Games Without Pure Strategy Nash Equilibria

It’s pretty easy to write down games without PSNE’s. For example,

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But it seems like the “right answer” is that players should use each of their pure strategies half the time. How do we make sense of this idea?

Probability

A probability model \((E, pr)\) is

- A set of events, \(E = \{e_1, e_2, ..., e_K\}\)
- A probability distribution over events, \(pr(e_k)\), giving the probability of each event \(e_k\), where (i) \(0 \leq pr(e_k) \leq 1\), (ii) the probability that one of the \(K\) events occurs is 1,
  \[pr(e_1 \cup e_2 \cup ... \cup e_K) = 1\]
  and (iii) for any subsets of events \(E_1 \subseteq E\) and \(E_2 \subseteq E\) that share nothing in common, so that \(E_1 \cap E_2\) is empty, we have
  \[pr(E_1 \cup E_2) = pr(E_1) + pr(E_2)\]

Example: Rolling a Die

Suppose we roll a six-sided die, which is “fair” in the sense that each of the possible sides is equally likely to come up.

- Events: 
  - \(\bullet, \bullet\bullet, \bullet\bullet\bullet, \bullet\bullet\bullet\bullet, \bullet\bullet\bullet\bullet\bullet, \bullet\bullet\bullet\bullet\bullet\bullet\)
  - \(pr(\bullet) = pr(\bullet\bullet) = pr(\bullet\bullet\bullet) = pr(\bullet\bullet\bullet\bullet) = pr(\bullet\bullet\bullet\bullet\bullet) = \frac{1}{6}\)

Since

\[pr(\bullet) + pr(\bullet\bullet) + pr(\bullet\bullet\bullet) + pr(\bullet\bullet\bullet\bullet) + pr(\bullet\bullet\bullet\bullet\bullet) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1\]

and \(0 \leq pr(e) \leq 1\) for each event, this is a valid probability model.

Example: Two Unfair Coins

Suppose we flip two coins, so we can potentially see \(HH, HT, TH, TT\). The first coin comes up heads \(p\) percent of the time, and the second comes up heads \(q\) percent of the time.

- Events: \(HH, HT, TH, TT\)
  - Probability Distribution: \(pr(HH) = pq, pr(HT) = p(1-q), pr(TH) = (1-p)q,\) and \(pr(TT) = (1-p)(1-q)\).

Since

\[pr(HH) + pr(HT) + pr(TH) + pr(TT) = pq + p - pq + q - pq + 1 - p - q + pq = 1\]

and all for all events \(xy\), \(0 \leq pr(xy) \leq 1\), this is a valid probability model.
Expected Utility

In situations with risk, how do we measure agents’ payoffs?

**Definition 1.** Let the set of events be $E = \{e_1, e_2, ..., e_K\}$ and the probability distribution be $pr(e_k)$. A *utility function* associates a payoff with each event $u(e)$. The agent’s *expected utility* is

$$\mathbb{E}[u] = pr(e_1)u(e_1) + pr(e_2)u(e_2) + \ldots + pr(e_K)u(e_K)$$

So we weight the payoff from each outcome ($u(e_k)$) by the likelihood that outcome occurs ($p(e_k)$), and sum over all the events. If all the events were equally likely, for example, we’d get

$$\mathbb{E}[u] = \frac{1}{K}(u(e_1) + u(e_2) + \ldots + u(e_K)) = \frac{1}{K}\sum_k u(e_k)$$

so this is the “average payoff” that the agent gets.

**Example: Risky Investment**

Suppose an agent can invest in a safe asset $s$ giving a return 1 per dollar invested, or a risky asset $a$ giving a return $g > 1$ (a gain) with probability $p$ and a return $l < 1$ (a loss) with probability $1 - p$. The agent has preferences $u(w) = \log(w)$ over final wealth, and total wealth $W$. Then his budget constraint is $W = a + s$, and his preferences are

$$\mathbb{E}[u(w)] = pu(s + ga) + (1 - p)u(s + la)$$

How much should the agent invest in the risky asset, $a$?

