Example: Cournot
Recall the Cournot game, where there are two firms \( a \) and \( b \) who each choose a quantity \( q_a \) or \( q_b \) from the set \( \{ 0, q_1, q_2, ..., q_N \} \), and the price is set by \( P = A - q_a - q_b \). Before, we solved this with a strategic form, such as

\[
\begin{array}{c|ccc}
1 & 2 & 3 \\
\hline
1 & 4,4 & 3.6 & 2.6 \\
2 & 6.3 & 4.4 & 2.3 \\
3 & 6.2 & 3.2 & 0,0 \\
\end{array}
\]

But what happens when \( N \) becomes large, so that the players have many possible strategies?

Games with Infinite Strategy Sets

Definition 1. A simultaneous-move game with complete information has infinite strategy sets if players choose their strategies from intervals, such as \([a, b]\), \([0, 5]\), or \([-3, \infty)\).

- For example, we might be interested in a strategy that tells an investor when to sell a particular asset: Time is a continuous variable, so we should let it take any value in \([0, \infty)\).
- Many quantities are essentially continuous: For example, pennies are such small denominations that money is essentially a continuous variable. If we’re considering how many fish to catch in a season, where the measurement is in millions of tons, the marginal fish is negligible.
- We’ve already seen some continuous games: Mixed strategies take a discrete choice (pure strategies) and smooth the situation out (mixed strategies). So choosing a probability \( p \) in \([0, 1]\) fits this class.

Best-Response Functions
A nice way of thinking about Nash equilibrium is by thinking about player \( i \)’s best-response function

\[
B_i(\sigma_{-i}) = \{ \text{Player } i \text{'s best-response to } \sigma_{-i} \} 
\]

Then a strategy profile \( \sigma^* = (\sigma^*_1, ..., \sigma^*_N) \) is a Nash equilibrium if, for each player \( i \),

\[
B_i(\sigma^*_{-i}) = \sigma^*_i 
\]

so that \( \sigma^* \) is a mutual best response.

Example: Quantity Competition
The best-response functions are the underlined entries. There are three “intersections”: (1, 3), (2, 2), (3, 1). These are the spots where $B_r(1, 3) = 1$ and $B_c(3, 1) = 3$, or $B_r(2, 2) = 2$ and $B_c(2, 2) = 2$, and $B_r(3, 1) = 3$ and $B_c(1, 3) = 1$.

Example: Battle of the Sexes

The intersection of the best-response functions gives all of the Nash equilibria.

**Nash Equilibrium**

**Definition 2.** For each player $i = 1, 2, ..., n$, let $B_i(\sigma)$ be player $i$’s best-response function. Then a Nash equilibrium is a strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, ..., \sigma_n^*)$ where, for each player $i$,

$$B_i(\sigma_{-i}^*) = \sigma_i^*$$

This is how we’re going to think about Nash equilibria for our next class of games.

**Nash Equilibrium**

In particular, suppose there are two players, $a$ and $b$, with best-response functions

$$B_a(\sigma_b)$$

and

$$B_b(\sigma_a)$$

Then to find Nash equilibria, we substitute $b$’s best-response function into $a$’s best-response function:

$$\sigma_a^* = B_a(B_b(\sigma_a^*))$$

See how this eliminates $\sigma_b$ from the problem? We find $a$’s best response to $b$ best responding to $a$, and we have an equilibrium.
The Cournot Game
There are two firms, \( a \) and \( b \). They simultaneously choose any quantity \( q_a, q_b \geq 0 \). The price in the market is given by \( p(q_a, q_b) = A - q_a - q_b \), and each firm’s total costs are \( C(q) = cq \). What quantities do the firms pick?

The Bertrand Game
There are two firms, \( a \) and \( b \). They simultaneously choose any price \( p_a, p_b \geq 0 \). If one firm chooses a strictly lower price than the other, it faces a market demand curve \( D(p) = A - p \) and the other firm gets no customers. If the two firms announce the same price, they split the market demand \( D(p) = (A - p)/2 \). Each firm’s total costs are \( C(q) = cq \). What price does each firm pick?

