Matching Through Position Auctions

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Abstract

This paper uses a mechanism design framework to characterize how an intermediary can design profit-maximizing matching markets when agents have private information about their quality. When match surpluses are supermodular, sufficient conditions are provided that ensure positive assortative matching is profit maximizing. Under these conditions, two-sided position auctions are characterized that can implement the optimal match and payments, but increasing, pure-strategy equilibria can fail to exist if agents only pay when they receive a partner. Restrictions on the matchmaker’s ability to price discriminate are shown to increase inefficiency and potentially shift the burden of price discrimination onto a single side.

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1 Introduction

Intermediaries and matchmakers play an important role in helping agents reveal information and coaxing markets towards equilibrium. However, the advantages of their position as an information broker gives them significant power to extract rents at the expense of social welfare. This paper clarifies how such agents engage in price discrimination when their clients hold private information about their quality as a partner, and explores the structure and limitations of bidding games that achieve profit maximization.

Consider a market that is split into two distinct sides, where each agent can produce surplus only by matching to a partner on the opposite side. If the match surplus is increasing in each agent’s quality and exhibits supermodularity, an intermediary could maximize profits and efficiency simply by ranking the agents on both sides, matching the highest-ranked agents together, the second-highest, and so on, and charging the matched agents up to their willingness to pay. This pattern of

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matching is called \textit{positive assortative matching}, and has long been recognized to be socially efficient in such an environment. If each agent privately knows his own quality, however, the matchmaker’s ability to price discriminate is limited by his ability to provide incentives for agents to reveal information honestly. The main results of the paper are necessary and sufficient conditions for a version of assortative matching to be profit-maximizing for the matchmaker, a characterization of bidding games that achieve the same results, and some results on how policies intended to reduce inefficiency due to price discrimination might lead to more ex ante exclusion.

As long as the virtual revenue accruing from the matches is supermodular, the matchmaker would find it profitable to match agents assortatively, but it turns out that further conditions are required to ensure incentive compatibility because of the interdependence of the match surplus on each party’s type. This arises because, in contrast to a one-sided, independent private values model, exclusion in a matching setting with interdependent values is characterized by a function of both agents’ types. Since reporting a higher type has the potential to improve a partner’s bargaining position against the matchmaker, a coordination problem can arise where agents find it profitable to misrepresent themselves to ensure their match is arranged.

Once sufficient conditions are characterized that avoid this problem, \textit{position auctions} are shown to implement the profit maximizing outcome. Broadly defined, position auctions are a class of indirect mechanism where agents submit a single bid to win one of a sequence of goods that are decreasing in value, the bids are ranked, and bidders receive the good associated with their rank. The payment can be defined in various ways, but two appealing choices are the \textit{winners-pay format}, where agents only make a payment to the matchmaker if they receive a partner, and the \textit{all-pay format}, where agents pay regardless of whether or not they receive a partner. Another alternative would be to charge agents who receive a partner the next highest bid, as in the generalized second price auction of Edelman et al. [7] or Gomes and Sweeney [9]. Since the allocation rule in the optimal direct revelation mechanism depends on the ranks of the agents’ reports, it turns out that position auctions can implement the profit-maximizing allocation. In the all-pay version of the game, the strategies are always increasing, but in the winners-pay version agents may prefer to settle for a lower-ranked partner to avoid making a higher payment, and non-monotonicities can occur. This threatens to greatly reduce the usefulness of many of the results since all-pay auctions
are infeasible in many environments. However, as the number of agents on both sides increases, the winners-pay bid functions are shown to be eventually increasing, so that winners-pay formats implement the profit-maximizing outcome in sufficiently thick markets.

The approach to indirect implementation builds on a common theme in many existing studies of matching: The analogy to market competition for partners. This goes back at least to the ideas of Spence [20] on costly signaling and the marriage market studied by Becker [2], and is developed in many subsequent papers, including the recent contributions of Bulow and Levin [4], who show that matching markets tend to lead to wage compression, and Hoppe, Moldovanu, and Sela [12], who show that markets that match assortatively on costly signals may be more inefficient than random matching. The contribution of the current paper is to explain how the competitive bidding analogy can be operationalized and exploited by an intermediary. Alternatives to assortative matching such as coarse matching — where agents are sorted into sets of ascending quality with increasing entry fees and randomly matched — have been explored in Damiano and Li [6], McAfee [13], and Hoppe, Moldovanu and Ozdenoren [11]. In particular, [6] assume the coarse matching institution as a constraint on the matchmaker’s ability to separate the market, and study when positive assortative matching is profitable in the limit as the number of matching sets grows large. The current paper begins instead with finding sufficient conditions for positive assortative matching to be profit-maximizing in small markets. When the sufficient conditions developed in the current paper are violated, however, coarse matching can turn out to be optimal for a profit-maximizing matchmaker. This provides a clearer explanation of how coarse matching can arise as an equilibrium phenomenon not because of practical or institutional constraints, but because it is profit-maximizing for the matchmaker to do so. Board [3] considers a one-sided matching model in which a monopolist arranges groups of agents together. That framework addresses a number of interesting questions about the motivations that face a profit-maximizing matchmaker in a similar matching environment, but has the added restriction that agents’ payoffs are multiplicatively separable in the agent’s type and a function that aggregates the quality of the rest of his group. Gomes [8] studies a framework similar to sponsored search, where an intermediary receives a signal affiliated with consumers’ preferences, and decides which firms to advertise, similar to how Google chooses sponsored search links.

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Matchmakers who engage in aggressive price discrimination and seemingly arbitrary exclusion run the risk of attracting attention from regulators. For example, Internet Service Providers have been under scrutiny by the FCC for the practice of service discrimination, where a subscriber’s access is restricted based on the volume of network traffic he generates. Advocates of “network neutrality” argue that this should be illegal, but the economic consequences of imposing policies that restrict the ISP’s ability to price discriminate are unclear. The current paper shows that restricting the matchmaker’s ability to price discriminate unambiguously increase ex ante exclusion, as well as raises the possibility that the matchmaker cuts the minimum bid or entry fees to zero on one side of the market and raises them on the other. This shows that regulation of matchmakers has the potential for unintended consequences that do not arise from a one-sided analysis, similar to many results in the two-sided markets literature (see Armstrong [1], for example). In the current paper, the details of the matchmaker’s price discrimination scheme can be expressed precisely, as well as the implications of regulation on exclusion. Similarly, Rayo [17] considers how a monopolist will restrict the variety of goods to profit from the signaling value that they provide to consumers in a model of conspicuous consumption. Here, the benefit arises directly from being matched to a better partner, but the papers share a similar theme of informational control.

