Recall that a *simultaneous-move game of complete information* is:

- A set of players $i = 1, 2, ..., n$
- A set of actions or strategies for each player
- A payoff or utility function for each player, expressing his payoff in terms of the decisions of all the participants,

$$u_i(s_1, s_2, ..., s_n) = u_i(s_i, s_{-i})$$

where $s_{-i}$ is “what everyone else is doing”,

$$s_{-i} = (s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_n)$$
**Definition**

A particular strategy $s_i^*$ is a *(weak) best-response for player $i$ to $s_{-i}$* if, for any other strategy $s_i'$ that player $i$ could choose,

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i})$$

It is a *strict best-response* if

$$u_i(s_i^*, s_{-i}) > u_i(s_i', s_{-i})$$
Strict vs. Weak Best-Responses

The first thing you should do when you see any game in a strategic form is to underline the players’ best responses. Consider the game:

\[
\begin{array}{c|cc}
 & l & r \\
\hline
u & 3,* & -2,* \\
m & 2,* & -5,* \\
d & 2,* & -2,* \\
\end{array}
\]
Strict vs. Weak Best-Responses

The first thing you should do when you see any game in a strategic form is to underline the players’ best responses. Consider the game:

\[
\begin{array}{cc}
  & l & r \\
u & 3, * & -2, * \\
m & 2, * & -5, * \\
d & 2, * & -2, *
\end{array}
\]

So \( u \) is a strict best-response to \( l \), and \( u \) and \( d \) are both weak best-responses to \( r \).
Strategy Dominance

**Definition**

A strategy $s_1$ *(weakly)* dominates a strategy $s_2$ for player $i$ if, for any $s_{-i}$ that player $i$’s opponents might use,

$$u_i(s_1, s_{-i}) \geq u_i(s_2, s_{-i})$$

Strategy $s_1$ *strictly dominates* $s_2$ if

$$u_i(s_1, s_{-i}) > u_i(s_2, s_{-i})$$
Going back to our example,

<table>
<thead>
<tr>
<th></th>
<th>l</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>u</td>
<td>3, *</td>
<td>−2, *</td>
</tr>
<tr>
<td>m</td>
<td>2, *</td>
<td>−5, *</td>
</tr>
<tr>
<td>d</td>
<td>2, *</td>
<td>−2, *</td>
</tr>
</tbody>
</table>

So $u$ strictly dominates $m$ for the row player, but $u$ only weakly dominates $d$ for the row player. (This distinction does end up mattering)
Dominant Strategies

Definition

A strategy $s_i^*$ is a (weakly) dominant strategy for player $i$ if, for any profile of opponent strategies $s_{-i}$ and any other strategy $s_i'$ that player $i$ could choose,

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i})$$

The strategy $s_i^*$ is strictly dominant for player $i$ if

$$u_i(s_i^*, s_{-i}) > u_i(s_i', s_{-i})$$
Second-Price Auctions

Suppose there are three buyers who want to purchase some property. The seller decides to use an auction to sell the land. The buyers’ values are $v_1 = 10$, $v_2 = 8$, and $v_3 = 5$. In particular, the rules of the auctions are as follows:

- Each buyer submits a bid $b_i$. The highest bidder wins the land. However, the winner pays the second-highest bid. For example, if the bids were 3, 5, and 7, the buyer who bid 7 would win, and pay a bid of 5.

Then the winner’s payoff is $v_i - b_{(2)}$ where $b_{(2)}$ is the second-highest bid, and the other buyers get a payoff of zero. The seller’s payoff is $b_{(2)} - c$, where $c$ is the value of the property to the seller.
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### Example

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>U</strong></td>
<td>0, -1</td>
<td>2, -3</td>
<td>1, 1</td>
</tr>
<tr>
<td><strong>M</strong></td>
<td>2, 4</td>
<td>-1, 1</td>
<td>2, 2</td>
</tr>
<tr>
<td><strong>A</strong></td>
<td>1, 2</td>
<td>0, 2</td>
<td>1, 4</td>
</tr>
</tbody>
</table>
So no strategies are strictly dominant for either player. But some strategies are certainly dominated. Maybe we can simplify the game by removing those strategies?
We often use a process to solve games called *Iterated Deletion of Strictly Dominated Strategies* (IDSDS):

- **Step 1:** For each player, eliminate all of his or her strictly dominated strategies.
- **Step 2:** If you deleted any strategies during Step 1, repeat Step 1. Otherwise, stop.

