The Coase Theorem
A popular idea in economics is called, “the Coase theorem”, and it goes like this:

- Suppose all the parties to a negotiation are completely informed and there are no costs to coming to an agreement. Then we should expect them to choose the most efficient outcome.

For example, suppose a railway goes by a corn field and tends to throw sparks, which set the crops on fire. A sparkguard costs $10 dollars. The profits of the railway are $100, and the profits of the farm are $X. When is the sparkguard installed? Does this depend on the legal or property rights of the agents?

Bargaining
In bargaining games, we are generally trying to model agents “haggling” back-and-forth over how to split the value or “surplus” from trade. They typically have the following ingredients:

- There’s a “pie”/value/surplus of fixed value. We usually normalize it to 1, so the players are trying to get the largest share they can. The players can choose any split, \((s, 1 - s)\).

- There’s an extensive form that describes who makes offers and counter-offers when, and under what circumstances bargaining “breaks down” and the players walk away from the trading opportunity.

- If trade breaks down, the players get their outside option payoffs, \(r_i\), which represent what the player can get on his own. We usually assume that \(r_1 + r_2 < 1\) in two-players bargaining games, so that trade would be efficient if it can be arranged.

The “Dictator” Game

- Player one proposes a split of the pie, \((t, 1 - t)\), with \(t\) going to player 1.

- Player two accepts or rejects. If player 2 accepts, they split the pie. Otherwise, player 2 gets \(r_2\) and player 1 gets \(r_1\), where \(r_1 + r_2 < 1\).

\[
\begin{array}{c}
1 & t & 2 \\
\text{Accept} & & \\
\text{Reject} & & r_1, r_2 \\
t, 1 - t & & \\
\end{array}
\]

What’s the subgame perfect Nash equilibrium, especially when \(r_1 = r_2 = 0\)? Is this a good prediction?

We solve for the SPNE through backwards induction:
• In the last subgame, player 2 is deciding between accept and reject. He should accept if

\[ 1 - t \geq r_2 \]

or \( 1 - r_2 \geq t \). He should reject otherwise.

• If player 1 wants player 2 to accept, player 1 must solve

\[ \max_t \]

subject to \( 1 - r_2 \geq t \). Since player 1 wants to maximize \( t \), he should pick \( t = 1 - r_2 \), since that is the largest value he can select where player 2 will still accept.

Then the SPNE is

• Player 1 offers \( t_1^* = 1 - r_2 \)

• Player 2 accepts if \( t_1^* \geq 1 - r_2 \), and rejects otherwise

Then we expect to see the players actually choose

• Player 1 offers \( t_1^* = 1 - r_2 \)

• Player 2 accepts

But if \( r_1 = r_2 = 0 \), this means that \( t_1^* = 1 \) and \( 1 - t_1^* = 0 \), so player 2 gets nothing.

**Discounting**

Suppose an agent gets a payoff \( u \) for each date \( t = 0, 1, 2, 3, ..., T \). Then the discounted payoff when an agent has discount factor \( \delta \) is given by the sum

\[ S = u + \delta u + \delta^2 u + ... + \delta^T u \]

where \( \delta < 1 \). If the interest rate is \( R \), then a dollar today is worth 1 and investing it gives \((1 + R)\) dollars tomorrow, so we can ask, "What is the discount factor that makes you indifferent between a dollar today and receiving \((1 + R)\) dollars tomorrow?"

\[ 1 = \delta (1 + R) \]

or

\[ \delta = \frac{1}{1 + R} \]

So if \( R = .05 \), which is a reasonable investment, the discount factor would be

\[ \delta = \frac{1}{1.05} = .95 \]
**Alternating Offer Bargaining**

But suppose that rather than accept $t^* = r_2$, the second player makes a *counteroffer*. To make it interesting, we imagine that there are costs to delay, so that the players’ payoffs after a counteroffer are discounted by $\delta$, giving us an extensive form:

In particular, if $r_1 = r_2 = 0$, what is the SPNE?