The maximization problem is

$$\max_{a,s} hu(s + ga) + (1 - p)u(s + la)$$

subject to $w = a + s$. Substituting the constraint into the objective yields

$$\max_x pu(w - a + ga) + (1 - p)u(w - a + la)$$

Maximizing yields

$$pu'(w + (g - 1)a)g - 1) = (1 - p)u'(w - (1 - l)a)(1 - l)$$

So if we have $u(x) = \log(x)$,

$$\frac{p(g - 1)}{w + (g - 1)a} = \frac{(1 - p)(1 - l)}{w - (1 - l)a}$$

Solving yields

$$a^* = \frac{p(g - 1) - (1 - p)(1 - l)}{(g - 1)(1 - l)}w$$

This honestly isn’t too far from what many quantitative financial firms do to compute optimal portfolios: They estimate $p$, $g$ and $R = l$ from data for a situation with many risky assets $a_k$, and instead maximize an objective function that looks like the mean minus the variance of the portfolio.
Strategies in Rock-Paper-Scissors

Let’s think about a “random strategy” in Rock-Paper-Scissors. Then we need

- Outcomes: R, P, S
- Probability distribution: \( \sigma_R + \sigma_P + \sigma_S = 1 \), with \( 0 \leq \sigma_R, \sigma_P, \sigma_S \leq 1 \)

This is what we call a “mixed strategy” in game theory, since the player is using a little bit of many of his pure strategies.

So we can think about a random strategy as a particular kind of probability distribution.

Mixed Strategies

Definition 2. Suppose a player has \( k = 1, 2, \ldots, K \) pure strategies. A mixed strategy for player \( i \) is a probability distribution over pure strategies, \( \sigma_i = (\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{iK}) \) with \( 0 \leq \sigma_{ik} \leq 1 \) and \( \sigma_{i1} + \sigma_{i2} + \ldots + \sigma_{iK} = 1 \).

The keys are (i) the sum of all the weights \( \sigma_{ik} \) on each of the pure strategies is one and (ii) each weight \( 0 \leq \sigma_{ik} \leq 1 \).

Expected Utility in Games

Let’s think about expected utility in Matching Pennies:

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Then a mixed strategy for the row player is \( \sigma_r = (\sigma_{rh}, \sigma_{rt}) \) and for the column player \( \sigma_c = (\sigma_{ch}, \sigma_{ct}) \). Then the row player’s expected utility is

\[
E[u_r(\sigma_r, \sigma_c)] = \sigma_{ch}\sigma_{rh}(1) + \sigma_{rt}\sigma_{ch}(-1) + \sigma_{rh}\sigma_{ct}(-1) + \sigma_{rt}\sigma_{ct}(1)
\]

How do we get this, in more detail? The strategic form of the game is

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So each strategy profile (and each box) has probability \( pr(s_r, s_c) \) of occurring. What’s the probability of each profile? Well, the probability the row player uses H and the column player uses H is just \( \sigma_{rh}\sigma_{ch} \), and the row player’s payoff is 1; the probability the row player uses H and the column player uses T is just \( \sigma_{rh}\sigma_{ct} \), and the row player’s payoff is –1; the probability the row player uses T and the column
player uses $H$ is $\sigma_r \sigma_{ch}$, and the row player’s payoff is $-1$; and the probability the row player uses $T$ and the column player uses $T$ is $\sigma_r \sigma_{ct}$, and the row player’s payoff is $1$. When we sum all the payoffs weighted by the probability they occur, we get

$$E[u_r(\sigma)] = pr(h, h)u_r(h, h) + pr(h, t)u_r(h, t) + pr(t, h)u_r(t, h) + pr(t, t)u_r(t, t)$$

or

$$\sigma_{rh}\sigma_{ch}(1) + \sigma_{rh}\sigma_{ct}(-1) + \sigma_{rt}\sigma_{ch}(-1) + \sigma_{rt}\sigma_{ct}(1)$$

Which is the row player’s expected utility.