Lastly, a number of papers in the mechanism design literature study similar markets, but in the vein of buyer-seller relationships or one-sided auctions. In particular, Myerson and Satterthwaite [16] study how a broker might arrange trade between a single buyer and seller in the presence of incomplete information. The current paper considers the case where there are many buyers and sellers, as well as externalities within the match. The implementation results also make a contribution to the literature on position auctions by showing how they might be profitably designed and implemented to maximize matchmaker profits.

2 Model Environment

There are two disjoint sets of agents, $I$ and $J$, with $K_I$ agents on the $I$ side and $K_J$ agents on the $J$ side. Let $K = \min\{K_I, K_J\}$, the largest number of matches that can be arranged in this market. Each $I$-side agent would like to match to one agent on the $J$ side, and each agent on the $J$ side
would like to match to one agent on the $I$ side. Since the two sides are symmetric up to the index $I$ or $J$ throughout the paper, details in definitions and results are generally given for the $I$ side, with the analogous statements holding mutatis mutandis for the $J$ side.

Each $I$-side agent $i$ has a privately known quality $q_i$ drawn from a probability density function $f_I(q)$ that is strictly positive on $[0, q\bar{I}]$ with cumulative distribution function $F_I(q)$, and likewise for the $J$ side. Let $q_I = (q_{I1}, q_{I2}, ..., q_{IK_I})$ and $q_I = (q_i, q_{I\backslash i})$. For any function $h(q_I, q_J)$, let $\mathbb{E}_i[h(q_i, q_{I\backslash i}, q_J)] = \mathbb{E}_{q_{I\backslash i}, q_J}[h(q_i, q_{I\backslash i}, q_J)|q_i]$; this is the expectation of $h(q_I, q_J)$ conditional on $i$’s information.

Agents $i$ and $j$ who are matched produce pairwise private surpluses $s_I(q_i, q_j)$ for $i$ and $s_J(q_j, q_i)$ for $j$. This can be thought of as truly private surplus, in the case of a marriage between two individuals who derive value from each others’ company, or as the equilibrium payoff from a complete information, non-cooperative game played after the matching game that satisfies appropriate restrictions. Surplus is increasing in both arguments and thrice differentiable.

Agents have quasi-linear preferences, so an agent $i$ paying $t$ to match to agent $j$ receives a payoff of

$$s_I(q_i, q_j) - t$$

Each agent’s outside option is to take a payoff of zero. Depending on the application, the payments $t$ might be positive or negative. For example, potential employees may pay to join a headhunter service while firms bid competitively, or a broker might pay firms for a block of shares and simultaneously sell the assets to investors to make a profit. There is a constant marginal cost of arranging matches $c$, independent of the realized qualities and reports.

For a vector $X = (x_1, x_2, ..., x_i, ..., x_K)$, let $\rho_X(x_i)$ be the rank of $x_i$ in $X$:

$$\rho_X(x_i) = |\{x_k \in X : x_k \geq x_i\}|$$

where $|A|$ is the number of elements in a set $A$. Let

$$w_{I,k}(q) = \frac{(K_I - 1)!}{(K_I - k)! (k - 1)!} \frac{F_I(q)^{K_I-k}(1 - F_I(q))^{k-1}}{F_I(q)^{K_I}(k - 1)!}$$
This is the probability of coming in rank $k$ out of $I$ draws from distribution $F_I$ with value $q$. Let the density of the $k$-th of $I$ order statistics be given by $f_{I,(k)}(q)$.

3 Profit-Maximizing Matching Rule and Sustaining Transfers

A direct revelation mechanism is a set of functions $\{m_{ij}(q_I,q_J), t_i(q_I,q_J), t_j(q_I,q_J)\}_{i,j}$ whose domains are the type spaces of the agents, $m_{ij}(q_I,q_J)$ is the probability that $i$ is matched to $j$, $t_i(q_I,q_J)$ is the amount paid by agent $i$, and $t_j(q_I,q_J)$ is the amount paid by agent $j$. A well-known consequence of the revelation principle is that the matchmaker can restrict attention to the class of incentive compatible direct revelation mechanisms without loss of generality.

The matchmaker seeks to maximize expected profits,

$$\max_{t_i,t_j,m_{ij}} \mathbb{E} \left[ \sum_{i=1}^{K_I} t_i(q_I,q_J) + \sum_{j=1}^{K_J} t_j(q_J,q_I) - \sum_{i=1}^{K_I} \sum_{j=1}^{K_J} m_{ij}(q_I,q_J)c \right]$$

subject to feasibility constraints that for all $i$ and $j$, $0 \leq \sum_{i=1}^{K_I} m_{ij}(q_I,q_J) \leq 1$, $0 \leq \sum_{j=1}^{K_J} m_{ij}(q_I,q_J) \leq 1$, $0 \leq m_{ij}(q_I,q_J) \leq 1$, and individual rationality and incentive compatibility constraints. The mechanism is incentive compatible if, for all agents $i$ on the $I$ side, for all $q'$ not equal to the agent’s true type $q_i$,

$$\mathbb{E}_i \left[ \sum_{j=1}^{K_J} m_{ij}(q_i,q_{I\setminus i},q_J)s_I(q_i,q_J) - t_i(q_i,q_{I\setminus i},q_J) \right] \geq \mathbb{E}_i \left[ \sum_{j=1}^{K_J} m_{ij}(q',q_{I\setminus i},q_J)s_I(q_i,q_J) - t_i(q',q_{I\setminus i},q_J) \right]$$

The mechanism is individually rational if, for all agents $i$ on the $I$ side with true quality $q_i$,

$$\mathbb{E}_i \left[ \sum_{j=1}^{K_J} (q_i,q_{I\setminus i},q_J)s_I(q_i,q_J) - t_i(q_i,q_{I\setminus i},q_J) \right] \geq 0$$

Similar constraints incentive compatibility and individual rationality constraints apply for the $J$-side agents.
Using the standard approach of deriving a participant’s indirect utility function in any incentive compatible mechanism and equating it with the direct utility function, the expected profits of the matchmaker can be expressed as

$$\max_{m_{ij}} \mathbb{E} \left[ \sum_{i=1}^{K_I} \sum_{j=1}^{K_J} m_{ij}(q_I, q_J)(\psi_I(q_i, q_j) + \psi_J(q_j, q_i) - c) \right]$$

where the virtual surplus is given by

$$\psi_I(q_i, q_j) = s_I(q_i, q_j) - \frac{1 - F_I(q_i)}{f_I(q_i)} \frac{\partial s_I(q_i, q_j)}{\partial q_i}$$

This can be interpreted as the marginal revenue accruing to the matchmaker from agent $i$ when he is matched to agent $j$.