The Hotelling Game
There are two firms, \( a \) and \( b \). They simultaneously choose any price \( p_a, p_b \geq 0 \). Firm \( a \) is located at the left endpoint of the interval \([0, 1]\), and firm \( b \) is located at the right endpoint. Consumers are uniformly distributed on \([0, 1]\), so for any \( 1 \geq b \geq a \geq 0 \), there are \( b - a \) consumers in \([a, b]\). The consumer living at address \( 0 \leq x \leq 1 \) visiting firm \( f \) gets a payoff

\[
    u(x, f) = \begin{cases} 
    v - p_a - tx, & \text{visit firm } f = a \\
    v - p_b - t(1-x), & \text{visit firm } f = b \\
    0, & \text{make no purchase}
    \end{cases}
\]

Each firm’s total costs are \( C(q) = cq \). What price does each firm pick?

Our Problem:
Since the players in these games have strategy sets like \( s_i > 0 \), we can’t use a strategic form with rows and columns to solve the game. We’re going to have to approach these as maximization problems, and use calculus.

General Approach:

- Step 1: Write down the players’ payoff functions. Are they differentiable? (If not, we can’t use calculus.)

- Step 2: Maximize each player’s payoff with respect to his own strategy, taking the behavior of the other players as given. Solving this problem gives that player’s best-response function.

- Step 3: Now that we have a best-response function for each player, we can look for an intersection, which is a Nash equilibrium. In particular, this is the exact spot where we use the idea

\[
    \sigma^*_i = B_i(B_{-i}(\sigma^*_i))
\]

to solve for equilibria
Example: The Cournot Game, Step 1
In the Cournot game, the players’ payoff functions are

\[ \pi_a(q_a, q_b) = (A - q_a - q_b)q_a - cq_a \]  

Price

\[ \pi_b(q_a, q_b) = (A - q_a - q_b)q_b - cq_b \]  

Total costs

(These payoff functions are differentiable, so we are free to use calculus.)

Example: The Cournot Game, Step 2
Maximize with respect to \( q_a \),

\[ \frac{\partial \pi_a(q_a, q_b)}{\partial q_a} = A - q_a - q_b - q_a - c = 0 \]

and solve to get

\[ q_a = \frac{A - c - q_b}{2} \]

This is firm a’s best-response function. Doing the same work for firm b gives

\[ q_b = \frac{A - c - q_a}{2} \]

Example: The Cournot Game, Step 3
Now we’ve got two best-response functions:

\[ q_a = \frac{A - c - q_b}{2}, \quad q_b = \frac{A - c - q_a}{2} \]

We need to solve these simultaneously. Meaning, we have two equations in two unknowns, and the entire system determines their values, not just one equation at a time.

One way to tackle this is to insert the second equation into the first to get

\[ q_a = \frac{A - c - (A - c - q_a)/2}{2} = A - c + \frac{q_a}{4} \]

Then

\[ q_a = \frac{A - c}{4} + \frac{q_a}{4} \]

and taking the \( q_a/4 \) to the left-hand side,

\[ \frac{3}{4} q_a = \frac{A - c}{4} \]

And solving yields

\[ q_a^* = \frac{A - c}{3} \]
If we insert this back into firm $b$’s best-response function, we get

$$q_b^* = \frac{A - c - (A - c)/3}{2} = \frac{A - c}{3}$$

So the Nash equilibrium is $q_a^* = \frac{A - c}{3}$ and $q_b^* = \frac{A - c}{3}$.

**Example: The Cournot Game, Equilibrium**

Then the Nash equilibrium is:

$$q_a^* = \frac{A - c}{3}, \quad q_b^* = \frac{A - c}{3}$$

If we were in an econometrics class, we might try to estimate $A$ or $c$ from data, now that we know how the players’ behavior is determined. If we were in an industrial organization class, we would want to know how the players’ behavior varies as we increase or decrease $A$ and $c$. If we were in a public policy class, we might ask how a tax on one firm but not the other affects competition (tarriffs or environmental policy, for example).

Note that the equilibrium quantities do not depend on the other player’s strategy. For example, $q_b^*$ does not appear in $q_a^*$, and vice versa. We have determined the players’ behavior from the features of the market ($A$ and $c$) and “solved out” the strategic aspects.

**Graphing Best-Response Functions**

Consider the two functions

$$q_a(q_b) = \frac{A - c - q_b}{2}, \quad q_b(q_a) = \frac{A - c - q_a}{2}$$

We might want to get a better sense of what competition “looks like” in these games.