### General Expected Utility in Games

Suppose we’ve got two players, $r$ and $c$. Then row’s expected utility is

$$E[u_r(\sigma_r, \sigma_c)] = \sum_{\text{All pure row strategies } k} \sum_{\text{All pure column strategies } j \sigma_{rk}\sigma_{cj}u_r(s_{rk}, s_{cj})}$$

In general, we just add more summations and probability terms as we add more players and strategies.

### Games with Mixed Strategies

A simultaneous-move game of complete information is

- A set of players $i = 1, 2, ..., N$
- A set of pure strategies for each player, and the corresponding mixed strategies
- A set of expected utility functions for each player,

$$\mathbb{E}[u_i(\sigma_i, \sigma_{-i})]$$

### Mixed-Strategy Nash Equilibrium

**Definition 3.** A set of mixed strategies $\sigma^* = (\sigma_1^*, \sigma_2^*, ..., \sigma_n^*)$ is a Nash equilibrium if, for every player $i$ and any other mixed strategy $\sigma_i'$ that $i$ could choose,

$$\mathbb{E}[u_i(\sigma_i^*, \sigma_{-i}^*)] \geq \mathbb{E}[u_i(\sigma_i', \sigma_{-i}^*)]$$

Note that a pure strategy is a special case of a mixed strategy, so the definition generalizes our earlier one.
Mixed-Strategy Nash Equilibrium for Matching Pennies

A set of strategies \( \sigma^*_r \) and \( \sigma^*_c \) is a mixed strategy equilibrium of matching pennies if, for any \( \sigma'_r = \sigma'_{rh} + \sigma'_{rt} \) for the row player,

\[
E[u_r(\sigma^*_r, \sigma^*_c)] = \sigma^*_r \sigma'^*_{rh}(1) + \sigma^*_r \sigma'^*_{ch}(1) + \sigma^*_r \sigma'^*_{ct}(1) \geq \sigma^*_r \sigma'^*_{rh}(1) + \sigma^*_r \sigma'^*_{ch}(1) + \sigma^*_r \sigma'^*_{ct}(1) = E[u_r(\sigma'_r, \sigma^*_c)]
\]

and for any \( \sigma'_c = \sigma'_{ch} + \sigma'_{ct} \) for the column player,

\[
E[u_c(\sigma^*_c, \sigma^*_r)] = \sigma^*_c \sigma'^*_{rh}(-1) + \sigma^*_c \sigma'^*_{ch}(1) + \sigma^*_c \sigma'^*_{ct}(1) \geq \sigma^*_c \sigma'^*_{rh}(-1) + \sigma^*_c \sigma'^*_{ch}(1) + \sigma^*_c \sigma'^*_{ct}(1) = E[u_c(\sigma'_c, \sigma^*_r)]
\]

This is kind of a mess though. How do we solve for the equilibrium strategies?

**How do we solve for mixed-strategy Nash equilibria?**

There’s a systematic method to solving these games:

- **Step 1:** Do iterated deletion of weakly dominated strategies
- **Step 2:** Choose the mix such that each player is making his opponents indifferent over their pure strategies

Why?

- **Step 1:** If you are putting any weight on a weakly dominated strategy, you can move that weight to the strategy dominating it and improve your payoff
- **Step 2:** Suppose your opponent strictly preferred one of his strategies to all the others. Then he should just use that strategy for sure, right? So your opponent will only play randomly if you make him indifferent, and you will play randomly only if he makes you indifferent.

**Step 1 : IDWDS**

Suppose an agent is placing some weight on a strategy \( k \) that is weakly dominated by a strategy \( l \).