If the process eliminates all but one strategy profile $s^*$, we call it a *dominant strategy equilibrium* or we say it is the *outcome of iterated deletion of strictly dominated strategies*. 
Example

Suppose it is first and ten. What should offense and defense should football teams use?

<table>
<thead>
<tr>
<th>Offense</th>
<th>Defend Run</th>
<th>Defend Pass</th>
<th>Blitz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run</td>
<td>3, −3</td>
<td>7, −7</td>
<td>15, −15</td>
</tr>
<tr>
<td>Pass</td>
<td>9, −9</td>
<td>8, −8</td>
<td>10, −10</td>
</tr>
</tbody>
</table>

(If you really like football, think of these numbers as the average number of yards for the whole drive, given a particular strategy profile chosen above.)
Example

This can even work for large, complicated games.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>63,−1</td>
<td>28,−1</td>
<td>−2,0</td>
<td>−2,45</td>
<td>−3,19</td>
</tr>
<tr>
<td>B</td>
<td>32,1</td>
<td>2,2</td>
<td>2,5</td>
<td>33,0</td>
<td>2,3</td>
</tr>
<tr>
<td>C</td>
<td>54,1</td>
<td>95,−1</td>
<td>0,2</td>
<td>4,−1</td>
<td>0,4</td>
</tr>
<tr>
<td>D</td>
<td>1,−33</td>
<td>−3,43</td>
<td>−1,39</td>
<td>1,−12</td>
<td>−1,17</td>
</tr>
<tr>
<td>E</td>
<td>−22,0</td>
<td>1,−13</td>
<td>−1,88</td>
<td>−2,−57</td>
<td>−2,72</td>
</tr>
</tbody>
</table>
Quantity Competition

Suppose there are two firms, $a$ and $b$, who choose to produce quantities $q_a$ and $q_b$ of their common product. Each firm can choose to produce either 1, 2, or 3 units. They have no costs, and the market price is $p(q_a, q_b) = 6 - q_a - q_b$. The firm’s payoffs, then, are

$$\pi_A(q_a, q_b) = p(q_a, q_b)q_a = (6 - q_a - q_b)q_a$$

and

$$\pi_B(q_b, q_a) = p(q_a, q_b)q_b = (6 - q_a - q_b)q_b$$

Does the game have a dominant strategy equilibrium?
Then we get a strategic form:

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 4,4 & 3,6 & 2,6 \\
2 & 6,3 & 4,4 & 2,3 \\
3 & 6,2 & 3,2 & 0,0 \\
\end{array}
\]
Then we get a strategic form:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4,4</td>
<td>3,6</td>
<td>2,6</td>
</tr>
<tr>
<td>2</td>
<td>6,3</td>
<td>4,4</td>
<td>2,3</td>
</tr>
<tr>
<td>3</td>
<td>6,2</td>
<td>3,2</td>
<td>0,0</td>
</tr>
</tbody>
</table>

This game, however, isn’t solvable by IDSDS. But it does look like some strategies are weakly dominated.
Iterated Deletion of Weakly Dominated Strategies

- Step 1: For each player, eliminate all of his weakly dominated strategies.
- Step 2: If you deleted any strategies during Step 1, repeat Step 1. Otherwise, stop.

If the process eliminates all but one strategy profile $s^*$, we call it a weak dominant strategy equilibrium or we say it is the outcome of iterated deletion of weakly dominated strategies.
The difference between IDSDS and IDWDS

If IDSDS succeeds, the solution is unique: No matter what order in which you delete strategies, you will always get that result.
The difference between IDSDS and IDWDS

If IDSDS succeeds, the solution is unique: No matter what order in which you delete strategies, you will always get that result. For IDWDS, however, you can get *different* outcomes, depending on the order in which you delete strategies. For example,