**Best-Response Functions in the Cournot Game**
The Bertrand Game, Step 1

The firm’s payoffs in the Bertrand game are

\[
\pi_a(p_a, p_b) = \begin{cases} 
(A - p_a)(p_a - c), & p_a < p_b \\
\frac{A - p_a}{2}(p_a - c), & p_a = p_b \\
0, & p_a > p_b
\end{cases}
\]

Is this function continuous? Differentiable?

First, note that charging strictly more than the monopoly price is a weakly dominated strategy — if my opponent is charging something greater than the monopoly price, I can undercut him and get the same profits as if he never existed. So let’s find the monopoly price:

\[
\max_{p_m} (A - p_m)(p_m - c)
\]

\[
A - p_m - (p_m - c) = 0 \rightarrow p_m = \frac{A + c}{2}
\]

Second, note that charging strictly less than marginal cost, \(c\), is a weakly dominated strategy. Its profit would then be negative (if it’s undercutting its opponent) or zero (if its opponent is charging an even lower price).

So we should restrict attention to prices in the interval \([c, p_m]\). But on this interval, we have a problem: as \(p_a \to p_b\), there is a discontinuity in firm \(a\)’s profit function, right at the maximum (see the next graph). This is a discontinuous function, and we can’t use calculus.
For $c \leq p_b \leq p_m$, our profit function is, in general, not differentiable. So firm a’s best-response function doesn’t exist.

Without calculus, there isn’t a general method of solving games, since a player’s best-response function fails to exist. However, sometimes we can use economics to (intelligently) guess the solution and then defend it.

The Bertrand Game, Equilibrium

It is a Nash equilibrium of the Bertrand Game for both firms to choose $p_a^* = p_b^* = c$. Why? Suppose firm b uses the strategy $p_b = c$. If firm a charges strictly more than $p_b^* = c$, firm a makes no sales. If firm a charges strictly less than $p_b^* = c$, firm a makes $(A - p_a)(p_a - c) < 0$, since $p_a < p_b^* = c$; since firm a is making losses and could instead be making zero. So if firm b chooses $p_b = c$, firm a has no strictly profitable deviation from $p_a = c$. Since all the above reasoning applies to firm b when firm a chooses $p_a = c$, the strategy profile $(c, c)$ is a Nash equilibrium.

The Hotelling Game

There are two firms, a and b. They simultaneously choose any price $p_a, p_b \geq 0$. Firm a is located at the left endpoint of the interval $[0, 1]$, and firm b is located at the right endpoint. Consumers are uniformly distributed on $[0, 1]$, so for any $1 \geq b \geq a \geq 0$, there are $b - a$ consumers in $[a, b]$. The consumer living at address $0 \leq x \leq 1$ visiting firm $f$ gets a payoff

$$u(x, f) = \begin{cases} v - p_a - tx & \text{, consumer } x \text{ visits firm } f = a \\ v - p_b - t(1 - x) & \text{, consumer } x \text{ visits firm } f = b \\ 0 & \text{, make no purchase} \end{cases}$$

Their total costs are $C(q) = cq$. 
Hotelling, Step 1: Payoffs

What are the firms’ payoffs?

\[ \pi_a(p_a, p_b) = \left( \frac{1}{2} + \frac{p_b - p_a}{2t} \right) \ast (p_a - c) \]

and

\[ \pi_b(p_b, p_a) = \left( \frac{1}{2} + \frac{p_a - p_b}{2t} \right) \ast (p_b - c) \]

In this kind of price competition game, the firm’s payoffs are determined by the demand for their product, which is a function of their opponent’s costs. So let firm a’s demand be \( D_a(p_a, p_b) \), and then we can write

\[ \pi_a(p_a, p_b) = D_a(p_a, p_b) \ast (p_a - c) \]

or

\[ \pi_a(p_a, p_b) = D_a(p_a, p_b) \ast (p_a - c) \]

and

\[ \pi_b(p_b, p_a) = D_b(p_b, p_a) \ast (p_b - c) \]

But what are the firms’ demands? We need to derive them from the game somehow.

