **??** the solution to the ODE (4.17) has a radius of convergence no less than 42σ for consumption growth.

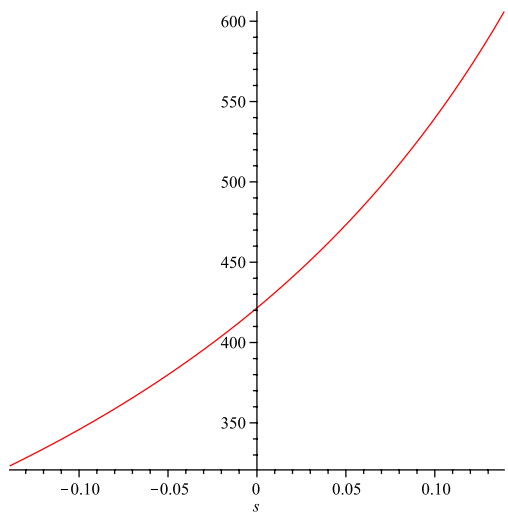


Figure 6

Figure 6 displays the 475th approximation of $\frac{1}{c_2(s)}$ centered on 0. The parameter values are $\gamma = 2.001$, $\phi = 0.9896$, $b = 0$, $\bar{r} = 0.00783$, $\sigma_r = 0.005514$, $\bar{x} = 0.00157$, $\sigma = 0.00323$, $p_0 = 219.60$, $p_1 = 111.173$, and $\bar{S} = 0.04483$. The x-axis gives the log surplus consumption ratio on the support of $[-43\sigma, 43\sigma] = [-0.1390, 0.1390]$. The y-axis records the 475th approximation of the forcing term $\frac{1}{c_2(s)}$ in the ODE for p. The function appears to converge over this interval.

5 Conclusion

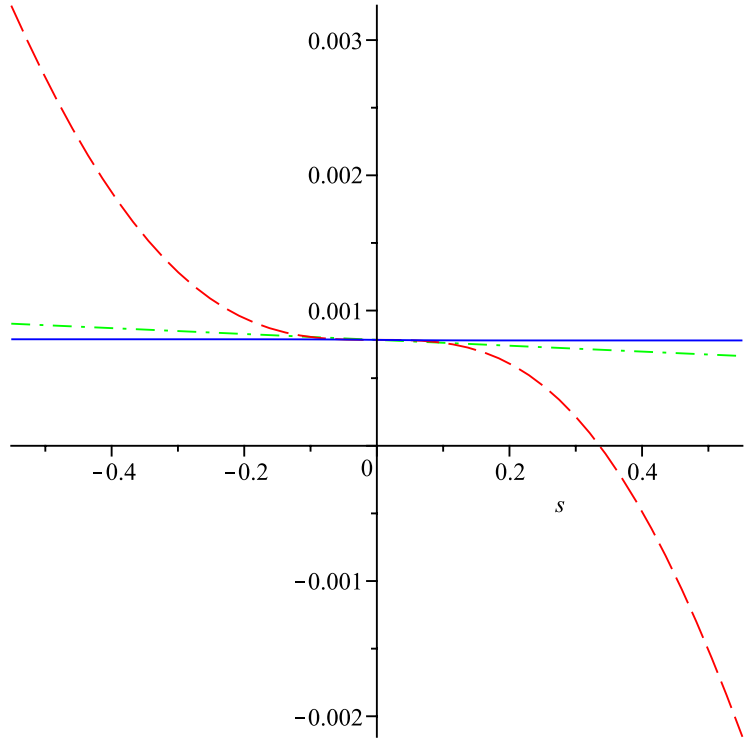


Figure 7

Figure 7 displays the CC, CCH, and new monthly risk-free rates as bold, dashed, and dot-dashed lines, respectively, centered on 0. The parameter values are $\gamma = 2.001$, $\phi = 0.9896$, $b = 0$, $\bar{r} = 0.00783$, $\sigma_r = 0.005514$, $\bar{x} = 0.00157$, $\sigma = 0.00323$, $p_0 = 219.60$, $p_1 = 111.173$, and $\bar{S} = 0.04483$. The x-axis gives the log surplus consumption ratio on the support of $[-100\sigma_r, 100\sigma_r] = [-0.5514, 0.5514]$. The y-axis records the CC, CCH, and new monthly risk-free rates as bold, dashed, and dot-dashed lines, respectively, centered on 0.

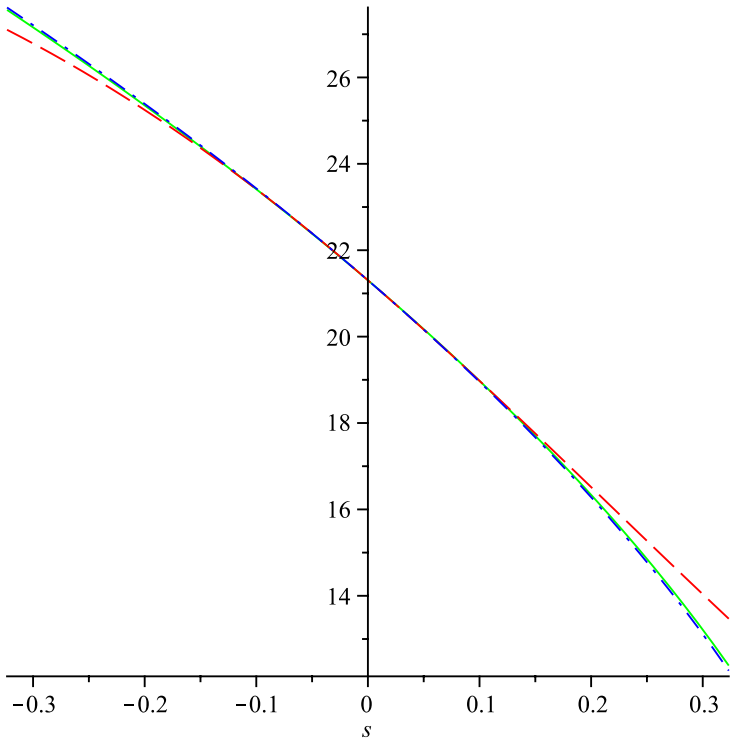


Figure 8

Figure 8 displays the CC, CCH, and new sensitivity functions as bold, dashed, and dot-dashed lines, respectively, centered on 0. The parameter values are $\gamma = 2.001$, $\phi = 0.9896$, $b = 0$, $\bar{r} = 0.00783$, $\sigma_r = 0.005514$, $\bar{x} = 0.00157$, $\sigma = 0.00323$, $p_0 = 219.60$, $p_1 = 111.173$, and $\bar{S} = 0.04483$. The x-axis gives the log surplus consumption ratio on the support of $[-100\sigma, 100\sigma] = [-0.3233, 0.3233]$. The y-axis records the CC, CCH, and new sensitivity functions as bold, dashed, and dot-dashed lines, respectively, centered on 0.

Above, we see that the new risk-free rate and sensitivity function match both CC's and CCH's attempts at the steady state. More than that, the new equations we have derived for each correct some of the economic problems we pinpointed in the Introduction, especially those concerning the risk-free rate. Instead of diverging or becoming negative, the new risk-free rate is both bounded and nonnegative.

Figure 9 displays the 475th Taylor approximation of the new annual price-dividend ratio, centered on $\bar{S}$. The parameter values are $\gamma = 2.001$, $\phi = 0.9896$, $b = 0$, $\bar{r} = 0.00783$, $\sigma_r = 0.005514$, $\bar{x} = 0.00157$, $\sigma = 0.00323$, $p_0 = 219.60$, $p_1 = 111.173$, and $\bar{S} = 0.04483$. The x-axis gives the surplus consumption ratio on the support $[\bar{S}e^{-44\sigma}, \bar{S}e^{44\sigma}] = [0.03889, 0.05169]$. The y-axis records the 475th approximation of the new annual price-dividend ratio.