That means that \( u_i(s_{ik}, s_{-i}) \geq u_i(s_{il}, s_{-i}) \), for any \( s_{-i} \) that might occur. But then the expected payoff given the opponents’ mixed strategies is

\[
\sum_{s_{-i}} p_r(s_{-i}) u_i(s_{ik}, s_{-i}) \geq \sum_{s_{-i}} p_r(s_{-i}) u_i(s_{il}, s_{-i})
\]

So if the agent is placing \( \sigma_{ik} \) weight on strategy \( k \), and \( \sigma_{il} \) weight on strategy \( l \), shifting it all to strategy \( k \) gives a higher payoff:

\[
(\sigma_{ik} + \sigma_{il}) \left( \sum_{s_{-i}} p_r(s_{-i}) u_i(s_{ik}, s_{-i}) \right) \geq \sigma_{ik} \left( \sum_{s_{-i}} p_r(s_{-i}) u_i(s_{ik}, s_{-i}) \right) + \sigma_{il} \left( \sum_{s_{-i}} p_r(s_{-i}) u_i(s_{il}, s_{-i}) \right)
\]
So weakly dominated strategies should be eliminated from consideration.

**Step 2: Make Your Opponents Indifferent Between Their Pure Strategies**

Note that we can factor a player’s strategy, as follows:

\[
E[u_r(\sigma^*)] = \sigma^*_{ch} \sigma^*_{rh}(1) + \sigma^*_{rt} \sigma^*_{ch}(-1) + \sigma^*_{ch} \sigma^*_{ct}(1)
\]

Notice that the terms with underbraces are the expected utility the row player gets from *just playing heads* and *just playing tails*, respectively, where no randomization is involved. Also note that if one is strictly bigger than the other, the row player should just play the strategy with the better expected return.

→ A player is only willing to play a mixed strategy if she is indifferent between the expected payoffs coming from her pure strategies.

The argument on the above slide can be generalized easily (the general case is done in the notes from class if you don’t believe me). The idea is that you can think of each pure strategy as a kind of lottery, giving a return of \(\sum_{s_{-i}} p_r(s_{-i}) u_i(s_{ik}, s_{-i})\) for strategy \(s_{ik}\). If one of your pure strategies gives a strictly higher return, it must weakly dominate the others, and you should put all the weight on that strategy. Consequently, players must be indifferent between their pure strategies if they are willing to use mixed strategies, and any mixed strategy equilibrium will involve this feature: All players are indifferent between their pure strategies, and choose their own strategy to make their opponents indifferent.

**Mixed Strategy Equilibrium in Battle of the Sexes**

Recall

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Does this game have a mixed-strategy Nash equilibrium? Then we can write player \(A\)'s expected utility as

\[
E[u_a(\sigma_a, \sigma_b)] = \sigma_{au} \sigma_{bl} + \sigma_{au} \sigma_{br} + \sigma_{ad} \sigma_{bl} + \sigma_{ad} \sigma_{br}^2
\]

and \(b\)'s expected utility as

\[
E[u_b(\sigma_a, \sigma_b)] = \sigma_{au} \sigma_{bl} + \sigma_{au} \sigma_{br} + \sigma_{ad} \sigma_{bl} + \sigma_{ad} \sigma_{br}
\]
But then player a’s expected utility can be rewritten as

\[ E[u_a(\sigma_a, \sigma_b)] = \sigma_{au}(\sigma_{bl}(1) + \sigma_{br}(0)) + \sigma_{ad}(\sigma_{bl}(0) + \sigma_{br}(2)) \]

So the return to playing \( u \) is just \( \sigma_{bl}(1) + \sigma_{br}(0) \), and the return to playing \( d \) is just \( \sigma_{bl}(0) + \sigma_{br}(2) \). If one of these is strictly larger than the other, then, player a should shift all of the probability weight to that strategy. For example, if \( \sigma_{bl} = \frac{1}{3} \) and \( \sigma_{br} = \frac{2}{3} \), we have \( \frac{1}{3} + \frac{2}{3}(0) = \frac{1}{3} \) for strategy \( u \) and \( \frac{1}{3}(0) + \frac{2}{3}(2) = \frac{4}{3} \) for strategy \( d \). In this case, strategy \( d \) is the clear winner, and player a has no reason to randomize.