Consider the following candidate matching function:

**Definition** Truncated Assortative Matching (TAM)

$$m^{TAM}_{ij}(q_I, q_J) = \begin{cases} 1 & \text{if } \rho_{q_I}(q_i) = \rho_{q_J}(q_j) = k \text{ and } \psi_I(q_i, q_j) + \psi_J(q_j, q_i) \geq c \\ 0 & \text{otherwise} \end{cases}$$

To warrant matching, the joint virtual surplus of two potential partners must be greater than the marginal cost of the match, or $\psi_I(q_i, q_j) + \psi_J(q_j, q_i) \geq c$. This divides the set $[0, \bar{q}_I] \times [0, \bar{q}_J]$ into a set of matches that are arranged by the matchmaker, and a set that are denied (See Figure 1). The reserve function $R_I(q)$ is defined as the minimum of $\bar{q}_J$ and the solution to

$$\psi_I(q_i, R_I(q_i)) + \psi_J(R_I(q_i), q_i) = c$$

This gives the lowest-quality partner that the matchmaker will allocate to an agent with type $q_i$, and plays a similar role to a reserve price in an auction. The worst-off type $q_{I_{\text{worst}}}$ is the lowest-quality agent on the $I$-side for whom

$$\psi_I(q_{I_{\text{worst}}}, \bar{q}_J) + \psi_J(\bar{q}_I, q_{I_{\text{worst}}}) \geq c$$
This is the lowest-quality agent on the I-side who does as well participating as dropping out of the market.

**Proposition 3.1** (i) Suppose the virtual surplus functions $\psi_I(q_i, q_j)$ and $\psi_J(q_j, q_i)$ are increasing in both arguments and supermodular. Then truncated assortative matching is profit-maximizing and the interim expected transfers equal

$$\mathbb{E}_i[TAM_i(q_i, q_{I\setminus i}, q_J)] = \sum_{k=1}^{K} w_{I,k}(q_i) \int_{R_I(q_i)}^{q_j} s_I(q_i, y) f_{J,(k)}(y) dy - \int_{q_j}^{q_i} \sum_{k=1}^{K} w_{I,k}(z) \int_{R_I(z)}^{q_j} \frac{\partial s_I(z, y)}{\partial q_i} f_{J,(k)}(y) dy dz$$

and similarly for the J side.

(ii) The following conditions guarantee that the virtual surplus functions satisfy the properties of part (i): $s_I(q_i, q_j)$ is increasing in both arguments, concave in $q_i$,

$$\frac{\partial^2 s_I(q_i, q_j)}{\partial q_i \partial q_j} \geq 0,$$  

$$\frac{\partial^3 s_I(q_i, q_j)}{\partial q_i^2 \partial q_j} \leq 0,$$  

$$\frac{f_I(q_i)}{1 - F_I(q_i)} \geq \frac{\partial}{\partial q_i} \log \left( \frac{\partial s_I(q_i, q_j)}{\partial q_j} \right),$$

the distribution function $F_I(q)$ is log-concave, and similar conditions hold mutatis mutandis for the J side.

The conditions in Eq. (1) and Eq. (2) of Proposition 4.1 (ii) are standard, but the hazard rate bound in Eq. (3) is a consequence of allowing for externalities within the match. When one party to a match raises his report, there are two effects: The match becomes more valuable, and his partner’s bargaining position improves against the matchmaker. In particular,

$$\frac{\partial \psi_I(q_i, q_j)}{\partial q_j} = \frac{\partial s_I(q_i, q_j)}{\partial q_j} - \frac{1 - F_I(q_i)}{f_I(q_i)} \frac{\partial^2 s_I(q_i, q_j)}{\partial q_i \partial q_j}$$

is ambiguous in sign. In such a case, agent $i$ might want to lower his report to offset the effect.
of a higher report by his partner to ensure a match occurs, leading to informational coordination problems within each match and potential violations of incentive compatibility. Note that the hazard rate bound is independent of supermodularity of the virtual surplus and instead serves to guarantee that higher types will be allowed to match to a larger set of partners, so that the reserve function $R_I(q)$ is decreasing.

4 Indirect Implementation with Position Auctions

This section shows that position auctions can achieve the profit-maximizing allocation of the optimal direct revelation mechanism characterized in the previous section.

Consider the all-pay (winners-pay) position auction:

1. The matchmaker announces a bid-reservation schedule, $\phi_{AP}^I(b_i) (\phi_{WP}^I(b_i))$, giving the lowest bid a $J$-side agent can make and still be eligible to match to an $I$-side agent making a bid of $b_i$.

2. Agents submit a sealed bid $b_i$ to the matchmaker.

3. The matchmaker opens all bids, and ranks the bids from greatest to least on each side, tentatively matching the highest-bidding agent on the $I$ side with the highest-bidding agent on the $J$ side, the second-highest bidder on the $I$ side with the second-highest bidder on the $J$ side, and so on.

4. The matchmaker checks that the bids satisfy the bid-reservation schedule for all tentative matches. If so, he arranges that the match, but blocks it otherwise. All agents pay their bids (All matched agents pay their bids).

Theorem 4.1 (Profit-Maximizing Implementation) The all-pay position auction has a Bayesian Nash equilibrium that implements the profit-maximizing matches and payments, with a symmetric bidding strategy given by

$$b_{AP}^I(q) = \sum_{k=1}^{K} w_{I,k}(q) \int_{R_I(q)}^{\bar{q}_I} s_I(q,y) f_{J,(k)}(y) dy - \int_{R_J(z)}^{\bar{q}_J} \sum_{k=1}^{K} w_{I,k}(z) \int_{R_I(z)}^{\bar{q}_I} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(k)}(y) dy dz$$
where the bid-reservation schedule is given by \( \phi^\text{AP}_I(b_i) = b^\text{AP}_I(R_I((b^\text{AP}_I)^{-1}(b_i))) \), and similarly for the \( J \) side. Let

\[
b^\text{WP}_I(q) = b^\text{AP}_I(q) \frac{1}{\sum_{k=1}^{K} w_{I,k}(q)(1 - F_{I,(k)}(R_I(q)))}
\]

If \( b^\text{WP}_I(q) \) is increasing in \( q \), then there is a symmetric Bayesian Nash equilibrium in the winner-pay position auction that implements the profit-maximizing matches and payments, with bid-reservation schedule \( \phi^\text{WP}_I(b_i) = b^\text{WP}_J(R_I((b^\text{WP}_I)^{-1}(b_i))) \), and similarly for the \( J \) side.