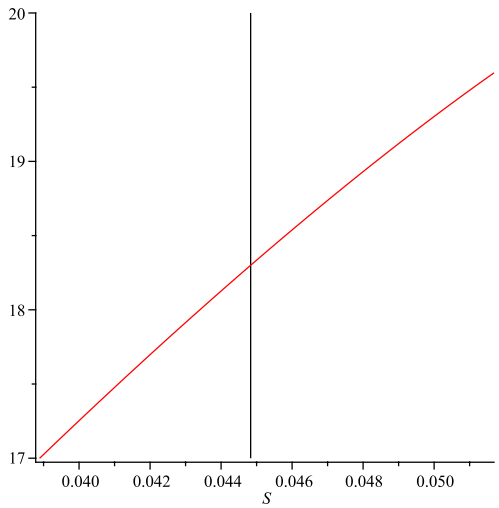


Figure 9

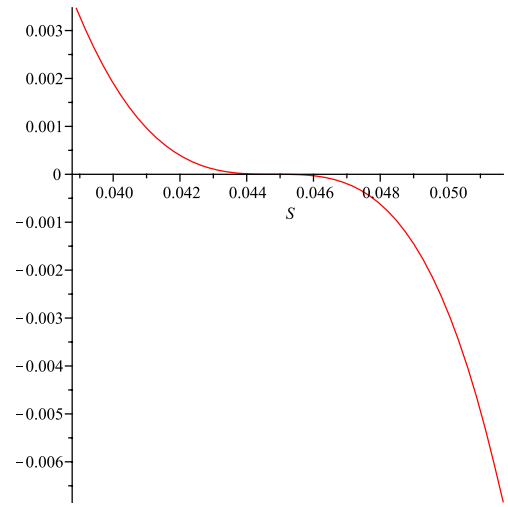


Figure 10

Figure 10 displays the difference between the 475th approximation of the price-dividend ratio in the CC model and the 475th approximation of the new price-dividend ratio. The parameter values are $\gamma = 2.001$, $\phi = 0.9896$, $b = 0$, $\bar{r} = 0.00783$, $\sigma_r = 0.005514$, $\bar{x} = 0.00157$, $\sigma = 0.00323$, $p_0 = 219.60$, $p_1 = 111.173$, and $\bar{S} = 0.04483$. The x-axis gives the surplus consumption ratio on the support $[\bar{S}e^{-44\sigma}, \bar{S}e^{44\sigma}] = [0.03889, 0.05169]$. The y-axis records the difference between 475th approximations of the CC price-dividend ratio and 475th approximation of the price-dividend from the new model.

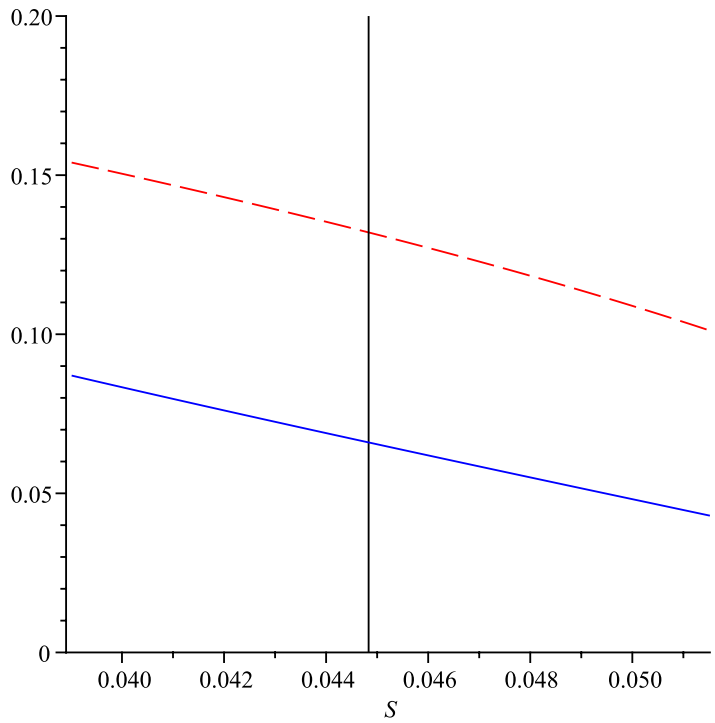


Figure 11

Figure 11 displays the expected annual return on equity as a solid line and the standard deviation on that return as a dashed line, centered on $\bar{S}$. The parameter values are $\gamma = 2.001$, $\phi = 0.9896$, $b = 0$, $\bar{r} = 0.00783$, $\sigma_r = 0.005514$, $\bar{x} = 0.00157$, $\sigma = 0.00323$, $p_0 = 219.60$, $p_1 = 111.173$, and $\bar{S} = 0.04483$. The x axis gives the surplus consumption ratio on the support $[\bar{S}e^{-43\sigma}, \bar{S}e^{43\sigma}] = [0.03901, 0.05152]$. The y-axis records the expected annual return on equity as a solid line and the standard deviation on that return as a dashed line.

Table 2: Computed Values

Statistic	New Simulation	CCH Simulation
$E_t(R^e)$	0.0754	0.075
$\sigma(R^e)$	0.132	0.133
$E_t(R^b)$	0.0094	0.009
$E_t(R^e - R^b)$	0.066	0.066
$Sharpe$	0.500	0.56
p	18.3	18.3

Notes: R^e is the real return on stocks, $R^b = 1 + r^f$ is the real return on bonds, and p is the price-dividend ratio. E_t is the conditional expectations operator, and σ is the standard deviation operator. We compute these data at the historic steady-state for the surplus consumption ratio $S_t = \bar{S}$. The parameters are $\gamma = 2.001$, $\phi = 0.9896$, $b = 0$, $\bar{r} = 0.00783$, $\sigma_r = 0.005514$, $\bar{x} = 0.00157$, $\sigma = 0.00323$, $p_0 = 219.60$, $p_1 = 111.173$, and $\bar{S} = 0.04483$.

This paper has introduced a new model of the risk free interest rate which closely mimics the empirical property of this rate. In addition, the model maintains the empirical properties for stock returns demonstrated by CC [**?**] and CCH [**?**]. In particular, Table 1 compares the stationary values produced by the analytic models generated by CCH [**?**] and the new attempt in this paper. Note that all the values are very similar, and in fact the Sharpe ratio in the new model is economically better than the one produced in the CCH attempt. So the new risk-free rate, which describes the economy better than the constant function of Campbell and Cochrane [**?**] or the linear function of Wachter [**?**], yields reliable economic data following the analytic method of CCH [**?**]. Thus, this change improves the existing continuous-time model of CC without altering its economic characteristics.

6 The Black-Scholes Equation

We will now present the Black-Scholes Equation for pricing European call options and discuss an example of how a firm would use it. This formula is often called "the most beautiful formula in all of finance," and no discussion of financial economics is complete without it. Our setup and proof of the Black-Scholes-Merton PDE for pricing European options closely follows [**?**]. For the original Black-Scholes European option pricing paper, see [**?**]. For a text in partial differential equations, see [**?**].

Setup of the Model The model begins with a European call option. The purchaser of a European call option is buying the right, but not the obligation, to purchase a stock at a determined price, the strike price K, at a determined time in the future, the maturity date T. We are concerned with determining the value of this option at time $t \leq T$ in continuous time. It seems reasonable to assume that the price of the option at time t, $P(t)$, is a function of both time t and the value of the stock at time t, $S(t)$. In other words,

$$P(t) = V(t, S(t)).$$

Moreover, to be consistent with modern approaches to the continous-time motion of stock prices, we assume that $S(t)$ is a Ito process:

$$dS = \alpha S dt + \sigma S d\omega,$$

where α and σ are positive constants and ω is a Brownian motion, or in other words

$$d\omega \sim N(0; dt).$$

In a simple situation, we present an investor with three decisions: how much to save, how much to spend, and of that amount he spends, how much to spend on stocks and how much on options for the stock. The amount he saves at time t, which we call $Q(t)$, he puts in a risk-free government bonds at a riskless (constant) rate r. Also, we assume no dividends and free exchange of commodities. We now define

$$A_1(t) = \text{number of stocks investor owns at time } t \tag{6.1}$$

$$A_2(t) = \text{number of options for stock investor owns at time } t. \tag{6.2}$$

We are now able to evaluate the value of the investor's portfolio of riskless bonds, stocks, and options at any time t:

$$\Pi(t) = Q(t) + A_1(t)S(t) + A_2(t)P(t).$$

The driving equation of the above valuation equation is $P(t)$, so we turn our attention back to it. Using Ito's Lemma, we differentiate this equation:

$$dP = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S \partial S}(dS)^2.$$