We can write out the player a’s expected pay-offs of playing \( u \) surely or \( d \) surely:

\[ E[u(\sigma_b)] = \sigma_{bl}1 + \sigma_{br}0 \]
\[ E[u(\sigma_b)] = \sigma_{bl}0 + \sigma_{br}2 \]

Then that \( u \) is strictly better than \( d \) if \( \sigma_{bl} > 2\sigma_{br} \). Since \( \sigma_{bl} + \sigma_{br} = 1 \), we can substitute in to get \( 1 - \sigma_{br} > 2\sigma_{br} \), or that \( \sigma_{br} < \frac{1}{3} \). So \( \sigma_{br} = \frac{1}{3} \), a is indifferent between \( u \) and \( d \); if \( \sigma_{br} > \frac{1}{3} \), player a strictly prefers \( d \); if \( \sigma_{br} = \frac{1}{3} \), player a is exactly indifferent. These observations are summarized in Figure 1.

The important insight of these calculations is that you control your opponent’s preferences over her strategies. To get an opponent to randomize, he needs to be indifferent over his options. If he strictly preferred one strategy over all others, he would just use that one. As a result, a mixed strategy equilibrium will require that players make their opponents indifferent over their pure strategies. We already solved for that condition above: \( (\sigma^*_{br} = \frac{1}{3}, \sigma^*_{bl} = \frac{2}{3}) \) succeeds in making player a indifferent over her pure strategies. Repeating this and switching the roles of the two players gives the other half of the equilibrium strategies, for player a.

Mixed Strategy Equilibrium in Matching Pennies

Recall

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Does this game have a mixed-strategy Nash equilibrium? The payoff to player a from using \( u \) surely is

\[ E[u_a(u, \sigma_b)] = \sigma_{bl}(1) + \sigma_{br}(-1) \]

and the payoff to using \( d \) surely is

\[ E[u_a(d, \sigma_b)] = \sigma_{bl}(-1) + \sigma_{br}(1) \]
Then \( u \) is strictly better than \( d \) if
\[
\sigma_{bl}(1) + \sigma_{br}(-1) > \sigma_{bl}(-1) + \sigma_{br}(1)
\]
or
\[
\sigma_{bl} > \sigma_{br}
\]
If \( \sigma_{br} > \sigma_{bl} \), then \( d \) is strictly better than \( u \). If \( \sigma_{br} = \sigma_{bl} \), player \( a \) is exactly indifferent. Then to ensure that player \( a \) is indifferent in the mixed strategy equilibrium, we can use \( \sigma_{br} + \sigma_{bl} = 1 \) and \( \sigma_{br} = \sigma_{bl} \) to get \( \sigma_{*br} = \sigma_{*bl} = \frac{1}{2} \). (Note that we used the row player’s payoff to compute the column player’s strategy).

The calculations are essentially the same for the column player. Consequently, \( \sigma_{*au} = \sigma_{*ad} = \frac{1}{2} \).

Then our mixed-strategy Nash Equilibrium is \( \sigma_{*br} = \sigma_{*bl} = \sigma_{*au} = \sigma_{*ad} = \frac{1}{2} \). The best-response functions for this game are graphed in Figure 2.

Example

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<tr>
<td>m</td>
<td>1.7</td>
<td>5.0</td>
<td>1.2</td>
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<tr>
<td>d</td>
<td>2.1</td>
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Solve for all Nash equilibria.

First, we do iterated deletion of weakly dominated strategies. For the column player, $b$ is weakly dominated by $a$. Once $b$ is gone, then $m$ can be eliminated for the row player. This leaves the game

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To find the mixed strategies, we need to choose each player’s strategy to make his opponent indifferent over her pure strategies.

