Mixed Strategy Nash Equilibrium

Econ 400

University of Notre Dame
Games Without Pure Strategy Nash Equilibria

It’s pretty easy to write down games without PSNE’s. For example,

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But it seems like the “right answer” is that players should use each of their pure strategies half the time. How do we make sense of this idea?
A probability model \((E, \text{pr})\) is
- A set of events, \(E = \{e_1, e_2, ..., e_K\}\)
- A probability distribution over events, \(\text{pr}(e_k)\), giving the probability of each event \(e_k\), where (i) \(0 \leq \text{pr}(e_k) \leq 1\), (ii) the probability that one of the \(K\) events occurs is 1,

\[
\text{pr}(e_1 \cup e_2 \cup ... \cup e_K) = 1
\]

and (iii) for any subsets of events \(E_1 \subset E\) and \(E_2 \subset E\) that share nothing in common, so that \(E_1 \cap E_2\) is empty, we have

\[
\text{pr}(E_1 \cup E_2) = \text{pr}(E_2) + \text{pr}(E_2)
\]
Example: Rolling a Die

Suppose we roll a six-sided die, which is “fair” in the sense that each of the possible sides is equally likely to come up.
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- Events: ♦, ♣, ♤, ♦♦, ♦♣, ♦♦♣, ♦♣♣, ♦♦♣♣

\[
\begin{align*}
pr(♦) &= pr(♣♣) = pr(♦ ♦) = pr(♣ ♦ ♦) = pr(♦ ♦ ♦ ♦) = \frac{1}{6} \\
pr(♦ ♦ ♦ ♦ ♦) &= \frac{1}{6}
\end{align*}
\]
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- Events: •, ••, • ••, • • •, • • ••, • • • •

  \[
  \begin{align*}
  p_r(\bullet) &= p_r(\bullet\bullet) = p_r(\bullet\bullet\bullet) = p_r(\bullet\bullet\bullet\bullet) = p_r(\bullet\bullet\bullet\bullet\bullet) = p_r(\bullet\bullet\bullet\bullet\bullet\bullet) = \frac{1}{6}
  \end{align*}
  \]

Since

\[
  p_r(\bullet) + p_r(\bullet\bullet) + p_r(\bullet\bullet\bullet) + p_r(\bullet\bullet\bullet\bullet) + p_r(\bullet\bullet\bullet\bullet\bullet) + p_r(\bullet\bullet\bullet\bullet\bullet\bullet) = 6 \times \frac{1}{6} = 1
\]

and \(0 \leq p_r(e) \leq 1\) for each event, this is a valid probability model.
Example: Two Unfair Coins

Suppose we flip two coins, so we can potentially see $HH$, $HT$, $TH$, and $TT$. The first coin comes up heads $p$ percent of the time, and the second comes up heads $q$ percent of the time.
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- Events: $HH$, $HT$, $TH$, $TT$
- Probability Distribution: $pr(HH) = pq$, $pr(HT) = p(1 - q)$, $pr(TH) = (1 - p)q$, and $pr(TT) = (1 - p)(1 - q)$. 


Example: Two Unfair Coins

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- **Events:** $HH$, $HT$, $TH$, $TT$
- **Probability Distribution:** $pr(HH) = pq$, $pr(HT) = p(1-q)$, $pr(TH) = (1-p)q$, and $pr(TT) = (1-p)(1-q)$.

Since

$$pr(HH) + pr(HT) + pr(TH) + pr(TT) = pq + p - pq + q - pq + 1 - p - q + pq = 1$$

and all for all events $xy$, $0 \leq pr(xy) \leq 1$, this is a valid probability model.
In situations with risk, how do we measure agents’ payoffs?