From our definition of dS above, and using the stochastic calculus rules $dtdt = d\omega dt = 0$ and $d\omega d\omega = dt$, we see that

$$(dS)^2 = \sigma^2 S^2 dt.$$

If we substitute our definitions for dS and $(dS)^2$ into our expression for dP, we see that

$$dP = \frac{\partial V}{\partial t}dt + \alpha S\frac{\partial V}{\partial S}dt + \sigma S\frac{\partial V}{\partial S}d\omega + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S \partial S}dt \tag{6.3}$$

$$= \Big(\frac{\partial V}{\partial t} + \alpha S\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S \partial S}\Big)dt + \sigma S\frac{\partial V}{\partial S}d\omega. \tag{6.4}$$

We make another assumption: the investor makes no instantaneous decisions about what types of assets he holds. In other words, at every time t the quantity of stock and options is fixed, so that $dA_1(t) = dA_2(t) = 0$. Moreover, since the risk-free investment has return r, we have $dQ = rQdt$. Thus, differentiating our expression for the value of the investor's portfolio, we have

$$d\Pi = rQdt + A_1 dS + A_2 dP.$$

We have expressions for dS and dP, and inserting these we have

$$d\Pi = A_1\Big(\alpha S dt + \sigma S d\omega\Big) + A_2\Big(\frac{\partial V}{\partial t} + \alpha S\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S \partial S}\Big)dt + A_2\sigma S\frac{\partial V}{\partial S}d\omega + rQdt \tag{6.5}$$

$$= \Big(\alpha A_1 S + A_2(V_t + \alpha S V_S + \frac{\sigma^2 S^2}{2}V_{SS}) + rQ\Big)dt + \sigma S(A_1 + V_S A_2)d\omega. \tag{6.6}$$

Next, we make the asssumption that the investor wants a risk-free portfolio. In other words, he wants to balance his assets with his liabilities in such a way that his portfolio earns only the risk-free rate r. Put in yet another way, the investor's goal is to remove the stochasticity from his portfolio. To do this, he chooses $A_1(t)$ and $A_2(t)$ such that $A_1 + A_2 V_S = 0$, or $A_1 = -A_2 V_S$. By this choice, he ensures that his portfolio earns the risk-free rate:

$$d\Pi = r\Pi dt.$$

In our portfolio valuation, we now have

$$\Pi = A_2 V - A_2 S V_S + Q.$$

Thus, in our equation for $d\Pi$ we see

$$d\Pi = \Big(A_2(V_t + \frac{\sigma^2 S^2}{2}V_{SS}) + rQ\Big)dt.$$

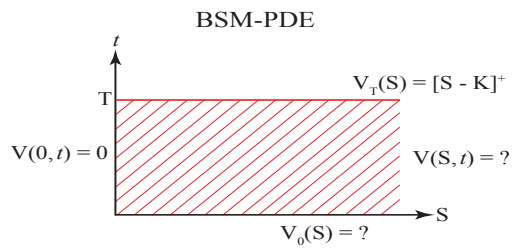
If we then insert these two equations into the expression $d\Pi = r\Pi dt$, we then have

$$\Big(A_2(V_t + \frac{\sigma^2 S^2}{2}V_{SS}) + rQ\Big)dt = (A_2 V - A_2 S V_S + Q)rdt.$$

We can cancel the dt's and rearrange the terms, arriving at the partial differential equation

$$V_t + rSV_S + V_{SS} - rV = 0.$$

Solution of the PDE



For the PDE above, we have the boundary conditions

$$V(T,S) = V_T(S) = \lim_{t\to T} V(t,S) = [S-K]^+ = max(0, S-K).$$

This condition seems natural, for if the value of the stock at the maturity date is less than the strike price, there is no way that an investor would exercise his option and buy it. Rather, to him the option is worthless, and he will let it expire. We also know that if the value of the stock is zero, then the price of the option is also zero. We write this as

$$V(t,0) = \lim_{S\to 0} V(t,S) = 0 \text{ for all } t.$$

We now have a boundary-value partial differential equation. In fact, this PDE is linear, second-order, and parabolic. In the study of PDE's, it is well understood that any parabolic PDE can be written in the form $\frac{\partial^2 G}{\partial x \partial x} = \frac{\partial G}{\partial y}$ for some change of coordinates. Thus, let us assume that $V(t,S)$ is of the form

$$V(t,S) = e^{-r\tau} G(x,y)$$

where $\tau = T - t$ and $x = x(t,S), y = y(t,S)$ are the appropriate change of coordinates. We have the multiplier function $e^{-r\tau}$ so that the value of the option goes to 0 as we draw closer to the exercise date. For the PDE, we will need three different partial derivatives of F:

$$\frac{\partial V}{\partial S} = e^{-r\tau}\left(\frac{\partial G}{\partial x}\frac{\partial x}{\partial S} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial S}\right) \tag{6.7}$$

$$\frac{\partial V}{\partial t} = re^{-r\tau}G + e^{-r\tau}\left(\frac{\partial G}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial t}\right) \tag{6.8}$$

$$\frac{\partial^2 V}{\partial S \partial S} = e^{-r\tau}\Big(\frac{\partial^2 G}{\partial x \partial x}\left(\frac{\partial x}{\partial S}\right)^2 + \frac{\partial^2 G}{\partial x \partial y}\frac{\partial x}{\partial S}\frac{\partial y}{\partial S} + \frac{\partial G}{\partial x}\frac{\partial^2 x}{\partial S \partial S} + \frac{\partial^2 G}{\partial y}\partial x\frac{\partial x}{\partial S}\frac{\partial y}{\partial S} + \frac{\partial^2 G}{\partial y \partial y}\left(\frac{\partial y}{\partial S}\right)^2 + \frac{\partial G}{\partial y}\frac{\partial^2 y}{\partial S \partial S}\Big). \tag{6.9}$$

If we insert these expressions into our PDE, cancelling the factor $e^{-r\tau}$, we first have

$$\big(rG + G_x x_t + G_y y_t\big) + rS\big(G_x x_S + G_y y_S\big) \tag{6.10}$$

$$+ \frac{\sigma^2 S^2}{2}\big(G_{xx}x_S^2 + G_{xy}x_S y_S + G_x x_{SS} + G_{yx}x_S y_S + G_{yy}y_S^2 + G_y y_{SS}\big) - rG = 0. \tag{6.11}$$

If we rearrange, grouping according to derivatives of G, we then have

$$\left(\frac{\sigma^2 S^2}{2} x_S^2\right) G_{xx} + \left(\sigma^2 S^2 x_S y_S\right) G_{xy} + \left(\frac{\sigma^2 S^2}{2} y_S^2\right) G_{yy} \tag{6.12}$$

$$+ \left(x_t + rSx_S + \frac{\sigma^2 S^2}{2} x_{SS}\right) G_x + \left(y_t + rSy_S + \frac{\sigma^2 S^2}{2} y_S S\right) G_y = 0. \tag{6.13}$$

Recall that our goal is to write our PDE, with an appropriate change of coordinates, into the form $G_{xx} = G_y$. Searching for this change of coordinates, we have from the equation above that for the PDE to take the desired form, we must have

$$\frac{\sigma^2 S^2}{2} x_S^2 + \frac{\sigma^2 S^2}{2} y_S S + rSy_S + y_t = 0 \tag{6.14}$$

$$\sigma^2 S^2 x_S y_S = 0 \tag{6.15}$$

$$\frac{\sigma^2 S^2}{2} x_{SS} + rSx_S + x_t = 0 \tag{6.16}$$

$$\frac{\sigma^2 S^2}{2} y_S^2 = 0. \tag{6.17}$$

Since S is non-negative, we can see from our last equation that y is independent of S. We use this condition to simplify the set of three conditions above:

$$\frac{\sigma^2 S^2}{2} x_S^2 + y_t = 0 \tag{6.18}$$

$$\frac{\sigma^2 S^2}{2} x_S S + rSx_S + x_t = 0. \tag{6.19}$$

We can write the first condition above as

$$\frac{\sigma^2 S^2}{2} x_S^2 = -y_t.$$

Since we know that y is independent of S, and thus so is y_t, we can conclude that Sx_S is independent of S as well. If we make the transformation

$$x(t, S) = \beta ln(S) + \gamma\tau,$$

with β, γ constants, we see that

$$x_S = \frac{\beta}{S} \tag{6.20}$$

$$x_{SS} = \frac{-\beta}{S^2} \tag{6.21}$$

$$Sx_S = \beta. \tag{6.22}$$

This is the simplest transformation that makes x a function of both t and S, yet makes Sx_S independent of S. Substituting our expressions for x and x_S into the remaining two conditions, we have

$$\frac{\sigma^2\beta^2}{2} + y_t = 0 \tag{6.23}$$

$$\frac{-\sigma^2\beta}{2} + r\beta - \gamma = 0. \tag{6.24}$$

For simplicity's sake we choose $\beta = 1$. This leads us to choose $\gamma = r - \frac{\sigma^2}{2}$. Finally, we see that if $y_t = \frac{-\sigma^2}{2}$, then the simplest antiderivative is $y(t) = \frac{\sigma^2}{2}\tau$. Restating our results, the simplest change of variables that makes the PDE of the form $G_{xx} = G_y$ is