The consumers who are nearest to 0 go to firm a (depending on the prices), and the consumers who are closest to 1 go to firm b (depending on the prices). So there must be a consumer at address \( \hat{x} \) who is exactly indifferent between going to 0 or to 1:

\[ v - t\hat{x} - p_a = v - t(1 - \hat{x}) - p_b \]

or

\[ \hat{x} = \frac{1}{2} + \frac{p_b - p_a}{2t} \]

So all the consumers to the left of \( \hat{x} \) go to firm a, and all the consumers to the right of \( \hat{x} \) go to firm b. This gives us demands

\[ D_a(p_a, p_b) = \hat{x} = \frac{1}{2} + \frac{p_b - p_a}{2t} \]

and

\[ D_b(p_b, p_a) = 1 - \hat{x} = \frac{1}{2} + \frac{p_a - p_b}{2t} \]

Substituting these into the profit functions above, we get

\[ \pi_a(p_a, p_b) = \left( \frac{1}{2} + \frac{p_b - p_a}{2t} \right) \ast (p_a - c) \]

and

\[ \pi_b(p_b, p_a) = \left( \frac{1}{2} + \frac{p_a - p_b}{2t} \right) \ast (p_b - c) \]
Hotelling, Step 2: Best-Response Functions (1)

We maximize each player’s payoff with respect to the strategy that player controls:

\[
\frac{\partial \pi_a(p_a, p_b)}{\partial p_a} = \frac{1}{2} + \frac{p_b - p_a}{2t} - \frac{1}{2t} (p_a - c) = 0
\]

\[
\frac{\partial \pi_b(p_b, p_a)}{\partial p_b} = \frac{1}{2} + \frac{p_a - p_b}{2t} - \frac{1}{2t} (p_b - c) = 0
\]

Remember the product rule, \( D_p[f(p)g(p)] = f'(p)g(p) + f(p)g'(p) \):

\[
\frac{d}{dp} \left[ D(p)(p - c) \right] = D'(p)(p - c) + D(p)
\]

Hotelling, Step 2: Best-Response Functions (2)

Solving each player’s first-order condition in terms of that player’s strategy gives a best-response function:

\[
p_a(p_b) = \frac{t + p_b + c}{2}
\]

and

\[
p_b(p_a) = \frac{t + p_a + c}{2}
\]

For firm \( a \),

\[
\frac{1}{2} + \frac{p_b - p_a}{2t} - \frac{1}{2t} (p_a - c) = 0
\]

If we move all the \( p_a \) terms to the right-hand side, we get

\[
\frac{1}{2} + \frac{p_b}{2t} + \frac{1}{2} c = \frac{1}{2t} p_a + \frac{1}{2t} p_a = \frac{p_a}{t}
\]

Multiplying by \( t \),

\[
\frac{t}{2} + \frac{p_b}{2} + \frac{1}{2} c = p_a
\]

yielding

\[
p_a(p_b) = \frac{t + p_b + c}{2}
\]

Doing the same work for firm \( b \) gives

\[
p_b(p_a) = \frac{t + p_a + c}{2}
\]
Hotelling, Step 3: Equilibrium

Now we substitute one best-response function into the other to solve them simultaneously, giving the Nash equilibrium prices:

\[ p_a^* = t + c \]

and

\[ p_b^* = t + c \]

So our Nash equilibrium is that both players charge \( t + c \), which is the marginal cost \( c \) plus a term that depends on product differentiation, \( t \).

Substituting firm \( b \)'s best-response function into firm \( a \)'s best-response function gives

\[ p_a(p_b(p_a)) = \frac{t + t + p_a + c}{2} + c \]

Which is an equation just in \( p_a \),

\[ p_a = \frac{3}{4}(t + c) + \frac{1}{4}p_a \]

And solving for \( p_a \) gives

\[ p_a^* = t + c \]

Doing the same work for firm \( b \) yields

\[ p_b^* = t + c \]

Graphing Best-Response Functions

Consider the two functions

\[ p_a(p_b) = \frac{t + p_b + c}{2}, \quad p_b(p_a) = \frac{t + p_a + c}{2} \]

Again, we might want to get a better sense of what competition “looks like” in these games.

Best-Response Functions in the Hotelling Game
Strategic Complements and Strategic Substitutes

**Definition 3.** A game exhibits *strategic substitutes* if the players’ best-response functions are downward sloping in their opponents’ strategies (as in Cournot competition), and a game exhibits *strategic complements* if the players’ best-response functions are upward sloping in their opponents’ strategies (as in Hotelling competition).

In Cournot, player $a$’s best-response function is

$$ q_a(q_b) = \frac{A - c - q_b}{2} $$

How is $q_a(q_b)$ changing as $q_b$ changes?

$$ q'_a(q_b) = -\frac{1}{2} < 0 $$

So if $q_b$ increases, firm $a$ would prefer to decrease its strategy $q_a$. This is a case of *strategic substitutes*.