We use backwards induction:

- In the final subgame, player 1 is deciding between Accept and Reject. Player 1 should accept if
  \[ \delta t_2 \geq \delta r_1 \]
  or
  \[ t_2 \geq r_1 \]

  Player 1 should reject otherwise.

- Then player 2 is trying to maximize
  \[ \max_{t_2} \delta (1 - t_2) \]
  subject to $t_2 \geq r_1$. Since player 2’s payoff is decreasing in $t_2$, he should minimize it, choosing the lowest $t_2$ possible, or $t_2^* = r_1$.

- That means the payoffs following a rejection by player 2 will be $(\delta r_1, \delta (1 - r_1))$

- Now player 2 is deciding between accepting and getting $1 - t_1$, or rejecting and getting $\delta (1 - r_1)$. Player 2 should accept if
  \[ 1 - t_1 \geq \delta (1 - r_1) \]
  or
  \[ 1 - \delta (1 - r_1) \geq t_1 \]

  Player 2 should reject otherwise.
• Then player 1 is trying to solve

$$\max_{t_1} t_1$$

subject to $1 - \delta(1 - r_1) \geq t_1$. Since player 1’s payoff is increasing in $t_1$, he should pick the highest $t_1$ possible without causing a rejection, or $t_1^* = 1 - \delta(1 - r_1)$.

Then the SPNE is

• Player 1 offers $t_1 = 1 - \delta(1 - r_1)$
• Player 2 accepts if $t_1 \geq 1 - \delta(1 - r_1)$ and rejects otherwise.
• If a rejection occurs, player 2 offers $t_2 = r_1$.
• Player 1 accepts the offer if $t_2 \geq r_1$ and rejects otherwise.

Then we expect to see player 1 make an offer $t_1^* = 1 - \delta(1 - r_1)$ in the first round, player 2 accepts, and the game ends.

If $r_1 = r_2 = 0$, then $t_1^* = 1 - \delta$, and $t_2^* = \delta$. So the players DO split some of the profits, but not very equally.

### Alternating Offer Bargaining

Or...

![Game diagram](image)

Great practice!

### Alternating Offer Bargaining

If we set $r_1 = r_2 = 0$, a pattern begins to emerge: The game ends in the first round, with the first player offering

$$t_1^1 = 1$$

$$t_1^2 = 1 - \delta$$
\[ t_1^3 = 1 - \delta(1 - \delta) = 1 - \delta + \delta^2 \]
\[ t_1^4 = 1 - \delta(1 - \delta(1 - \delta)) = 1 - \delta + \delta^2 - \delta^3 \]

... 
\[ t_1^T = 1 - \delta + \delta^2 - \delta^3 + \delta^4 - \ldots \pm \delta^{T-1} = \sum_{t=0}^{T}(-\delta)^t \]

But these are somewhat confusing because of the \ldots terms.

**Geometric Sums**

But how do we compute a sum like \( S = y + xy + x^2y + \ldots + x^T y \)? First multiply \( S \) by \( x \):

\[ xS = xy + x^2y + x^3y + \ldots + x^{T+1}y \]

Then

\[
\begin{align*}
S &= y + xy + x^2y + x^3y + \ldots + x^T y \\
-xS &= -xy - x^2y - x^3y - \ldots - x^T y - x^{T+1}y
\end{align*}
\]

So

\[ S(1-x) = \frac{y - x^{T+1}y}{1-x} \]

Or

\[ S(T) = \frac{1 - x^{T+1}}{1-x} y \]

Note that

\[ S(\infty) = \frac{1}{1-x} y \]

By the way, I deliberately switched from \( \delta \) and \( u \) to \( x \) and \( y \) because later on, \( x \) will be \(-\delta\), not \( \delta \), and \( y \) will be one, so we get

\[ S(T) = \frac{1 - (-\delta)^{T+1}}{1-(-\delta)} \]

and

\[ S(\infty) = \frac{1}{1+\delta} \]

But we can always use this formula for geometric sums as long as \(-1 < x < 1\), so there’s nothing special about the discounted utility framework.