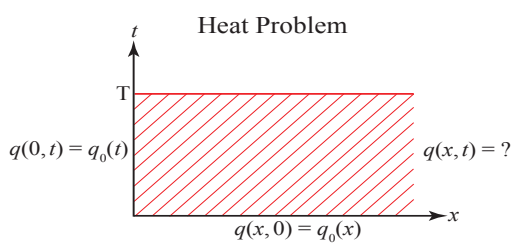
$$x(t, S) = ln(S) + \big(r - \frac{\sigma^2}{2}\big)\tau \tag{6.25}$$

$$y(t) = \frac{\sigma^2}{2}\tau. \tag{6.26}$$

With this change of variables, we have a new boundary condition. At time T, we are now concerned with when $\tau = T - T = 0$. In our change of variables we have that $y(T) = 0$ and $x(T, S) = ln(S) + 0$. Thus, we have the boundary condition

$$G(x(T, S), 0) = max(0, e^{x(T,S)} - K) \equiv f(x(T)).$$

We see that this change of variables has brought our partial differential equation under the conditions and into the form of the heat equation.



Because we have transformed this equation into this form, we are guaranteed the existence of a unique solution.

To begin solving this PDE, we make a standard assumption: we assume that G is separable. In other words, we can write G in the form

$$G(x, y) = X(x)Y(y),$$

thereby "separating" the variables x and y from each other. We then have the partial derivatives

$$G_{xx}(x,y) = X''(x)Y(y) \tag{6.27}$$
$$G_y(x,y) = X(x)Y'(y). \tag{6.28}$$

This gives us

$$X''(x)Y(y) = X(x)Y'(y) \tag{6.29}$$
$$\frac{X''}{X} = \frac{Y'}{Y} = k. \tag{6.30}$$

The expressions above are equal to the constant k as a consequence of the separation of the variables x, y. We immediately arrive at the two ordinary differential equations

$$X'' - kX = 0 \tag{6.31}$$
$$Y' - kY = 0 \tag{6.32}$$

We can quickly solve the differential equation for Y above, with the general solution of the form

$$Y(y) = Ae^{ky},$$

with A a constant. Using very minimal knowledge of differential equations, we can also solve the equation for X. It's general solution is

$$X(x) = Be^{i\sqrt{k}x},$$

where B is also a constant. Multiplying the two solutions we derived above, we have

$$G(x,y) = Ce^{ky+i\sqrt{k}x}.$$

Moreover, we know that if $c_1 f_1(x,y)$ and $c_2 f_2(x,y)$ are solutions to the same differential equation, then so is $c_1 f_1(x,y) + c_2 f_2(x,y)$. This result, called the superposition principle, holds for an arbitrary number of such solutions, so we can write the most general solution to the differential equation as a continuous sum as the constant varies:

$$G(x,y) = \int_{-\infty}^{\infty} C(\sqrt{k})e^{ky+i\sqrt{k}x} d\sqrt{k}.$$

But what of our boundary condition that will give us a unique solution? Recall, our boundary condition stated

$$G(x(T, S(T)), 0) = max(0, e^{x(T,S(T))} - K).$$

Moreover, at $\tau = T$, we know that $y = 0$.

$$f(x(T)) = \int_{-\infty}^{\infty} C(\sqrt{k})e^{\sqrt{k}xi} d\sqrt{k}.$$

This expression should look familiar. We know that the Fourier transform of a function $g(x)$ is

$$\widehat{g}(\zeta) = \int_{-\infty}^{\infty} e^{-ix\zeta} g(x) dx,$$

and from the Fourier inversion formula we also have that

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\zeta} \widehat{g}(\zeta) d\zeta.$$

From these formulae we see that $C = 2\pi \widehat{f}$. Using the Fourier transform, we rewrite

$$C(\sqrt{k}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(w) e^{-i\sqrt{k}w} dw.$$

We can now use this formula for C in our most general solution to the PDE:

$$G(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{i\sqrt{k}(x-u)+ky} du \; d\sqrt{k}.$$

If we integrate first with respect to $\sqrt{k}$, we then arrive at the integral

$$G(x,y) = \frac{1}{2\sqrt{\pi y}} \int_{-\infty}^{\infty} f(u) e^{-\frac{(u-x)^2}{4y}} du.$$

Recalling our boundary condition that defines f, we can restate the above integral expression:

$$G(x,y) = \frac{1}{2\sqrt{\pi y}} \int_{lnK}^{\infty} (e^u - K) e^{-\frac{(u-x)^2}{4y}} du,$$

where the lower limit of integration comes from our definition of f. Using this expression for G, along with the change of variables expressions for x and y, we can rewrite our equation for $F(t,S)$.

$$V(t,S) = \frac{e^{-rT}}{2\sqrt{\pi \frac{\sigma^2}{2} \tau}} \int_{lnK}^{\infty} (e^u - K) e^{-\frac{(u-(ln(S)+\left(r-\frac{\sigma^2}{2}\right)\tau))^2}{4\frac{\sigma^2}{2}\tau}} du \tag{6.33}$$

$$= \frac{e^{-rT}}{2\sigma\sqrt{2\pi\tau}} \int_{lnK}^{\infty} (e^u - K) e^{\frac{-(u-lnS-(r-\frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau}} du. \tag{6.34}$$

We can write the above integral as a sum:

$$V(t,S) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{lnK}^{\infty} e^{\frac{u-(u-lnS-(r-\frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau}} du - \frac{Ke^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{lnK}^{\infty} e^{\frac{-(u-lnS-(r-\frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau}} du.$$

We will focus on the first integral in the sum. We will first add and subtract the function $lnS + r\tau$ to the exponent of the exponential. We will then have in the exponent the quantity

$$-r\tau + u - \frac{(u - lnS - (r - \frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau} + [(lnS + r\tau) - (lnS + r\tau)] \tag{6.35}$$

$$= lnS - \frac{(u - lnS - (r + \frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau}, \tag{6.36}$$

which then gives us the integral equation

$$\frac{S}{\sigma\sqrt{2\pi\tau}} \int_{lnK}^{\infty} e^{-\frac{(u-lnS-(r+\frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau}} du.$$

We proceed with the change of variables

$$\Omega = \frac{u - lnS - (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \tag{6.37}$$

$$d\Omega = \frac{1}{\sigma\sqrt{\tau}} du. \tag{6.38}$$

$$\Omega(lnK) = \frac{lnK - lnS - (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \tag{6.39}$$

$$= \frac{ln\frac{K}{S} - (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \tag{6.40}$$

$$\Omega(\infty) = \infty. \tag{6.41}$$

Using the evenness of the function $e^{-\frac{\Omega^2}{2}}$, the change of variables gives us the integral

$$\frac{S}{\sqrt{2\pi}} \int_{\frac{lnK-lnS-(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{\Omega^2}{2}} d\Omega = \frac{S}{\sqrt{2\pi}} \int_{-\infty}^{\frac{ln\frac{S}{K}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}} e^{-\frac{\Omega^2}{2}} d\Omega = S\Phi\Big(\frac{ln\frac{S}{K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\Big),$$

where Φ is the cumulative distribution function for a standard normal variable.