**Definition**

Let the set of events be $E = \{e_1, e_2, ..., e_K\}$ and the probability distribution be $pr(e_k)$. A *utility function* associates a payoff with each event $u(e)$. The agent’s *expected utility* is

$$E[u(e)] = pr(e_1)u(e_1) + pr(e_2)u(e_2) + ... + pr(e_K)u(e_K)$$
Example: Risky Investment

Suppose an agent can invest in a safe asset \( s \) giving a return 1 per dollar invested, or a risky asset \( a \) giving a return \( g > 1 \) (a gain) with probability \( p \) and a return \( l < 1 \) (a loss) with probability \( 1 - p \). The agent has preferences \( u(w) = \log(w) \) over final wealth, and total wealth \( W \). Then his budget constraint is \( W = a + s \), and his preferences are

\[
E[u(w)] = pu(s + ga) + (1 - p)u(s + la)
\]

How much should the agent invest in the risky asset, \( a \)?
Strategies in Rock-Paper-Scissors

Let’s think about a “random strategy” in Rock-Paper-Scissors. Then we need

- **Outcomes**: $R$, $P$, $S$
- **Probability distribution**: $\sigma_R + \sigma_P + \sigma_S = 1$, with $0 \leq \sigma_R, \sigma_P, \sigma_S \leq 1$
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This is what we call a “mixed strategy” in game theory, since the player is using a little bit of many of his pure strategies.
Mixed Strategies

**Definition**

Suppose a player has \( k = 1, 2, ..., K \) pure strategies. A *mixed strategy for player* \( i \) *is a probability distribution over pure strategies, \( \sigma_i = (\sigma_{i1}, \sigma_{i2}, ..., \sigma_{iK}) \) with* \( 0 \leq \sigma_{ik} \leq 1 \) *and* \( \sigma_{i1} + \sigma_{i2} + ... + \sigma_{iK} = 1 \).

The keys are (i) the sum of all the weights \( \sigma_{ik} \) on each of the pure strategies is one and (ii) each weight \( 0 \leq \sigma_{ik} \leq 1 \).
Expected Utility in Games

Let’s think about expected utility in Matching Pennies:

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Then a mixed strategy for the row player is $\sigma_r = (\sigma_{rh}, \sigma_{rt})$ and for the column player $\sigma_c = (\sigma_{ch}, \sigma_{ct})$. Then the row player’s expected utility is

$$
E[u_r(\sigma_r, \sigma_c)] = \sigma_{ch}\sigma_{rh}(1) + \sigma_{rt}\sigma_{ch}(-1) + \sigma_{rh}\sigma_{ct}(-1) + \sigma_{rt}\sigma_{ct}(1)
$$
Suppose we’ve got two players, $r$ and $c$. Then row’s expected utility is

$$E[u_r(\sigma_r, \sigma_c)] = \sum_{k \text{ all pure row strategies}} \sum_{j \text{ all pure column strategies}} \sigma_{rk} \sigma_{cj} u_r(s_{rk}, s_{cj})$$

In general, we just add more summations and probability terms as we add more players and strategies.
A simultaneous-move game of complete information is

- A set of players $i = 1, 2, ..., N$
- A set of pure strategies for each player, and the corresponding mixed strategies
- A set of expected utility functions for each player,

$$
\mathbb{E}[u_i(\sigma_i, \sigma_{-i})]
$$
A set of mixed strategies $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is a *Nash equilibrium* if, for every player $i$ and any other mixed strategy $\sigma'_i$ that $i$ could choose,

$$\mathbb{E}[u_i(\sigma_i^*, \sigma_{-i}^*)] \geq \mathbb{E}[u_i(\sigma'_i, \sigma_{-i}^*)]$$

Note that a pure strategy is a special case of a mixed strategy, so the definition generalizes our earlier one.
A set of strategies $\sigma^*_r$ and $\sigma^*_c$ is a mixed strategy equilibrium of matching pennies if, for any $\sigma'_r = \sigma'_{rh} + \sigma'_{rt}$ for the row player,

$$
\mathbb{E}[u_r(\sigma^*_r, \sigma^*_c)] = \sigma^*_{ch}\sigma^*_{rh}(1) + \sigma^*_{rt}\sigma^*_{ch}(-1) + \sigma^*_{rh}\sigma^*_{ct}(-1) + \sigma^*_{rt}\sigma^*_{ct}(1) \\
\geq \sigma^*_{ch}\sigma'_{rh}(1) + \sigma'_{rt}\sigma^*_{ch}(-1) + \sigma'_{rh}\sigma^*_{ct}(-1) + \sigma'_{rt}\sigma^*_{ct}(1) = \mathbb{E}[u_r(\sigma'_r, \sigma^*_c)]
$$