In Hotelling, player $a$’s best-response function is

$$ p_a(p_b) = \frac{t + p_b + c}{2} $$

How is $p_a(p_b)$ changing as $p_b$ changes?

$$ p'_a(p_b) = \frac{1}{2} > 0 $$

So if $p_b$ increases, firm $a$ would prefer to increase its strategy $p_a$. This is a case of *strategic complements*. 

11
Strategic Substitutes and Strategic Complements

- In games of strategic complements, if your opponent increases his strategy, you want to increase yours.
- In games of strategic substitutes, if your opponent increases his strategy, you want to decrease yours.

An Important Trick: Symmetry

**Definition 4.** A game is symmetric if any player’s payoff function \( u_i(s_i, s_j, s_{-i,j}) \) can be converted into any other player’s payoff function \( u_j(s_j, s_i, s_{-j,i}) \) simply by re-arranging the player’s “names” in the payoff functions.

**Theorem 5.** Any symmetric game has a symmetric equilibrium, where every player uses the same strategy.
For example, in Cournot,
\[ \pi_a(q_a, q_b) = (A - q_a - q_b)q_a - cq_a \]

By re-arranging the player’s identities \( a \leftrightarrow b \), we get
\[ \pi_a(q_b, q_a) = (A - q_a - q_b)q_b - cq_b = \pi_b(q_b, q_a) \]

So \( \pi_a(q_b, q_a) = \pi_b(q_b, q_a) \), and the game is symmetric.

Then when we maximize player \( a \)’s payoff function, we get
\[ A - 2q_a - q_b - c = 0 \]

But since there’s a symmetric equilibrium \( q_a^* = q_b^* = q^* \), we can substitute to get
\[ A - 2q^* - q^* - c = 0 \quad \rightarrow \quad q^* = \frac{A - c}{3} \]

Which is the same answer with 1/3 the work.

Note, however, that we use the symmetric equilibrium idea after we differentiate/maximize each player’s profit function, taking the other players’ strategies as given. For example, if you INCORRECTLY did this too early, and substituted the symmetric equilibrium into payoff function (AGAIN, INCORRECT!), you’d get
\[ \pi(q^\dagger) = (A - 2q^\dagger)q^\dagger - cq^\dagger \]

and maximizing gives
\[ A - 4q^\dagger - c = 0 \]
\[ q^\dagger = \frac{A - c}{4} \]

which does not agree with any of the equilibrium calculations we’ve done for the Cournot game. So don’t make this mistake (which is easy to do if you’re not careful).

A Simple Partnership Model

Suppose there are two (engineers, lawyers, doctors, etc.) who start a firm. They each exert effort \( e_1, e_2 > 0 \), and equally split the profits of the firm,
\[ \pi(e_1, e_2) = se_1e_2, \quad s > 0 \]

The cost of effort is \( \frac{c}{2}e^2 \) for both agents, where \( c > 0 \). If they work separately and do not monitor each other, what is the Nash equilibrium effort level if \( s < c \)? Do the results change if agent 1 gets a payoff
\[ u_1(e_1, e_2) = se_1e_2 + e_1 - \frac{c}{2}e_1^2 \]

and agent 2 gets
\[ u_2(e_2, e_1) = se_2e_1 + e_2 - \frac{c}{2}e_2^2 \]

13
“The Tragedy of the Commons” and Taxes

There are three farmers in a market for beef. The cows all graze on a common pasture. The more cows that are put on the pasture, the lower the value of each cow, since the cows then compete more aggressively for food, for a price \( p(q_1, q_2, q_3) = A - q_1 - q_2 - q_3 \). The cows are otherwise costless.

- What is the Nash equilibrium of the game where the farmers simultaneously and non-cooperatively decide how many cows to graze?
- If the town placed a tax \( t \) per cow grazing on the common, could the efficient number of cows be achieved? What would the tax be?

\( N \)-player Cournot

Suppose there are \( I = 1, 2, ..., N \) players in a Cournot market, where price \( p(q_1, q_2, ..., q_N) = A - q_1 - q_2 - ... - q_N \) and \( C(q) = cq \) for all firms.

- What is the equilibrium for \( I \) players, in general?
- What happens to the equilibrium price and quantity as \( I \) grows?