We solve the other integral $-\frac{Ke^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{lnK}^{\infty} e^{\frac{-(u-lnS-(r-\frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau}} du$ using a similar change of variables.

$$\Delta = \frac{u - lnS - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \tag{6.42}$$

$$d\Delta = \frac{1}{\sigma\sqrt{\tau}} du \tag{6.43}$$

$$\Delta(lnK) = \frac{lnK - lnS - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = \frac{ln\frac{K}{S} - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \tag{6.44}$$

$$\Delta(\infty) = \infty. \tag{6.45}$$

Again using the evenness of $e^{-\frac{\Omega^2}{2}}$, the integral becomes

$$-\frac{Ke^{-r\tau}}{\sqrt{2\pi}}\int_{\frac{ln\frac{K}{S}-(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{\Delta^2}{2}}d\Delta = -\frac{Ke^{-r\tau}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{ln(\frac{S}{K})+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}} e^{-\frac{\Delta^2}{2}}d\Delta = -Ee^{-r\tau}\Phi\Big(\frac{ln\frac{S}{K}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\Big).$$

Summing our two solutions above, we have solved the PDE for the Black-Scholes model for a European call option:

$$V(t,S) = S\Phi\Big(\frac{ln\frac{S}{K}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\Big) - Ee^{-r\tau}\Phi\Big(\frac{ln\frac{S}{K}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\Big).$$

Discussing the Black-Scholes Equation in the context of the CC model for stock price motion leads us to an open question worth presenting: is it possible to implement the surplus-consumption ratio, the key component and revolutionary innovation of CC, in a new call option pricing formula? Using the work of CC and modifications made to their model here and elsewhere could lead to a pricing equation for a European call option that better represents the market. This would be a truly revolutionary contribution.

7 Appendix

7.1 An Introduction to Complex Analysis

To prove the radius of convergence for analytic solutions to the price-dividend differential equation, we would need to make use of complex analysis (CCH first proposed this method in [**?**].) We here present the minimal amount required for our purposes. For a complete presentation of complex analysis, see the classic [**?**].

Definition 1. *A complex-valued function $f(z) = f(x,y) = u(x,y) + iv(x,y)$ is holomorphic, or, equivalently, analytic, at z_0 if*

$$f'(z_0) \equiv \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

exists. $f(z)$ is holomorphic if the above is true for all z_0 where $f(z)$ is defined.

By following this definition of holomorphicity, we arrive at an important result of complex

analysis.

$$
\begin{aligned}
f'(z_0) &\equiv \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \\
&= \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0+h+iy_0)-f(x_0+iy_0)}{h} = u_x + iv_x \\
&= \lim_{h\to 0} \frac{f(x_0+i(y_0+h))-f(x_0+iy_0)}{ih} \\
&= \lim_{h\to 0} \frac{[u(x_0,y_0+h)+iv(x_0,y_0+h)]-[u(x_0,y_0)+iv(x_0,y_0)]}{ih} \\
&= \frac{1}{i}\lim_{h\to 0} \frac{u(x_0,y_0+h)-u(x_0,y_0)}{h} + \lim_{h\to 0} \frac{v(x_0,y_0+h)-v(x_0,y_0)}{h} = -iu_y + v_y. \qquad (7.1)
\end{aligned}
$$

Since our definition did not define in which direction z is to approach z_0, by approaching in both the purely real and purely imaginary directions, we arrive at both $f'(z_0) = u_x + iv_x$ and $f'(z_0) = -iu_y + v_y$. From these two equalities we see that for any holomorphic function, we must have

$$u_x = v_y \tag{7.2}$$

$$u_y = -v_x. \tag{7.3}$$

The above are called the Cauchy-Riemann equations, and any holomorphic function must satisfy them. Put in another form we can write that for a holomorphic function we must have

$$\bar{\partial} f = 0, \tag{7.4}$$

where $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$.

With the definition of a holomorphic function, the Cauchy-Riemann equations, and Green's theorem, we can state and prove an important theorem of complex analysis that will can be utilized later.

Theorem 12. *(**Cauchy's Integral Theorem**) If we have that* f *is a holomorphic function in a simply connected region* D *and* C *is a simple closed curve inside* D *oriented positively, then we have that*

$$\int_C f(z)dz = 0. \tag{7.5}$$

Proof. First, we define the interior of the curve C as G. Then we have

$$\begin{aligned}
\int_C f(z)dz &= \int_{\partial G} (u+iv)(dx+idy) \\
&= \int\int_G df \wedge dz \text{ (by Green's Theorem)} \\
&= \int\int_G (\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy) \wedge (dx+idy) \\
&= \int\int_G \frac{\partial f}{\partial x}dx \wedge idy + \frac{\partial f}{\partial y}dy \wedge dx \\
&= \int\int_G (i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y})dx \wedge dy = i\int\int_G (\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})dx \wedge dy \\
&= i\int\int_G 0 \text{ (by the Cauchy-Riemann equations)} \\
&= 0. \qquad (7.6)
\end{aligned}$$

□

Along with his Integral Theorem, Cauchy also discovered a formula for computing the value of the line integral of a holomorphic function f along a simple closed curve C with positive orientation, where f is holomorphic inside the simply connected region G and C is contained in G.

We will consider the case that $C = C_R$ is a circle with center z_0 and radius R. We define the circle C_r to have center z_0 and radius $r < R$ with negative orientation. We also define L_1 and L_2 to be line segments connecting C_R and C_r with positive orientation. This leads to defining C_R^1 and C_R^2, the left- and right-hand portions of C_R, and C_r^1 and C_r^2, defined in the same way. Since we stay away from z_0, we know that the function $\frac{f(z)}{z-z_0}$ is holomorphic in the regions bounded by $C_r^1 + L_2 - C_R^1 + L_1 = D$ and by $C_r^2 - L^2 - C_R^2 - L_2 = E$. By Cauchy's Integral Theorem, we see that

$$\int_D \frac{f(z)}{z-z_0}dz = \int_E \frac{f(z)}{z-z_0}dz = 0. \qquad (7.7)$$

If we add these two integrals together, we have

$$\int_{C_R - C_r} \frac{f(z)}{z-z_0}dz = 0,$$

or

$$\begin{aligned}
\int_{C_R} \frac{f(z)}{z-z_0}dz &= \int_{C_r} \frac{f(z)}{z-z_0}dz \\
& f(z_0)\int_{C_r} \frac{1}{z-z_0}dz + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz. \qquad (7.8)
\end{aligned}$$

Now as $r \to 0$, we know that $\frac{f(z)-f(z_0)}{z-z_0} \to f'(z_0)$ since f is holomorphic by assumption. Moreover, as $r \to 0$ we see that $\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} dz \to 0$ since $C_r \to z_0$. Finally, we compute

$$\int_{C_r} \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{re^{i\theta} i d\theta}{re^{i\theta}} = 2\pi i \tag{7.9}$$

where $z - z_0 = re^{i\theta}$ and $dz = re^{i\theta} i d\theta$.

So we can conclude that as $r \to 0$, $\int_{C_R} \frac{f(z)}{z-z_0} dz = f(z_0) \cdot 2\pi i$.

We now consider the radius of convergence of a Taylor series for a function f holomorphic inside a simply connected region G. We consider the Taylor series about $z_0 \in G$. This result is extremely important for determining the maximal radius of convergence for a solution to our ODE.

Theorem 13. *The radius of convergence of a power series for $f(z)$ at z_0 is equal to the distance from z_0 to the boundary of the domain of holomorphicity.*

Proof. We can assume without loss of generality that $z_0 = 0$. Let the distance from 0 to the boundary of holomorphicity be r. Then C_r is a circle with positive orientation, center 0, and radius r. Let z be in the interior of C_r and ζ on C_r. We will assume that f is continuous on C_r. If this is not the case, we would instead use $r - \epsilon$ rather than r.

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \int_{C_r} \frac{1}{\zeta} \cdot \frac{1}{1 - \frac{z}{\zeta}} f(\zeta) d\zeta = \frac{1}{2\pi i} \int_{C_r} \frac{1}{\zeta} \sum_{k=0}^{\infty} \left(\frac{z}{\zeta}\right)^k f(\zeta) d\zeta \\ &\text{since } \left|\frac{z}{\zeta}\right| < 1 \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta \right) z^k \end{aligned} \tag{7.10}$$

since the integral has no k argument.

Now if we differentiate our first expression above, we see

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta. \tag{7.11}$$

Evaluating at 0, we have

$$f^{(k)}(0) = \frac{k!}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta. \tag{7.12}$$

Inserting this in our work above, we have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k. \tag{7.13}$$

□

We did make use of the Cauchy-Kovalevski Theorem for linear second-order ordinary differential equations. We restate this theorem now and follow it with a proof, which very closely follows the proof given in [**?**]. The proof, as well as other parts of the thesis, use some tools of real analysis. For complete presentations of this subject, see [**?**] or [**?**].