**Row player strategy:** Since we are solving for Row’s strategy, we need to make the Column player indifferent, so

$$E[u_c(\sigma_r, a)] = E[u_c(\sigma_r, b)]$$

or

$$\sigma_{ru}(3) + \sigma_{rd}(1) = \sigma_{ru}(0) + \sigma_{rd}(2)$$

Simplifying this yields $\sigma_{rd} = 3\sigma_{ru}$, and we also have $\sigma_{ru} + \sigma_{rd} = 1$ (or $\sigma_{rd} = 1 - \sigma_{ru}$). Combining these two equations gives $1 - \sigma_{ru} = 3\sigma_{ru}$, or $\sigma_{ru}^* = 1/4$, which means $\sigma_{rd}^* = 3/4$. 

Figure 2: Best-Response Functions in Matching Pennies
Column player strategy: Since we are solving for Column’s strategy, we need to make the Row player indifferent, so
\[ E[u_r(u, \sigma_c)] = E[u_r(d, \sigma_c)] \]
or
\[ \sigma_{ca}(3) + \sigma_{cc}(2) = \sigma_{ca}(2) + \sigma_{cc}(3) \]

Simplifying yields \( \sigma_{ca} = \sigma_{cc} \); combining this with the equation \( \sigma_{ca} + \sigma_{cc} = 1 \) implies that \( \sigma_{ca}^* = \sigma_{cc}^* = \frac{1}{2} \).

Then the mixed-strategy Nash equilibrium is \( \sigma_{ru}^* = \frac{1}{4}, \sigma_{rd}^* = \frac{3}{4}, \sigma_{ca}^* = \sigma_{cc}^* = \frac{1}{2} \).

Notice that if you plot the best-response functions, this game is “similar” to Battle of the Sexes: There are two pure-strategy Nash equilibria, and one mixed-strategy Nash equilibrium.

Existence of Nash Equilibrium

**Theorem 4.** *In any finite game with a finite number of pure strategies, a (mixed-strategy) Nash equilibrium is guaranteed to exist.*

So we’ve finally found a solution concept that works in any situation: Give me any finite game, and I know that a Nash equilibrium exists, unlike IDDS or pure-strategy Nash equilibrium which only works for special kinds of games.

Interpreting Mixed Nash Equilibrium

- **As Randomization:** The players simply attempt to play randomly, as in rock-paper-scissors.

- **Pure Strategies in Large Populations:** No one actually randomizes. There’s simply a large population of players who use pure strategies, and on average no player has an incentive to switch. For example, think of drivers using congested routes to work: Some proportion take one route and some another, and each driver does so deliberately, but on average they balance out to equilibrium proportions

- **Purification:** Here, we imagine the players have a small amount of private information, and this causes what appears to be randomization, but it is really just uncertainty (this is a somewhat technical idea).

Bank Runs

Suppose there are two agents who put a dollar in a bank. On the news, there’s a story that there’s been some event that may adversely effect the economy. They might now withdraw their funds, fearing that the bank will fail.

- If they both leave the money in the bank, it gives a return \( 1 + R \). If they both withdraw, they keep their dollar.
If one withdraws, that agent gets a dollar, but the bank folds and the other agent loses his dollar.

What are the Nash equilibria of the game?

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<td>W</td>
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<td>1+R</td>
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If $R$ goes up, is a bank run where both players withdraw more or less likely?

There are two pure-strategy Nash equilibria: $(S, S)$ and $(W, W)$. But there's also a mixed equilibrium. Let's solve for the row player's strategy:

There are no weakly dominated strategies, so now we choose the row player's strategy $(\sigma_{rs}, \sigma_{rw})$ for "probability the row player stays" and "the probability the row player withdraws", respectively, to make the column player indifferent between staying and withdrawing. The column player's expected payoff from these two strategies are

$$E[u_c(s, \sigma_r)] = \sigma_{rs}(1 + R) + \sigma_{rw}(-1)$$

We set these equal to make the column player indifferent between staying and withdrawing, or

$$1 = \sigma_{rs}(1 + R) - \sigma_{rw}$$

We also know that $\sigma_{rs} + \sigma_{rw} = 1$; if we substitute $\sigma_{rs} = 1 - \sigma_{rw}$ in, we get

$$1 = (1 - \sigma_{rw})(1 + R) - \sigma_{rw}$$

Solving for the probability of withdrawal by the row player gives

$$1 = (1 + R) - \sigma_{rw}(2 + R)$$

or

$$\sigma_{rw}^* = \frac{R}{2 + R}$$

and the probability the row player stays is

$$\sigma_{rs}^* = 1 - \sigma_{rw}^* = \frac{2}{2 + R}$$

Since these calculations would be exactly the same for the column player, we know they'll adopt the same strategy.