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>U 1,2</td>
<td>3,3</td>
<td>3,4</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>1,2</td>
<td>3,5</td>
<td>3,2</td>
<td></td>
</tr>
</tbody>
</table>
Games that aren’t dominance solvable

But recall the Battle of the Sexes game:

<table>
<thead>
<tr>
<th></th>
<th>l</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>u</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>d</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

This game isn’t dominance solvable, strictly or weakly. What do we do now?
A strategy profile \( s^* = (s_1^*, s_2^*, ..., s_n^*) \) is a pure-strategy Nash equilibrium (PSNE) if, for every player \( i \) and any other strategy \( s'_i \) that player \( i \) could choose,

\[
u_i(s_i^*, s_{-i}^*) \geq u_i(s'_i, s_{-i}^*)\]
Pure-Strategy Nash Equilibrium

Definition
A strategy profile \( s^* = (s_1^*, s_2^*, ..., s_n^*) \) is a pure-strategy Nash equilibrium (PSNE) if, for every player \( i \) and any other strategy \( s'_i \) that player \( i \) could choose,

\[
u_i(s_i^*, s_{-i}^*) \geq u_i(s'_i, s_{-i}^*)
\]

A strategy profile is a Nash equilibrium if all players are using a “mutual-best response”, or no player can change what he is doing and get a strictly higher payoff.
How to find PSNE’s in Strategic Form Games

Finding Nash equilibria in strategic form can quickly be done in two ways:

- **(Elimination)** Pick a box. If any player can switch strategies and get a better payoff, it is not a Nash equilibrium, so cross it out. Check all boxes. If any box is left, it is a Nash equilibrium.

- **(Construction)** Pick a row. Underline the best payoff the column player can receive. Check all rows. Pick a column. Underline the best payoff the row player can receive. Check all columns. If any box has both pay-offs underlined, it is a Nash equilibrium.
Consider the following game:

\[
\begin{array}{ccc}
B & L & R \\
A & 2,1 & 1,0 \\
D & 1,-1 & 3,3 \\
\end{array}
\]
Nash Equilibria in our Classic Games

Prisoners’ Dilemma:

<table>
<thead>
<tr>
<th></th>
<th>$s$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>$-3, -3$</td>
<td>$-7, -1$</td>
</tr>
<tr>
<td>$c$</td>
<td>$-1, -7$</td>
<td>$-5, -5$</td>
</tr>
</tbody>
</table>
Prisoners’ Dilemma:

\[
\begin{array}{c|cc}
 & s & c \\
\hline
s & -3, -3 & -7, -1 \\
c & -1, -7 & -5, -5 \\
\end{array}
\]

So our new tool — PSNE — agrees with our prediction from IDSDS.
Battle of the Sexes:

<table>
<thead>
<tr>
<th></th>
<th>F</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
<th></th>
<th>F</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>B</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

There are two PSNE: $(F, F)$ and $(B, B)$. So PSNE can make useful predictions where IDDSNS cannot.
Nash Equilibria in our Classic Games

Rock-Paper-Scissors:

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>P</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>0,0</td>
<td>-1,1</td>
<td>1, -1</td>
</tr>
<tr>
<td>P</td>
<td>1, -1</td>
<td>0,0</td>
<td>-1, 1</td>
</tr>
<tr>
<td>S</td>
<td>-1,1</td>
<td>1, -1</td>
<td>0,0</td>
</tr>
</tbody>
</table>
Nash Equilibria in our Classic Games

Rock-Paper-Scissors:

\[
\begin{array}{ccc}
R & P & S \\
R & 0,0 & -1,1 & 1,-1 \\
P & 1,-1 & 0,0 & -1,1 \\
S & -1,1 & 1,-1 & 0,0 \\
\end{array}
\]

And we have at least one “class” of games that don’t have pure-strategy Nash equilibria: No strategy profile is underlined twice, so there are no pure-strategy Nash equilibria.
Guess Half the Average