Theorem 14. *(**Cauchy-Kovalevksi Theorem**) The initial value problem*

$$y''(x) + a(x)y'(x) + b(x)y(x) = g(x), y(x_0) = y_0, y'(x_0) = y_1$$

with $a(x), b(x), g(x)$ analytic near x_0, has a unique analytic solution $y(x)$ near x_0 with radius of convergence equal to at least the smallest radius of convergence of $a(x), b(x), g(x)$.

Proof. We take $a(x), b(x)$, and $g(x)$ to be analytic about $x_0 = 0$ (WLOG), and we let r_0 be the smallest radii of convergence for these functions. So we can write

$$a(x) = \sum_{k=0}^{\infty} a_k s^k \tag{7.14}$$

$$b(x) = \sum_{k=0}^{\infty} b_k s^k \tag{7.15}$$

$$g(x) = \sum_{k=0}^{\infty} g_k s^k \tag{7.16}$$

for any x such that $|x| < r_0$. Moreover, since these series converge for such x (they are analytic), for any $0 < r < r_0$ we can find $M_a, M_b, M_g > 0$ such that

$$|a_k| r^k \leq M_a \tag{7.17}$$

$$|b_k| r^k \leq M_b \tag{7.18}$$

$$|g_k| r^k \leq M_g \tag{7.19}$$

for each k.

Consider a (for now, formal) analytic solution to the original differential equation:

$$y(x) = \sum_{k=0}^{\infty} c_k x^k \tag{7.20}$$

with $c_0 = y_0$ and $c_1 = y_1$. Differentiating, we have

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1} = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k y'' \quad = \sum_{k=2}^{\infty} k(k-1) x^{k-2} = \sum_{k=0}^{\infty} (k+1)(k+2) c_{k+2} x^k. \tag{7.21}$$

Inserting these into our differential equation, we have

$$\sum_{k=0}^{\infty}(k+1)(k+2)c_{k+2}x^k + \Big(\sum_{k=0}^{\infty}a_k s^k\Big)\Big(\sum_{k=0}^{\infty}(k+1)c_{k+1}x^k\Big) + \Big(\sum_{k=0}^{\infty}b_k s^k\Big)\Big(\sum_{k=0}^{\infty}c_k x^k\Big) = \sum_{k=0}^{\infty}g_k s^k$$

$$\sum_{k=0}^{\infty}(k+1)(k+2)c_{k+2}x^k + \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k}a_{k-j}(j+1)c_{j+1}\Big)x^k + \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k}b_{k-j}c_j\Big)x^k = \sum_{k=0}^{\infty}g_k x^k. \tag{7.22}$$

Solving for the coefficient of the kth power of x, we have

$$(k+1)(k+2)c_{k+2} = g_k - \sum_{j=0}^{k}a_{k-j}(j+1)c_{j+1} - \sum_{j=0}^{k}b_{k-j}c_j. \tag{7.23}$$

Taking absolute values:

$$\begin{aligned}(k+2)(k+1)|c_{k+2}| &\le |g_k| + \sum_{j=0}^{k}|a_{k-j}|(j+1)|c_{j+1}| + \sum_{j=0}^{k}|b_{k-j}||c_j| \\ &\le \frac{M_g}{r^k} + \sum_{j=0}^{k}\frac{M_a}{r^{k-j}}(j+1)|c_{j+1}| + \sum_{j=0}^{k}\frac{M_b}{r^{k-j}}|c_j| \\ &\le \frac{M_g}{r^k} + \frac{\bar{M}}{r^k}\sum_{j=0}^{k}\Big((j+1)|c_{j+1}| + |c_j|\Big)r^j \\ &\le \frac{M_g}{r^k} + \frac{\bar{M}}{r^k}\sum_{j=0}^{k}\Big((j+1)|c_{j+1}| + |c_j|\Big)r^j + \bar{M}|c_{k+1}|r, \end{aligned} \tag{7.24}$$

where $\bar{M} = \max\{M_a, M_b\}$.

We define $C_0 \equiv |c_0|$ and $C_1 \equiv |c_1|$. Moreover, we define C_k for $k \ge 2$ using the recurrence relation above:

$$(k+2)(k+1)C_{k+2} \equiv \frac{M_g}{r^k} + \frac{\bar{M}}{r^k}\sum_{j=0}^{k}\Big((j+1)C_{j+1} + C_j\Big)r^j + \bar{M}C_{k+1}r \text{ for } k \ge 0. \tag{7.25}$$

In this way we have bounded the original $|c_k|$'s: $|c_k| \le C_k$ for all k. Thus if we can show that $\sum_{k=0}^{\infty}C_k x^k$ converges, then we know that $\sum_{k=0}^{\infty}c_k x^k$ converges as well.

We show that $\sum_{k=0}^{\infty}C_k x^k$ converges for all $|x| < r$ using the Ratio Test: if $\limsup_{k\to\infty}\frac{C_{k+1}}{C_k} \le \frac{1}{r}$, then the series converges. Shifting indices in (7.25), we have the two relations:

$$(k+1)kC_{k+1} = \frac{M_g}{r^{k-1}} + \frac{\bar{M}}{r^{k-1}}\sum_{j=0}^{k-1}\Big((j+1)C_{j+1} + C_j\Big)r^j + \bar{M}C_k r \text{ for } k \ge 1 \tag{7.26}$$

$$k(k-1)C_k = \frac{M_g}{r^{k-2}} + \frac{\bar{M}}{r^{k-2}}\sum_{j=0}^{k-2}\Big((j+1)C_{j+1} + C_j\Big)r^j + \bar{M}C_{k-1}r \text{ for } k \ge 2. \tag{7.27}$$

If we multiply the first relation by r, we then have

$$\begin{aligned}
(k+1)kC_{k+1}r &= \frac{M_g}{r^{k-2}} + \frac{\bar{M}}{r^{k-2}}\sum_{j=0}^{k-1}\Big((j+1)C_{j+1}+C_j\Big)r^j + \bar{M}C_k r^2 \\
&= \frac{M_g}{r^{k-2}} + \frac{\bar{M}}{r^{k-2}}\Big(\sum_{j=0}^{k-2}\Big((j+1)C_{j+1}+C_j\Big)r^j + (kC_k + C_{k-1})r^{k-1}\Big) + \bar{M}C_k r^2 \\
&= \frac{M_g}{r^{k-2}} + \frac{\bar{M}}{r^{k-2}}\sum_{j=0}^{k-2}\Big((j+1)C_{j+1}+C_j\Big)r^j + \bar{M}kC_k r + \bar{M}C_{k-1})r + \bar{M}C_k r^2 \\
&= k(k-1)C_k + \bar{M}kC_k r + \bar{M}C_k r^2, \qquad (7.28)
\end{aligned}$$

where the last equality follows from our second relation above. Associating like terms and then dividing, we then have

$$\begin{aligned}
r(k+1)kC_{k+1} &\leq \Big(k(k-1) + \bar{M}kr + \bar{M}r^2\Big)C_k \\
\frac{C_{k+1}}{C_k} &\leq \frac{(k-1)}{r(k-1)} + \bar{M}\frac{k+r}{(k+1)k}. \qquad (7.29)
\end{aligned}$$

Then in the limit we see that $\limsup_{k\to\infty}\frac{C_{k+1}}{C_k} \leq \frac{1}{r}$. Moreover, this holds for any $0 < r < r_0$. So we see that the formal power series we purported for y converges with a radius of convergence r_0. (More than that, we have legitimized the multiplication and differentiation of power series that we performed in the above proof.) □

7.2 Proofs for Theorems Presented in Introduction

Before we begin proving the theorems on which the mathematical viability of our modeling problem rests, we will introduce a piece of notation and prove a lemma that will be helpful. First, for any $x \in \Phi(x(0))$, we write

$$\bar{U}(x) \equiv \sum_{t=0}^{\infty}\beta^t U(x(t), x(t+1))$$

for the sum of lifetime rewards if the investor were to choose this plan. Recall that by Assumption 1, we know that $\bar{U}(x)$ exists and is finite.