In the all-pay format, the agent constructs his optimal bid by computing the expected surplus under TAM, and then shading his bid to keep some of the surplus for himself. By trading off an explicitly higher payment with a better lottery over partners (the distributional effect) and a larger range of eligible partners (the support effect), he arrives at the necessary condition

\[
b^\text{WP}I(q) = \sum_{k=1}^{K} w_{I,k}'(q) \int_{R_I(q)}^{q_I} s_I(q,y)f_{I,(k)}(y)dy - \sum_{k=1}^{K} w_{l,k}(q)s_I(q,R_I(q))f_{I,(k)}(R_I(q))R_I'(q)
\]

Using standard tools from auction theory, solving this differential equation yields the all-pay strategies. To derive the winners-pay bid from the all-pay bid, he simply conditions on the event that he receive a partner at all. While certainly not a trivial series of thought experiments for a participant, the behavioral interpretations of the terms in the bid functions make it likely that the format could be used with suitably sophisticated participants.

The winners-pay implementation, however, may fail to implement the profit-maximizing outcome if \( b^\text{WP}_I(q) \) is non-increasing. In any symmetric, increasing equilibrium of the winners-pay format, the agent chooses his bid so that the distributional and support effects balance the ex-
expected marginal cost of increasing his bid:

\[
b^{WP}_I(q) = \left( \sum_{k=1}^{K} w_{Ik}(q)(1 - F_{J,(k)}(R_I(q))) \right) + \left( \sum_{k=1}^{K} w'_{Ik}(q)(1 - F_{J,(k)}(R_I(q))) \right) \left\{ \int_{R_I(q)}^{q_I} s_I(q, y) f_J(y) dy - b^{WP}_I(q) \right\} - R'_I(q) \sum_{k=1}^{K} w_{Ik}(q) f_{J,(k)}(R_I(q)) \left\{ s_I(q, R_I(q)) - b^{WP}_I(q) \right\} \]

Here however, the winners-pay feature of the game implies that the support effect can be negative (it is unambiguously negative at \( q_I \)). If the likelihood of achieving the \( k \)-th rank, \( w_{I,k}(q) \), is maximized at \( q_k \), then \( w'_{I,k}(q_k) \) will be zero, and the distributional effect will be relatively weak for that type. As a result, the negative support effect can dominate, and the bid function will decrease. This invalidates the proposed equilibrium strategies since they are derived under an assumption of monotonicity. Intuitively, this corresponds to a situation in which an agent who “knows” his rank focuses on winning a lower-quality partner at a lower cost, rather than increasing his explicit payment when he is unlikely to receive a better partner. This possibility can be confirmed through computational examples where non-monotonicities arise even in simple settings\(^1\).

However, as the market becomes “thicker” on both sides, agents have more precise assessments of both the probability they will match and the likely quality of that partner, but a less precise assessment of the exact rank of the partner they will get. Consequently, the non-monotonicity problems disappear, as confirmed in the next result.

**Proposition 4.2** Let \( b^{WP}_{I,n}(q) \) be the proposed equilibrium of the winners-pay format when \( K_I = K_I = n \). Then for all \( q \) and \( q' \) such that \( q \geq q' \), there exists an \( N \) such that \( n \geq N \) implies \( b^{WP}_{I,n}(q) \geq b^{WP}_{I,n}(q') \).

This suggests that intermediaries have incentives to pursue thick markets for transactions, since a winners-pay format has a number of obvious advantages. A similar phenomenon appears

\(^1\)For \( K_I = 5, K_J = 2, c = 0, s_I(q_i, q_j) = q_i^2 q_j^2 \) and \( s_J(q_j, q_i) = q_j^2 q_i^2 \) with types distributed uniformly on \([0, 1]\), the bid function is increasing for \( \alpha = .5, \beta = .2 \) but non-monotone for \( \alpha = .2 \) and \( \beta = .5 \).
in Gomes and Sweeney (2011), which considers Bayesian Nash equilibria of sealed-bid position auctions. There, the authors are interested in implementing the efficient outcome in a generalized second-price auction, but the nature of the problem is the same: If a higher bid fails to substantially improve the distribution of partners, it can be better to respond by cutting one’s bid in the hopes of getting a lower-ranked, but cheaper, partner.

5 Restricted Mechanisms

This section shows that inefficiency will generally increase when the matchmaker’s ability to price discriminate is restricted, and the burden of price discrimination may shift entirely onto one side of the market.

Consider restricting the matchmaker to pick a lowest eligible type on each side, \( q_I \) or \( q_J \), and arrange all possible matches among agents who report a higher type. In this restricted direct revelation mechanism, the matchmaker’s ability to price discriminate is reduced, but it is unclear whether the policy will improve or harm welfare. Under the assumptions of Proposition 3.1, truncated assortative matching will still be profit-maximizing within the class of mechanisms that satisfy this restriction, and truth-telling will be incentive compatible. The restricted match function takes the form:

\[
m_{ij}^R(q_I, q_J) = \begin{cases} 
1 & \text{if } \rho_{q_i}(q_i) = \rho_{q_j}(q_j) \text{ and } q_i \geq q_I, q_j \geq q_J \\
0 & \text{otherwise}
\end{cases}
\]

and the matchmaker’s expected profits can be shown to be

\[
\sum_{k=1}^{K} \int_{q_I}^{q} \int_{q_J}^{q} \{\psi_I(q_i, q_j) + \psi_J(q_j, q_i) - c_k\} f_{I,(k)}(q_j)f_{J,(k)}(q_i) dq_j dq_i
\]

To implement the same outcome as the restricted direct revelation mechanism, consider the restricted all-pay (winners-pay) position auction:

1. The matchmaker announces a minimum bid for each side, \((\underline{b}_I, \underline{b}_J)\), giving the lowest bids that agents can make and still be eligible to participate.