What happens to the likelihood of a run if $R$ goes up?

$$\frac{\partial \sigma_{rw}^*}{\partial R} = \frac{2 + R - 2}{(2 + R)^2} = \frac{R}{(2 + R)^2} > 0$$

The equilibrium likelihood of a run increases in the interest rate $R$. (Why?)
The Bystander Effect

Someone who can’t swim falls into a river at a park. There are \( i = 1, 2, \ldots, N \) people at the park who witness the accident. All the people receive a positive payoff \( v \) if drowning person is saved, and each person assess the risk of personal injury and other costs at \( c \) if they act. What are some of the pure and mixed Nash equilibria of the game?

If \( c > v \), it is a pure strategy not to help. The payoff to the person who jumps in is \( v - c < 0 \), and no one is willing to incur the personal cost to provide the public good of saving the person.

If \( c < v \), there are pure strategy equilibria where someone helps with probability 1. If a single person helps with probability 1, no one else has an incentive to help. The person who helps gets a payoff of \( v - c \), while the others get \( v \). Can anyone deviate and get a strictly higher payoff (“no” means this is a Nash equilibrium). If the person helping stops, his payoff goes from \( v - c \) to zero, which is not a profitable deviation. If anyone who isn’t currently helping jumps in to help, their payoff goes from \( v \) to \( v - c \). Therefore, there are no profitable deviations, and this is a Nash equilibrium.

Finally, suppose each person plans to help with probability \( p \). Then we choose \( p \) to make all the agents indifferent between helping as a pure strategy and not helping as a pure strategy.

\[
\frac{v - c}{\text{Help surely}} = \frac{(1 - (1 - p)^{N-1})v}{\text{Don't help surely}}
\]

The term \( 1 - (1 - p)^{N-1} \) works like this: Suppose I refuse to help. There are \( N - 1 \) other people who might all help with probability \( 1 - p \) each. Then the probability that no one helps among \( N - 1 \) people is the probability that no single person helps, or \( (1 - p)(1 - p) \ldots (1 - p) \) a total \( N - 1 \) times. Then the probability that someone helps among \( N - 1 \) people (and possibly more than one person) is just \( 1 - (1 - p)^{N-1} \).

Solving for \( p \) gives

\[
p^* = 1 - \left( \frac{c}{v} \right)^{1/(N-1)}
\]

Now, as we raise \( N \), the second term on the right hand side grows, since \( v > c \). Therefore, \( p^* \) is decreasing in \( N \) — more people implies a lower probability that any given one of them will help. In particular, as \( N \to \infty \),

\[
p^* = 1 - \left( \frac{c}{v} \right)^0 = 1 - 1 = 0
\]

When you think about it, there are actually many, many equilibria to the game. For example, the last three people don’t help for sure, and the previous \( N - 3 \) people all randomize as we did above. Or the last four people don’t help for sure, and the previous \( N - 4 \) people all randomize. And so on. So it looks like there are at least \( \sum_{k=1}^{N} \frac{N!}{(N-k)!k!} \) mixed equilibria where the \( k \) players who might help choose the same mixed strategy. Then we potentially have equilibria where the players use
asymmetric strategies, where they each adopt a different probability of helping. So this game actually seems to be pretty complex.