Lemma 1. *Under Assumption 1, we have that for any initial state $x(0) \in X$ and any feasible plan $x \in \Phi(x(0))$,*

$$\bar{U}(x) = U(x(0), x(1)) + \beta\bar{U}(x(2), x(3), \cdots).$$

Proof of Lemma 1. As we mentioned above, Assumption 1 tells us that $\bar{U}(x)$ exists and

is finite. Rewriting, we have

$$\bar{U}(x) = \sum_{t=0}^{\infty} \beta^t U(x(t), x(t+1)) \tag{7.30}$$

$$= U(x(0), x(1)) + \beta \sum_{t=0}^{\infty} \beta^t U(x(t+1), x(t+2)) \tag{7.31}$$

$$= U(x(0), x(1)) + \beta \bar{U}(x(2), x(3), \cdots). \tag{7.32}$$

□

Our first goal is to prove Theorem 1, the Equivalence of Values, which states that any solution to (1.9) is a solution to (1.10), and vice versa. But what must occur for these solutions to coincide? Insight into this question comes from the definitions of these value functions as suprema. In (1.9), for any initial state $x(0) \in X$ a solution must satisfy

$$V^*(x(0)) = \sup_{x \in \Phi(x(0))} \bar{U}(x).$$

Invoking Assumption 1 again, we know that all $\bar{U}(x)$ are bounded, so that, by virtue of being the supremum,

$$V^*(x(0)) \geq \bar{U}(x) \text{ for all } x \in \Phi(x(0)). \tag{7.33}$$

Moreover, using the definition of supremum again, for any $\epsilon > 0$ there must exist $y \in \Phi(x(0))$ such that

$$V^*(x(0)) \leq \bar{U}(y) + \epsilon. \tag{7.34}$$

The case is very similar for (1.10). By virtue of being a supremum, we have that for any $x(0) \in X$,

$$V(x(0)) \geq U(x(0), y) + \beta V(y) \tag{7.35}$$

for all $y \in G(x(0))$. Again using the definition of supremum, for any $\epsilon > 0$ there must exist $y \in X$ such that

$$V(x(0)) \leq U(x(0), y) + \beta V(y) + \epsilon. \tag{7.36}$$

We will make use of the inequalities in the proof of Theorem 1.

Proof of Theorem 1. Since $\beta = 0$ produces a triviality, we deal only with the case where $\beta > 0$. Starting with initial state $x(0) \in X$ and $x(1) \in G(x(0))$, we know from (7.34) that, given $\epsilon > 0$, there exists $x'_\epsilon \in \Phi(x(1))$ with $\bar{U}(x_\epsilon) \geq V^*(x(1)) - \epsilon$. Using (7.33), we have also that for any $x \in \Phi(x(0))$, and particularly $x_\epsilon = (x(0), x'_\epsilon)$, $\bar{U}(x_\epsilon) \leq V^*(x(0))$. We now have

$$V^*(x(0)) \geq \bar{U}(x_\epsilon) = U(x(0), x(1)) + \beta \bar{U}(x'_\epsilon) \geq U(x(0), x(1)) + \beta V^*(x(1)) - \beta\epsilon,$$

where the equality follows from Lemma 1. ϵ is arbitrary, so we conclude

$$V^*(x(0)) \geq \bar{U}(x_\epsilon) = U(x(0), x(1)) + \beta \bar{U}(x'_\epsilon) \geq U(x(0), x(1)) + \beta V^*(x(1)).$$

So we see that V^* satisfies (7.35).

If we can also show that V^* satisfies (7.36), then we know that any V^* that satisfies (1.9) satisfies (1.10). We again take $\epsilon > 0$. From (7.34) we know that we can find a plan $x'_\epsilon = (x(0), x'_\epsilon(1), x'_\epsilon(s), \cdots) \in \Phi(x(0))$ with

$$\bar{U}(x'_\epsilon) \geq V^*(x(0)) - \epsilon.$$

Since $x'_\epsilon = (x(0), x'_\epsilon(1), x'_\epsilon(s), \cdots) \in \Phi(x(0))$, we know that $x''_\epsilon \equiv (x'_\epsilon(1), x'_\epsilon(s), \cdots) \in \Phi(x'_\epsilon(1))$. We have

$$V^*(x(0)) - \epsilon \leq \bar{U}(x'_\epsilon) = U(x(0), x'_\epsilon(1)) + \beta\bar{U}(x''_\epsilon) \leq U(x(0), x'_\epsilon(1)) + \beta V^*(x'_\epsilon(1)),$$

where the equality again follows from Lemma 1. So we have

$$V^*(x(0)) \leq U(x(0), x'_\epsilon(1)) + \beta V^*(x'_\epsilon(1)) + \epsilon,$$

and we see that V^* satisfies (7.36).

We now must show that any V that is a solution of (1.10) is a solution of (1.9). First we note that (7.35) implies that for any $x(1) \in \Phi(x(0))$

$$V(x(0)) \geq U(x(0), x(1)) + \beta V(x(1)).$$

We also see that

$$V(x(t)) \geq U(x(t), x(t+1)) + \beta V(x(t+1)).$$

Substituting, we have

$$V(x(0)) \leq \sum_{t=0}^{n} U(x(t), x(t+1)) + \beta^{n+1} V(x(n+1)) \text{ for all } n.$$

We let $x \equiv (x(0), x(1), x(2), \cdots)$ and have

$$\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t U(x(t), x(t+1)) = \bar{U}(x). \tag{7.37}$$

From Assumption 1, we know that $\bar{U}(x)$ is finite. Moreover,

$$\lim_{n\to\infty} \beta^{n+1} V(x(n+1)) = \lim_{n\to\infty} [\beta^{n+1} \lim_{m\to\infty} \sum_{t=n}^{m} \beta^t U(x(t), x(t+1))] = 0.$$

The first equality comes from the definition of $V(x(n+1))$; the second equality follows from (7.37). So we now have

$$V(x(0)) \leq \sum_{t=0}^{\infty} U(x(t), x(t+1)) = \bar{U}(x).$$

Note that our result follows for any $x \in \Phi(x(0))$. So we conclude that V satisfies (7.33). We need only show that V satisfies (7.34) to complete the proof.

Given $\epsilon > 0$, we know from (7.36) that for any $\epsilon' = \epsilon(1-\beta) > 0$ we can find an $x_{\epsilon'}(1) \in G(x(0))$ satisfying

$$V(x(0)) \leq U(x(0), x_{\epsilon'}(1)) + \beta V(x_{\epsilon'}(1)) + \epsilon'.$$

We now choose a plan $x_{\epsilon'} = (x_{\epsilon'}(0), x_{\epsilon'}(1), x_{\epsilon'}(2), \cdots)$ such that $x_{\epsilon'}(0) = x(0)$ and $x_{\epsilon'}(t) \in G(x(t-1))$. We thus have

$$V(x_{\epsilon'}(0)) \leq U(x_{\epsilon'}(t), x_{\epsilon'}(t+1)) + \beta V(x_{\epsilon'}(t+1)) + \epsilon'.$$

Substituting, we have

$$V(x(0)) \leq \sum_{t=0}^{n} U(x_{\epsilon'}(t), x_{\epsilon'}(t+1)) + \beta^{n+1} V(x_{\epsilon'}(n+1)) + \epsilon' + \epsilon'\beta + \cdots + \epsilon'\beta^n \leq \bar{U}(x_{\epsilon'}).$$

The last inequality follows from $\lim_{n\to\infty} \sum_{t=0}^{n} U(x_{\epsilon'}(t), x_{\epsilon'}(t+1)) = \bar{U}(x_\epsilon)$ and that $\epsilon = \frac{\epsilon'}{1-\beta} = \epsilon' \sum_{t=0}^{\infty} \beta^t$, so $\epsilon' = \frac{\epsilon}{\sum_{t=0}^{\infty} \beta^t}$. We conclude that V satisfies (7.34). □

Proof of Theorem 2. We begin by assuming that $x^* = (x(0), x^*(1), x^*(2), \cdots)$ is an optimizing plan for (1.9). In other words, x^* attains $V^*(x(0))$. From the theorem we see immediately that we are concerned with the relationship of one decision with those around it: $x^*(t)$ and its relationship to $x^*(t-1)$ and $x^*(t+1)$. We thus define the truncation of our optimizing plan.

$$x_t^* \equiv (x^*(t), x^*(t+1), \cdots).$$

Reviewing, the first part of the theorem states

$$V^*(x^*(t)) = U(x^*(t), x^*(t+1)) + \beta V^*(x^*(t+1)).$$

To prove this, we must show that the lifetime utility from x_t^* is optimizing given initial state $x^*(t)$. In other words, we must show that $\bar{U}(x_t^*) = V^*(x^*(t))$. We prove this inductively, and notice that our base case $t = 0$ is true by definition.