2. Agents submit a sealed bid \( b_i \) to the matchmaker.
3. The matchmaker opens all bids $b_i$ and $b_j$ and ranks them from greatest to least on each side, matching the highest-bidding agent on the $I$ side with the highest-bidding agent on the $J$ side, the second-highest bidder on the $I$ side with the second-highest bidder on the $J$ side, and so on, until the supply of agents on one side is exhausted.

4. All agents are charged their bids (Any matched agent is charged his bid).

One way to compare inefficiency between the profit-maximizing and restricted direct revelation mechanisms is ex ante exclusion: The size of the set of agents $[0, q_I]$ and $[0, q_J]$ who expect a weakly negative payoff from participation. The restricted direct revelation mechanism has an interior solution if exclusion occurs on both sides of the market, so that both $q_I > 0$ and $q_J > 0$, and a corner solution if $q_I > 0$ but $q_J = 0$, or vice versa.

**Theorem 5.1 (Restricted Implementation)** At any interior solution, ex ante exclusion is higher on both sides in the restricted direct revelation mechanism than the profit-maximizing direct revelation mechanism. At any corner solution in which $q_J > 0$, ex ante exclusion is higher on the $I$ side than in the profit-maximizing mechanism or at any interior solution. The restricted all-pay position auction has a symmetric equilibrium that implements the restricted direct revelation mechanism, with bidding strategies

$$b_I^{AP}(q) = \sum_{k=1}^{K} w_{I,k}(q) \int_{q_I}^{q_J} s_I(q,y) f_{J,(k)}(y) dy - \int_{q_I}^{q_i} w_{I,k}(z) \int_{q_I}^{q_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(k)}(y) dy dz$$

If

$$b_I^{WP}(q) = \frac{b_I^{AP}(q)}{\sum_{k=1}^{K} w_{I,k}(q)(1 - F_{J,(k)}(q_J))}$$

is increasing, then the restricted winners-pay position auction has a symmetric equilibrium that implements the restricted direct revelation mechanism where players use $b_I^{WP}(q)$.

Figure 1 shows how exclusion varies between an interior solution of the restricted mechanism and the profit-maximizing mechanism. The dashed line corresponds to the profit-maximizing exclusion scheme, while the solid line is an interior solution. In particular, the matchmaker trades off the loss of some unlikely matches in regions I and III with the addition of more likely — but unprofitable —
matches in region II. The dotted line corresponds to a corner solution where there is no exclusion on the $I$ side, but more ex ante exclusion on the $J$ side than at the profit-maximizing solution or any interior solution. The practical implication is that by restricting the matchmaker’s attempts to price discriminate, a regulator might cause a large shift in market welfare. While an interior solution that is similar to the original exclusion scheme is possible, the matchmaker might also respond by dropping an entry fee or minimum bid on one side altogether and engaging in aggressive price discrimination on the other, similar to results from the two-sided markets literature. Indeed, a policy intended to restrict price discrimination might harm one side of the market disproportionately or increase ex ante exclusion overall.

6 Conclusion

This paper utilizes standard tools from mechanism design to study the structure of two-sided matching markets and indirect implementation of the profit-maximizing outcome through bidding games. A set of sufficient conditions is provided such that a version of positive assortative matching also maximizes matchmaker profits, avoiding the large-market or coarse matching approximations of previous papers. With these conditions, winners-pay and all-pay position auctions are shown to implement the profit-maximizing outcome as long as markets are sufficiently thick. These formats
might be useful in practice as services like EBay or Google become increasingly sophisticated, and branch out beyond search or merely providing a platform for buyers and sellers to interact. Similarly, regulation can be studied as a constraint on the space of mechanisms from which the matchmaker can select. With this framework, the unintended consequences — namely, higher ex ante exclusion — are illustrated in a manner useful for further investigation of financial regulation or net neutrality policies.

Extensions of this framework can allow a deeper analysis of price discrimination and competition in two-sided markets. By imposing an increasing cost function \((c_1, c_2, ..., c_K)\) on matchmakers, there is a scope for multiple matchmakers to operate in the same market at the same time due to network congestion. Such a model might provide insight into why, for example, multiple real estate agencies operate in the same market at once when it appears that centralization of buyers and sellers would lead to better outcomes.

Extending the matching beyond one-to-one to many-to-one or many-to-many would provide a useful framework for understanding the details of, for example, financial markets. In the many-to-one case, the main challenge is that the optimal matching function will no longer be rank-order, and agents will need more flexibility in expressing their willingness to pay for various packages of partners. In this situation, the proper generalization of the position auctions studied here is not obvious, but the recent literature on combinatorial auctions may yield helpful insights into this interesting question. A similar analysis of one-sided matching with incomplete information (the “roommates problem”) presents more complicated challenges because the matchmaker may find it profitable to match agents non-assortatively, even in the case of supermodular virtual surpluses. In any given realization of the market, there is a potential trade-off at the bottom between arranging two low-quality matches or a single good one. In the two-sided case, the matchmaker always prefers a single good match, but in the one-sided case, the matchmaker can prefer to match the \(k\)-th best agent with the \(k + 3\)-rd, and \(k + 1\)-st with the \(k + 2\)-nd. This yields another illustration of a general principle that two-sided matching markets are better behaved than their one-sided or many-sided counterparts.

Future work on matching with incomplete information would be of great value in continuing the synthesis of matching theory with auction theory, as advocated in Hatfield and Milgrom [10].
and elsewhere. The deep connections between auction theory and matching theory suggest a more satisfying understanding of how markets work, and what the limitations and challenges might be.

References


Proof of Proposition 3.1

Proof (i) Corollary 1 of Milgrom and Segal [14] provide necessary and sufficient conditions for incentive compatibility: For all $i$,

$$U_i(q) = U_i(q_i) + \int_{q_i}^{q} \mathbb{E}_i \left[ \sum_{j=1}^{K_j} m_{ij}(z, q_{I \backslash i}, q_J) \frac{\partial s_i(z, q_j)}{\partial q_i} \right] dz$$

where $U_i(q_i) \geq 0$, and for all $q' \neq q$,

$$\int_{q}^{q'} \mathbb{E}_i \left[ \sum_{j} \left\{ m_{ij}(z, q_{I \backslash i}, q_J) - m_{ij}(q', q_{I \backslash i}, q_J) \right\} \frac{\partial s_i(z, q_j)}{\partial q_i} \right] dz \leq 0$$

and similarly for the $J$ side. A profit-maximizing matchmaker will set the worst-off type’s payoff to zero. By equating the direct and indirect utility function $t_i(q_i, q_{I \backslash i}, q_J)$ can be isolated, and evaluating at the truncated assortative matching function provides the expression for the interim transfers in the proposition. By taking expectations with respect to $q_i$ and substituting the resulting expressions for the transfers into the matchmaker’s expected profit function, the revenue from a match becomes the sum of the two agents’ virtual surplus. It is well known that if the match surplus is supermodular, the optimal match function is positively assortative.