**Wars of Attrition**

There are two animals fighting over their territory. The value to each animal is \( v \), and the cost of continuing the battle is \( c \). At each moment in time, they decide whether to continue or stop. If they both continue, they both incur a cost of \( c \). If they both stop, they get payoffs of zero. If one animal continues but the other stops in the \( t \)-th period, the first gets \( v - (t-1)c \), while the other gets \(- (t-1)c\). What is the (symmetric) mixed Nash equilibria of the game? What is the probability that the game reaches the \( t \)-th period? Is it possible for the animals to fight long enough for their costs to outweigh the value of the prize?

Since the costs incurred in previous rounds are sunk, they actually don’t matter. For this reason, we just need to study the situation where each player is indifferent over stopping and continuing. Let \( p \) be the probability that either party decides to continue the battle.

\[
\begin{array}{c|cc}
0 & \text{Stop surely} & \text{Continue surely} \\
\hline
p(-c) & (1-p)(v) \\
\end{array}
\]

This gives a solution of

\[
p = \frac{v}{v + c}
\]

The probability of reaching the \( t \)-th round is then the probability that both animals fight at least \( t \) times, or

\[
(p^2)^t = \left( \frac{v}{v + c} \right)^{2t}
\]

If \( ct > v \), or \( t^* > v/c \), then the animals have exactly exhausted the value of the prize. The probability that they make it to \( t^* \) is

\[
\left( \frac{v}{v + c} \right)^{2v/c}
\]

For example, if \( c = 1 \) and \( v = 2c = 2 \), we have

\[
\left( \frac{2}{3} \right)^4 \approx .197
\]

So there’s a 20 percent chance that the animals exactly exhaust the value of the resource by fighting for two rounds.

Why can’t a player adopt the pure strategy, “fight forever”? We usually restrict attention to strategies that communicate meaningful information, and this honestly doesn’t. Think about it for a second: If I say “fight forever”, can you ever decide whether I have fought forever or not? It is not decidable by any finite date, so no one can ever know if you have actually used this strategy. Another example of this is a statement like, “Let \( A \) be the set of all sets”. It turns out that whatever \( A \) is, it is not a set.
Wars of Attrition and “Swoopo”

From Wikipedia:

- Swoopo was a bidding fee auction site where purchased credits were used to make bids.

- In order to participate in an auction, registered users had to first buy bids (called credits, and henceforth referred to as "Bid-credits") before entering into an auction. For the US version of the site, Bid-credits cost $0.60 apiece and were sold in lots (called BidPacks) of 40, 75, 150, 400, and 1,000. Each credit is good for one bid. Standard auctions begin with an opening price of $0.12 and every time someone bids the price increases by $0.12. Other auction types use different values, penny auctions use $0.01, 6 cent auctions $0.06, etc. The price of bids and the incremental values vary depending on the regional version of the site used.

Wars of Attrition and “Swoopo”

- The method of selling employed by Swoopo is controversial and has been criticized. The company, responding to claims that Swoopo is a type of gambling, stated that winning auctions involves skill and is not reliant upon chance.[3] Ted Dziuba writing for The Register stated that Swoopo “does not amount to a hustle, it’s simply a slick business plan”, and that while it might be close to gambling, “the non-determinism comes directly from the actions of other users, not the randomness of a dice roll or a deck of cards.”

- Nevertheless, the argument about “skill game” is put down by MSN Money: “Chris Bauman [[director of Swoopo in the US]] told one blogger: ‘Winning takes two things: money and patience. Every person has a strategy.’ Indeed, he undoubtedly does. The problem is that, as with the gambling systems peddled by countless books, none of those strategies will actually work. Just remember that no matter how many times you bid, your chance of winning does not increase”.

Wars of Attrition and “Swoopo”

- Ian Ayres writing for New York Times blog called Swoopo a “scary website that seems to be exploiting the low-price allure of all-pay auctions”. MSN Money has called Swoopo “The crack cocaine of online auction websites”, and stated that “in essence, what your 60 bidding fee gets you at Swoopo is a ticket to a lottery”. The New York Times has called the process ”devilish.”

See quibids, zeekler