We assume that $\bar{U}(x_t^*) = V^*(x^*(t))$, with our goal being to show $\bar{U}(x_{t+1}^*) = V^*(x^*(t+1))$. We can rewrite our induction step:

$$\bar{U}(x_t^*) = V^*(x^*(t)) = U(x^*(t), x^*(t+1)) + \beta \bar{U}(x_{t+1}^*). \tag{7.38}$$

Now if we consider any feasible (and not necessarily optimizing) plan starting with $x^*(t+1)$, say $x_{t+1} \equiv (x^*(t+1), x(t+2), \cdots) \in \Phi(x^*(t+1))$. Then since this plan is feasible starting with $x^*(t+1)$, we have obviously that $x_t \equiv (x^*(t), x_{t+1}) \in \Phi(x^*(t))$. $V^*(x^*(t))$ is a supremum, we have that

$$V^*(x^*(t)) \geq \bar{U}(x_t) = U(X^*(t), x^*(t+1)) + \beta \bar{U}(x_{t+1}).$$

We see from the above inequality and (7.38) that

$$\bar{U}(x_{t+1}^*) \geq \bar{U}(x_{t+1}).$$

Since this follows for $x_{t+1} \in \Phi(x^*(t+1))$, we conclude that x^*_{t+1} achieves the supremum from initial state $x^*(t+1)$. So $\bar{U}(x^*_{t+1}) = V^*(x^*(t+1))$. Using this result, we see

$$V^*(x^*(t+1)) = \bar{U}(x^*_t) \tag{7.39}$$
$$= U(x^*(t), x^*(t+1)) + \beta\bar{U}(x^*_{t+1}) \tag{7.40}$$
$$= U(x^*(t), x^*(t+1)) + \beta V^*(x^*(t+1)), \tag{7.41}$$

where we use the result in the final equality. Thus we have proved the first part of the theorem.

The second part of the theorem claims that if a plan $x^* \in \Phi(x(0))$ satisfies $V^*(x^*(t)) = U(x^*(t), x^*(t+1)) + \beta V^*(x^*(t+1))$, then it also satisfies (1.9). So let's consider a feasible plan $x^* \in \Phi(x(0))$. We can then write

$$V^*(x(0)) = U(x^*(0), x^*(1)) + \beta V^*(x^*(1)) \tag{7.42}$$
$$= U(x^*(0), x^*(1)) + \beta(U(x^*(1), x^*(2)) + \beta V^*(x^*(2))) \tag{7.43}$$
$$= U(x^*(0), x^*(1)) + \beta U(x^*(1), x^*(2)) + \beta^2(U(x^*(2), x^*(3)) + \beta V^*(x^*(3))) \tag{7.44}$$
$$= \cdots \tag{7.45}$$
$$= \sum_{t=0}^{n} \beta^t U(x^*(t), x^*(t+1)) + \beta^{n+1} V^*(x^*(n+1)). \tag{7.46}$$

From Assumption 1 we know that $V^*(\cdot)$ is bounded, so we see

$$\bar{U}(x^*) = \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t U(x^*(t), x^*(t+1)) + \beta^{n+1} V^*(x^*(n+1)) \tag{7.47}$$
$$= \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t U(x^*(t), x^*(t+1)) \tag{7.48}$$
$$= V^*(x(0)). \tag{7.49}$$

This proves the second claim of Theorem 2. □

Proof of Theorem 4. Theorem 4 claims that unique V satisfying (1.10)is strictly concave. Let $\mathbf{C}(X)$ be the set of continuous functions over X, which is compact by Assumption 2. In fact, since X is compact, we have from basic topology that each $f \in \mathbf{C}'(X)$ is bounded. On this space we apply the sup norm $||f|| = \sup_{x\in X} |f(x)|$. It can be proved that this metric space with its metric defined by the norm above is complete. We then define $\mathbf{C}'(X) \subset \mathbf{C}(X)$ to be the set of bounded, continuous, weakly concave functions over X. Finally, we define $\mathbf{C}''(X) \subset \mathbf{C}'(X) \subset \mathbf{C}(X)$ to be the set of bounded, continuous, strictly concave functions over X. Note that weak concavity is a closed condition, but strict concavity is an open condition. Thus $\mathbf{C}'(X)$ is a closed subset whereas $\mathbf{C}''(X)$ is open. From Theorem 3, we have that the solution V to (1.10) is an element of $\mathbf{C}(X)$. We now define an operator on any $V \in \mathbf{C}(X)$:

$$TV(x) \equiv \max_{y\in G(x)} \{U(x,y) + \beta V(y)\}.$$

In the Proof of Theorem 3 we proved that T is a contraction, and thus the fixed point $TV = V$ is a solution to (1.10). We now introduce a very useful theorem.

Theorem 15. *Let $T : X \to X$ be a contraction mapping on a complete metric space (X, d). Let $x \in X$ satisfy $Tx = x$. Finally, let $X' \subset X$ is closed. If $T(X') \subset X'' \subset X'$, then $x \in X''$.*

With this theorem, if we can show that $T(\mathbf{C}'(X)) \subset \mathbf{C}''(X) \subset \mathbf{C}'(X)$, then we know that the solution and fixed point V is in $\mathbf{C}''(X)$. To prove this, let's consider a function $W \in \mathbf{C}'(X)$. Also, we consider two distinct points $x, x' \in X$ and define any point in their convex hull $x_\alpha \equiv= \alpha x + (1-\alpha)x'$, where $\alpha \in (0, 1)$. Then for the state vectors $x, x' \in X$ we choose $y \in G(x)$ and $y' \in G(x')$ so that y and y' satisfy the Bellman equation (1.10) for x and x', respectively. Then by virtue of their optimizing properties, we have

$$\begin{aligned} TW(x) &= U(x, y) + \beta W(y) \\ TW(x') &= U(x', y') + \beta W(y'). \end{aligned} \tag{7.50}$$

We now use Assumption 3: we have that G is convex, so we have that

$$y_\alpha \equiv \alpha y + (1-\alpha)y' \in G(x_\alpha).$$

Thus,

$$\begin{aligned} TW(x_\alpha) &\geq U(x_\alpha, y_\alpha) + \beta W(y_\alpha) && (7.51) \\ &> \alpha\Big(U(x, y) + \beta W(y)\Big) + (1-\alpha)\Big(U(x', y') + \beta W(y')\Big) && (7.52) \\ &= \alpha TW(x) + (1-\alpha)TW(x'). && (7.53) \end{aligned}$$

The above equalities and inequalities require some explanation. The weak inequality follows since y_α may not be maximizing for state x_α. The strict inequality follows from the assumption (Assumption 3) of strict concavity of U and concavity of W ($W \in \mathbf{C}'(X)$). Finally, the equality follows from (7.50). We conclude that $TW(x_\alpha) > \alpha TW(x) + (1-\alpha)TW(x')$, and thus $TW \in \mathbf{C}''(X)$. Since this follows for each $W \in \mathbf{C}'(X)$, Theorem 4 follows from Theorem 15. □

Proof of Theorem 5. We will use Theorem 15 again to prove Theorem 5. Here we alter our definitions for $\mathbf{C}''(X) \subset \mathbf{C}'(X) \subset \mathbf{C}(X)$. We now let $\mathbf{C}'(X)$ be the set of bounded, continuous, *nondecreasing* functions over X, and we let $\mathbf{C}''(X)$ be the set of bounded, continuous, *strictly increasing* functions over X. Again we observe that $\mathbf{C}'(X)$ is a closed subset while $\mathbf{C}''(X)$ is open. By Theorem 15, if we can show that $T(\mathbf{C}'(X)) \subset \mathbf{C}''(X)$, where T is as defined above, then we will have that the fixed point for T is in $\mathbf{C}''(X)$.

Consider $W \in \mathbf{C}'(X)$. From Assumption 4 we have that U is strictly increasing. Since $W \in\in \mathbf{C}'(X)$ and is this nondecreasing, we see that $\max_{y \in G(x)} \{U(x, y) + \beta W(y)\}$ is strictly increasing. But $TW(x) \equiv \max_{y \in G(x)} \{U(x, y) + \beta W(y)\}$, so we conclude that $TW \in \mathbf{C}''(X)$. This concludes the proof. □

For proofs of Theorems 3, 6, and 7 and Corollary 1, refer to Acemoglu [**?**].