(ii) To establish incentive compatibility, it must be shown that the monotonicity condition holds. This follows from standard mechanism design arguments as long as truncated assortative matching is used and $R_I(q)$ is decreasing, since then a higher type report yields an unambiguously better lottery over partner types.

Proof of Theorem 4.1 (Profit-Maximizing Implementation)
Proof Consider first the all-pay position auction. Suppose there is an increasing, symmetric equilibrium $b_I^{WP}(q)$ and $b_J^{WP}(q)$ for the $I$ side and $J$ side, respectively. Then some agent $i$ with type $q$ faces the maximization problem

$$
\max_b \sum_{k=1}^{K} w_{I,k}(b) \int_{\Phi_f^{AP}(b)}^{q_j} s_I(q,y)f_{J,(k)}(y)dy - b
$$

Since $b_j \geq \phi_f^{AP}(b)$ to match, this is equivalent to $b_j \geq b_I^{WP}(R_I[b_I^{WP}]^{-1}(b))$ or $q_j \geq R_I[b_I^{WP}]^{-1}(b)$. Substituting this into the objective yields

$$
\max_b \sum_{k=1}^{K} w_{I,k}(b) \int_{R_I[b_I^{WP}]^{-1}(b)}^{q_j} s_I(q,y)f_{J,(k)}(y)dy - b
$$

Differentiating yields a necessary condition for maximization

$$
b_I^{WP'}(q) = \sum_{k=1}^{K} w_{I,k}'(q) \int_{R_I(q)}^{q_j} s_I(q,y)f_{J,(k)}(y)dy - \sum_{k=1}^{K} w_{I,k}(q)s_I(q,R_I(q))f_{J,(k)}(R_I(q))R_I'(q) \tag{4}
$$

Noting that the right-hand side is equal to

$$
\frac{d}{dq} \left[ \sum_{k=1}^{K} w_{I,k}(q) \int_{R_I(q)}^{q_j} s_I(q,y)f_{J,(k)}(y)dy \right] - \sum_{k=1}^{K} w_{I,k}(q) \int_{R_I(q)}^{q_j} \frac{\partial s_I(q,y)}{\partial q_i} f_{J,(k)}(y)dy
$$

substitution and integration then yields a bid function that is determined up to an arbitrary constant, set so the payoff of the worst-off type is zero.

It must also be shown that $b_I^{AP}(q)$ is monotonically increasing, so that the quality rankings of the agents can be correctly inferred from their bids and the assumption of invertibility is satisfied. In the necessary condition (Eq. 4), the second term on the right-hand side is positive because $R_I'(q) < 0$. Note that by the binomial theorem, $\sum_{k=1}^{K_I} w_{I,k}(q) = 1$, so $\sum_{k=1}^{K_I} w_{I,k}'(q) = 0$. Also, the sequence $\int_{R_I(q)}^{q_j} s_I(q,y)f_{J,(k)}(y)dy$ is decreasing in $k$, since $F_{J,(k)}(y)$ first-order stochastically dominates $F_{J,(k+1)}(y)$ and $s_I(q_i,q_j)$ is increasing in $q_j$. Then for $k = 2, 3, ..., K_I - 1$,

$$
w_{I,k}'(q) = \frac{(K_I - 1)!}{(K_I - k)!(k - 1)!} f_I(q)F_I(q)^{K_I-k-1}(1 - F_I(q))^{k-2}$$

$$
[(K_I - k)(1 - F_I(q)) - (k - 1)F_I(q)]
$$
For \( k = 1 \) and \( k = K_I \), the slopes of \( w_{I,k}(q) \) are monotone increasing and decreasing, respectively. Then the sequence of maximizers of each of the \( w_{I,k}(q) \), \( \{\tilde{q}_k\}_{k=2}^{K_I-1} \), are given by

\[
\tilde{q}_k = F_I^{-1}\left(\frac{K_I - k}{K_I - 1}\right)
\]

This is a decreasing sequence in \( k \). For a given \( q \), find the interval \([\tilde{q}_k, \tilde{q}_{k+1}]\) and label the accompanying \( k \) as \( k^* \). This allows us to sign each of the \( w'_{I,k}(q) \) terms for each \( q \), since for all the terms \( w'_{I,k}(q) \) with \( k < k^* \), \( w'_{I,k}(q) \) is positive but for all terms \( w'_{I,k}(q) \) with \( k > k^* \), \( w'_{I,k}(q) \) is negative. Then

\[
b_{AP}^I(q) = \sum_{k=1}^{K_I} w_{I,k}(q) \int_{R_I(q)}^{q} s_I(q,y) f_{J,(k)}(y) dy - w_{I,k}(q) s_I(q,R_I(q)) f_{J,(k)}(R_I(q)) R_I'(q)
\]

\[
> \sum_{k=1}^{K_I} w_{I,k}(q) \int_{R_I(q)}^{q} s_I(q,y) f_{J,(k^*)}(y) dy
\]

\[
> \sum_{k=1}^{K_I} w_{I,k}(q) \int_{R_I(q)}^{q} s_I(q,y) f_{J,(k^*)}(y) dy
\]

\[
= \int_{R_I(q)}^{q} s_I(q,y) f_{J,(k^*)}(y) dy \left( \sum_{k=1}^{K_I} w_{I,k}(q) \right) = 0
\]

Where the second line follows since \( \int_{R_I(q)}^{q} s_I(q,y) f_{J,(k)}(y) dy \) is decreasing in \( k \); the third line follows by increasing the terms above \( k > k^* \) that appear with a negative \( w'_{I,k}(q) \), and decreasing the terms \( k < k^* \) that appear with a positive \( w'_{I,k}(q) \); and the last line follows from the binomial theorem. So the bid function is monotone increasing.