**Alternating Offer Bargaining**

Then using the geometric summation formula on

\[ t_1^T = 1 - \delta + \delta^2 - \delta^3 + \delta^4 - \ldots \pm \delta^{T-1} = \sum_{t=0}^{T}(-\delta)^t \]

yields

\[ t_1^T = \sum_{t=0}^{T-1}(-\delta)^t = \frac{1 - (-\delta)^T}{1+\delta} \]

If we take the limit as we allow the bargaining to go on indefinitely, \( T \to \infty \), and

\[ t_1^\infty = \frac{1}{1+\delta} \]
and the players’ payoffs are
\[ \frac{1}{1 + \delta} \frac{\delta}{1 + \delta} \]

### Alternating Offer Bargaining

So as \( \delta \to 1 \), we get

\[ \lim_{\delta \to 1} t^\infty = \lim_{\delta \to 1} \frac{1}{1 + \delta} = \frac{1}{2} \]

so the players split the value equally. Likewise, if \( \delta = e^{-R\Delta} \), so that interest compounds continuously at rate \( R \) over a short period of time of length, \( \Delta \)

\[ \lim_{\Delta \to 0} t^\infty = \lim_{\Delta \to 0} \frac{1}{1 + e^{-R\Delta}} = \frac{1}{2} \]

So if the players become very patient, or the time between offers goes to zero, the players split the pie equally: Bargaining power here comes from “agenda” control and the ability to waste resources credibly. If these are removed by allowing many offers and counter-offers, or the time between offers goes to zero, then the players split the pie equally in the SPNE.

### Alternating Offer Bargaining

What’s missing from this model of bargaining that is probably important in real applications?

### Bargaining with Investment

In many situations, opportunities to bargain and the inability to commit lead to inefficient outcomes, even when property rights are clearly defined.

- The student decides on an amount of education to get, \( e \geq 0 \), which costs \( \frac{c}{2}e^2 \)
- The firm makes the student a wage offer, \( w \geq 0 \)
- The student can accept the job, receiving a payoff of \( w - \frac{c}{2}e^2 \), or reject, and open his own business that makes profits \( be \). If the student accepts the job, the firm makes profits \( ae \), where \( a > b \), and otherwise the firm makes nothing.

Sketch an extensive form for this game and solve for the subgame perfect Nash equilibrium. What is the efficient level of education? Is the student over- or under-educated in the SPNE relative to the efficient outcome? If the firm could tie its hands and commit to making a wage offer \( w = a \cdot (a/c) \), would the problem be resolved?
• In the last subgame, the student accepts as long as
\[ w - \frac{c}{2}e^2 \geq be - \frac{c}{2}e^2 \]
or
\[ w \geq be \]
and the student rejects if \( w < be \).

• At the offer stage, the firm chooses the offer \( w \) to maximize
\[ \max_w ae - w \]
subject to the constraint that the student accept the offer, or \( w \geq be \). Since the firm’s profits are decreasing in \( w \), they want to pick the lowest \( w \) possible, or \( w = be \).

• At the education stage, the student chooses \( e \) to maximize
\[ \max_e be - \frac{c}{2}e^2 \]
Maximizing over \( e \) gives a condition
\[ b - ce = 0 \]
or
\[ e^* = \frac{b}{c} \]
So the SPNE is
• The student chooses \( e^* = b/c \)
• The firm offers a wage \( w(e) = be \)
• The student accepts the offer if \( w(e) \geq be \), but rejects otherwise

What we expect to see the players actually do is
• The student chooses \( e^* = b/c \)
• The firm offers a wage \( w = b^2/c \)
• The student accepts the offer

The efficient outcome is characterized by
\[ \max_e (w - \frac{c}{2}e^2) + (ae - w) = \max_e ae - \frac{c}{2}e^2 \]
So the efficient level of education is
\[ e^o = \frac{a}{c} \]
Since \( a > b \),
\[ e^o = \frac{a}{c} > \frac{b}{c} = e^* \]
So the student in the SPNE gets an inefficiently low level of education.