At the county fair, a farmer proposes the following game: The townspeople all guess the weight of a large pumpkin pie, and the person who is closest to half the average of the guesses gets her guess in pounds of pumpkin pie, and no one else gets anything. No one is quite sure how large the pie is, but they all have an estimate, $w_i$, where the expectation of $w_i$ is equal to the true weight of the pie.
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- Each townsperson $i = 1, 2, ..., N$ has a best estimate $w_i$ of the pie’s weight. They each get to submit a guess $g_i > 0$.
- The average guess is

$$\bar{g} = \frac{1}{N}(g_1 + g_2 + ... + g_N)$$

- The townsperson with the guess $g_i$ closest to $\bar{g}$ gets a payoff of $g_i$. Everyone else gets nothing

What is the pure-strategy Nash equilibrium of the game?
Average Bid Auctions

Suppose buyers $i = 1, 2, \ldots, N$ each have a value $v_i$ for a good, and these values are known to all the participants.

Desiring to be “fair”, the seller decides to use the following game to sell the good: Each buyer submits a sealed bid. The buyer whose bid is closest to the average of all the bids wins, and pays the minimum of the average bid and his bid.
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What is the equilibrium of the average bid auction?
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What is the equilibrium of the average bid auction? Is it fair?
Medicare Procurement Auctions

- As is standard in multi-unit procurement auctions, bids are sorted from lowest to highest, and winners are selected, lowest bid first, until the cumulative supply quantity equals the estimated demand.
Medicare Procurement Auctions

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—from Cramton and Ayres (2011)
Suppose there are customers uniformly distributed along Main Street, which is one mile long. Then on the interval \([0, 1]\), whenever \(1 \geq b \geq a \geq 0\), there are \(b - a\) customers in the subinterval \([a, b]\). There are two gas stations, \(a\) and \(b\) trying to decide where to locate their gas stations in \([0, 1]\); call these locations \(x_a\) and \(x_b\). All customers visit the closest gas station, and buy an amount of gasoline that gives the gas station profits of 1 per customer. What are the Nash equilibria of the game?
Dominant Strategy Equilibria and Pure Strategy Nash Equilibria

How are Dominant Strategy Equilibria and Nash Equilibria related?
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- If a game is solvable by iterated deletion of strictly dominated strategies, the outcome is a Nash equilibrium.
Dominant Strategy Equilibria and Pure Strategy Nash Equilibria

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- If a game is solvable by iterated deletion of strictly dominated strategies, the outcome is a Nash equilibrium.
- If a game is solvable by iterated deletion of weakly dominated strategies, the outcome is a Nash equilibrium, but other Nash equilibria may be eliminated from consideration along the way.
- A Nash equilibrium, in general, doesn’t have to be the outcome of iterated deletion.
How to interpret Nash Equilibria
How to interpret Nash Equilibria

- *The Outcome of Strategic Reasoning*: The logical end result of each player trying to reason about what their opponents will do, knowing the others are doing the same thing.
How to interpret Nash Equilibria

- **The Outcome of Strategic Reasoning**: The logical end result of each player trying to reason about what their opponents will do, knowing the others are doing the same thing.

- **Norms and Conventions**: The strategies that can be predicted as stable “norms” or “conventions” in society, where — given that a particular norm has been adopted — no single person can change the convention.
How to interpret Nash Equilibria

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- **Norms and Conventions**: The strategies that can be predicted as stable “norms” or “conventions” in society, where — given that a particular norm has been adopted — no single person can change the convention.

- **The Outcome of “Survival of the Fittest”**: Suppose we have a large population of players, and those who get low payoffs are removed from the game, while those who get high payoffs remain. As this game evolves, the stable outcomes of the dynamic process are Nash equilibria.
Criticisms of Nash and Dominant Strategy Equilibria

- Multiplicity of Equilibria: If a game has multiple equilibria, how do the players know which one to use?
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- **Multiplicity of Equilibria**: If a game has multiple equilibria, how do the players know which one to use?

- **Computability of Equilibria**: In very large or complicated games, how can players do IDDS or find pure-strategy Nash equilibria?
Criticisms of Nash and Dominant Strategy Equilibria

- **Multiplicity of Equilibria**: If a game has multiple equilibria, how do the players know which one to use?
- **Computability of Equilibria**: In very large or complicated games, how can players do IDDS or find pure-strategy Nash equilibria?
- **Plausibility of Equilibria**: In practice, many people don’t confess in prisoners’ dilemma games.