For the winners-pay format, a similar approach to the derivation of the all-pay strategies can be used. In any increasing symmetric equilibrium, the agents maximize

\[
\max_b \sum_{k=1}^{K} w_{I,k}(b_{WP}^{-1}(b)) \left[ \int_{R_I(b_{WP}^{-1}(b))}^{q_J} (s_I(q,y) - b) f_{J,(k)}(y) dy \right]
\]
Then a necessary condition for optimization is that

\[ b^W(q) \left\{ \sum_{k=1}^{K} w_{I,k}(q)(1 - F_{J,(k)}(R_I(q))) \right\} + b^W(q) \frac{d}{dq} \left\{ \sum_{k=1}^{K} w_{I,k}(q)(1 - F_{J,(k)}(R_I(q))) \right\} = \sum_{k=1}^{K} w'_I(q) \int_{R_I(q)}^{\bar{q}} s_I(q,y) f_{J,(k)}(y) dy - \sum_{k=1}^{K} w_{I,K}(q) s_I(q,R_I(q)) f_{J,(k)}(R_I(q)) R'_I(q) \]

Simplifying as in the analysis of the all-pay format yields

\[ \frac{d}{dq} \left[ b^W(q) \left\{ \sum_{k=1}^{K} w_{I,k}(q)(1 - F_{J,(k)}(R_I(q))) \right\} \right] = \frac{d}{dq} \left[ \sum_{k=1}^{K} w_{I,k}(q) \int_{R_I(q)}^{\bar{q}} s_I(q,y) f_{J,(k)}(y) dy \right] - \sum_{k=1}^{K} w_{I,K}(q) \int_{R_I(q)}^{\bar{q}} \frac{\partial s_I(q,y)}{\partial q_i} f_{J,(k)}(y) dy \]

This yields a candidate solution

\[ b^W_I(q) = \frac{\sum_{k=1}^{K} w_{I,k}(q) \int_{R_I(q)}^{\bar{q}} s_I(q,y) f_{J,(k)}(y) dy - \int_{R_I(z)}^{\bar{q}} w_{I,K}(q) \int_{R_I(z)}^{\bar{q}} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(k)}(y) dy dz}{\sum_{k=1}^{K} w_{I,k}(q)[1 - F_{J,(k)}(R_I(q))]} \]

Comparing the above strategy with the all-pay strategy, it is apparent that

\[ b^W_I(q) = \frac{b^AP_I(q)}{\sum_{k=1}^{K} w_{I,k}(q)[1 - F_{J,(k)}(R_I(k))] + b^AP_I(q)} \]

The numerator and denominator are both increasing in \( q \), so it is theoretically ambiguous whether the entire function is increasing or decreasing (see the discussion following Theorem 4.1). □

**Proof of Proposition 4.2**

**Proof** Throughout the proof, set \( b_I(q_I) = 0 \) to avoid taking with limits as \( q \to q_I \) where the probability that the worst-off type matches is zero. Note that this has no impact on the payoffs or behavior of any player, since the worst-off type receives a payoff of zero whether it participates or not. Let \( m_{i,j}^{n}(q_i, q_{I\setminus i}, q_I) \) be the truncated assortative matching function in the \( n \)-size market. Note that in the limit as \( n \) grows large, TAM converges to deterministic quantile matching, so that the limit matching function \( m_{i}^{\infty}(q) \) is defined by \( F_{I}(q) = F_J(m_{i}^{\infty}(q)) \), or \( m_{i}^{\infty}(q) = F_{J}^{-1}(F_{I}(q)) \). Additionally, letting \( \xi n = k \), \( w_{I,\xi}(q) \) converges to 1 if \( F_{I}(q) = \xi \) and zero otherwise, and \( f_{J,(\xi)}(q) \)
converges to 1 if \( F_j(q) = \xi \) and zero otherwise since the densities of order statistics whose parent distribution is log-concave are uni-modal. Then \( \mathbb{E}_{y,n}[y|n,q] \rightarrow F_j^{-1}(F_l(q)) \). In the limit market in which the sets \((0, \bar{q}_l)\) and \((0, \bar{q}_j)\) match assortatively, the bid function is \( b_{I,\infty}(q) = s_I(q, F_j^{-1}(F_l(q)) - \int_{\bar{q}_j}^q \frac{\partial s_I(z, F_j^{-1}(F_l(z))}{\partial q_l} dz \), which is strictly increasing. (The next part of the proof follows Bernstein’s proof of the Weierstrass Approximation theorem). The bid in the \( n \)-size market for an agent of type \( q \) is \( b_{I,n}^{WP}(q) = \mathbb{E}_y[s_I(q, y)| n, q] - \int_{\bar{q}_j}^q \mathbb{E}_y \left[ \frac{\partial s_I(z, y)}{\partial q_l} \right] n, z dz \), and |\( b_{I,n}^{WP}(q) - b_{I,\infty}(q) | \)

\[
\leq |\mathbb{E}_y[s_I(q, y)| n, q] - s_I(q, F_j^{-1}(F_l(q)))| + \int_{\bar{q}_j}^q \mathbb{E}_y \left[ \frac{\partial s_I(z, y)}{\partial q_l} \right] n, z dz - \int_{\bar{q}_j}^q \frac{\partial s_I(z, F_j^{-1}(F_l(z))}{\partial q_l} dz
\]

\[
\leq \mathbb{E}_y[|s_I(q, y) - s_I(q, F_j^{-1}(F_l(q)))| | n, q] + \int_{\bar{q}_j}^q \mathbb{E}_y \left[ \left( \frac{\partial s_I(z, y)}{\partial q_l} - \frac{\partial s_I(z, F_j^{-1}(F_l(z))}{\partial q_l} \right) \right] n, z dz
\]

(5)

Consider the first term in the last inequality (an analogous argument works for the second term).

Since \( s_I(q, y) \) is uniformly continuous in \( y \) on \((0, \bar{q}_j)\), for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( s_I(q, a) - s_I(q, b) < \epsilon/2 \) for all \( a, b \) in \((0, \bar{q}_j)\) with \(|a - b| < \delta \). Let \( A \) be the set of outcomes of \( y \) such that \(|y - F_j^{-1}(F_l(q))| < \delta \).

\[
\mathbb{E}_y |s_I(q, y) - s_I(q, F_j^{-1}(F_l(q))| = \int_A |s_I(q, y) - s_I(q, F_j^{-1}(F_l(q))| dy
\]

\[
+ \int_{A^c} |s_I(q, y) - s_I(q, F_j^{-1}(F_l(q))| dy
\]

\[
\leq \frac{\epsilon}{2} + 2 \sup_y |s_I(q, y)| Pr(A^c)
\]

By Chebyshev’s inequality, \( Pr(A^c) \leq \mathbb{E}_y[(y - F_j^{-1}(F_l(q)))^2| n, q]/\delta^2 \). Since \( \mathbb{E}_y[(y - F_j^{-1}(F_l(q)))^2| n, q] \rightarrow 0 \) in \( n \) because matching becomes deterministic, the last term can be made as small as desired by increasing \( n \). Applying this argument to both terms in Eq. 2 shows that \( b_{I,n}^{WP}(q) \) converges uniformly to \( b_{I,\infty}(q) \) on \((\bar{q}_l, \bar{q}_l)\)

Now suppose, by way of contradiction, that \( b_{I,n}^{WP}(q) \) is non-monotone for all \( n \). Pick any two points \( q > q' \), and let \( \epsilon > 0 \) satisfy \( \epsilon < b_{I,\infty}(q) - b_{I,\infty}(q') \). Let \( H \) be large enough that \( n \geq H \) implies that \(|b_{I,n}^{WP}(q) - b_{I,\infty}(q)| < \epsilon/2 \) and \(|b_{I,n}^{WP}(q') - b_{I,\infty}(q')| < \epsilon/2 \). Since \(|x| \leq c \) implies \(-c \leq x \leq c \), the two inequalities can be re-arranged to yield \( b_{I,\infty}(q') - b_{I,\infty}(q) + \epsilon > b_{I,n}^{WP}(q') - b_{I,n}^{WP}(q) \). The
left-hand side is unambiguously negative, but the right-hand side is positive if \( b_{I,n}^{WP} \) increases from \( q \) to \( q' \), leading to a contradiction. Therefore, since \( \varepsilon \) was an arbitrary positive number less than 

\[ 2(b_{I,\infty}(q) - b_{I,\infty}(q')) \]

and \( q \) and \( q' \) were arbitrary points in \((q_I, q_I']\), there must be some \( H \) at which 

\[ b_{I,n}^{WP}(q) > b_{I,n}^{WP}(q'). \]

**Proof of Theorem 5.1 (Restricted Implementation)**

**Proof** The matchmaker’s profits are

\[
\pi(q, q) = \sum_{k=1}^{K} \int_{q_I}^{q_J} \int_{q_I}^{q_J} \{ \psi_I(q, q_j) + \psi_J(q_j, q_i) - c_k \} f_{I,(k)}(q_i) f_{J,(k)}(q_j) dq_j dq_i
\]

which he maximizes subject to \( 0 \leq q_i \leq q_i \) and \( 0 \leq q_j \leq q_J \). As long as there is some set of matches that are denied in the profit-maximizing direct revelation mechanism, there are two relevant scenarios: An interior maximum with \( q_i \) and \( q_j \) both strictly positive, or a corner solution in which \( q_j \) is set to zero and \( q_i \) is strictly positive, or vice versa.

Maximizing over \( q_i \) (and likewise for \( q_J \)) yields a first-order necessary condition for maximization at any interior solution:

\[
0 = -\sum_{k=1}^{K} \int_{q_i}^{q_i} \{ \psi_I(q_i, q_j) + \psi_J(q_j, q_i) - c \} f_{I,(k)}(q_i) f_{J,(k)}(q_j) dq_j
\]

and a similar condition for \( q_J \). The sign of the derivative of \( f_{I,(k)}(q) \) is undetermined, so the objective function cannot be verified to be globally concave.

To compare exclusion at any interior solution to the profit-maximizing level of exclusion, perform an integration by parts to get

\[
\psi_I(q_i, q_j) + \psi_J(q_j, q_i) - c = \sum_{k=1}^{K} \int_{q_i}^{q_i} \left\{ \frac{\partial \psi_I(q_i, q_j)}{\partial q_j} + \frac{\partial \psi_J(q_j, q_i)}{\partial q_j} \right\} (F_{I,(k)}(q_j) - (F_{J,(k)}(q_j))) f_{I,(k)}(q_i) f_{J,(k)}(q_j) dq_j
\]

\[
= \sum_{k=1}^{K} f_{I,(k)}(q_i)(1 - F_{J,(k)}(q_j))
\]

The optimal level of ex ante exclusion in the profit-maximizing mechanism is achieved by setting the left-hand side of the equation to zero. Since the left-hand side is increasing in \( q_i \) and the
right-hand side is strictly positive, exclusion here is higher. To compare exclusion at an interior solution with \( q_I^i > 0 \) to the profit-maximizing reserve function, subtract the equation \( \psi(q_I^i, R_I(q_I^i)) + \psi_I(R_I(q_I^i), q_I^i) - c = 0 \) from the first-order condition for an interior \( q_I^i \),

\[
0 = \sum_{k=1}^{K} \int_{q_I^j}^{q_I^j} \left\{ \psi_I(q_I^i, y) + \psi_I(y, q_I^i) - \psi_I(q_I^i, R_I(q_I^i)) - \psi_I(R_I(q_I^i), q_I^i) \right\} f_{I,(k)}(y)f_{I,(k)}(q_I^i)dy
\]

So that for equality to hold, the term in braces must be negative for some values of \( y \), implying that \( q_I^i \leq R_I(q_I^i) \). To compare exclusion at a corner solution to an interior solution, compare the first-order necessary condition for a corner solution where \( q_C^c > 0 \) but \( q_J^c = 0 \) to the first-order necessary condition for an interior solution: The only difference is that the corner solution includes more negative terms corresponding to the \( J \)-types in the interval \([0, q_I^i]\). Therefore, exclusion is higher on the \( I \) side in a corner solution than in any interior solution. This provides an unambiguous ordering of ex ante exclusion.

The payments in the restricted mechanism can be computed similarly to the optimal mechanism to get

\[
\mathbb{E}_i[t_i(q_i, q_{I-i}, q_J)] = \sum_{k=1}^{K} w_{I,k}(q_i) \int_{q_I^j}^{q_J} s_I(q_i, y)f_{I,(k)}(y)dy - \int_{q_I^j}^{q_J} w_{I,k}(z) \int_{q_I^j}^{q_J} \frac{\partial s_I(z, y)}{\partial q_i} f_{I,(k)}(y)dydz
\]

Using similar arguments to Proposition 3.1 shows the monotonicity condition fails to bind for the restricted direct mechanism, so it is incentive compatible. The bid functions in a symmetric, increasing equilibrium can be derived by similar arguments to Theorem 4.1.