and for any $\sigma'_c = \sigma'_{ch} + \sigma'_{ct}$ for the column player,

$$
\mathbb{E}[u_c(\sigma^*_c, \sigma^*_r)] = \sigma^*_{ch}\sigma^*_{rh}(-1) + \sigma^*_{rt}\sigma^*_{ch}(1) + \sigma^*_{rh}\sigma^*_{ct}(1) + \sigma^*_{rt}\sigma^*_{ct}(-1) \\
\geq \sigma'_{ch}\sigma^*_r(-1) + \sigma^*_{rt}\sigma'_r(1) + \sigma^*_{rh}\sigma'_c(1) + \sigma^*_{rt}\sigma'_c(-1) = \mathbb{E}[u_c(\sigma'_c, \sigma^*_r)]
$$
A set of strategies $\sigma_r^*$ and $\sigma_c^*$ is a mixed strategy equilibrium of matching pennies if, for any $\sigma'_r = \sigma'_{rh} + \sigma'_{rt}$ for the row player,

$$E[u_r(\sigma_r^*, \sigma_c^*)] = \sigma_{ch}^*\sigma_{rh}^*(1) + \sigma_{rt}^*\sigma_{ch}^*(-1) + \sigma_{rh}^*\sigma_{ct}^*(-1) + \sigma_{rt}^*\sigma_{ct}^*(1)$$

$$\geq \sigma_{ch}^*\sigma'_{rh}(1) + \sigma'_{rt}\sigma_{ch}^*(-1) + \sigma'_{rh}\sigma_{ct}^*(-1) + \sigma'_{rt}\sigma_{ct}^*(1) = E[u_r(\sigma'_r, \sigma_c^*)]$$

and for any $\sigma'_c = \sigma'_{ch} + \sigma'_{ct}$ for the column player,

$$E[u_c(\sigma_c^*, \sigma_r^*)] = \sigma_{ch}^*\sigma_{rh}^*(-1) + \sigma_{rt}^*\sigma_{ch}^*(1) + \sigma_{rh}^*\sigma_{ct}^*(1) + \sigma_{rt}^*\sigma_{ct}^*(-1)$$

$$\geq \sigma'_{ch}\sigma_{rh}^*(-1) + \sigma_{rt}^*\sigma'_{ch}(1) + \sigma'_{rh}\sigma_{ct}^*(1) + \sigma'_{rt}\sigma_{ct}^*(-1) = E[u_c(\sigma'_c, \sigma_r^*)]$$

This is kind of a mess though. How do we solve for the equilibrium strategies?
How do we solve for mixed-strategy Nash equilibria?

There's a systematic method to solving these games:

- Step 1: Do iterated deletion of weakly dominated strategies
- Step 2: Choose the mix such that each player is making his opponents indifferent over their pure strategies
How do we *solve* for mixed-strategy Nash equilibria?

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**Why?**

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Why?

- **Step 1:** If you are putting any weight on a weakly dominated strategy, you can move that weight to the strategy dominating it and improve your payoff
- **Step 2:** Suppose your opponent strictly preferred one of his strategies to all the others. Then he should just use that strategy for sure, right? So your opponent will only play randomly if you make him indifferent, and you will play randomly only if he makes you indifferent.
Suppose an agent is placing some weight on a strategy $k$ that is weakly dominated by a strategy $l$.

That means that $u_i(s_{ik}, s_{-i}) \geq u_i(s_{il}, s_{-i})$, for any $s_{-i}$ that might occur.

But then the expected payoff given the opponents’ mixed strategies is

$$\sum_{s_{-i}} pr(s_{-i})u_i(s_{ik}, s_{-i}) \geq \sum_{s_{-i}} pr(s_{-i})u_i(s_{il}, s_{-i})$$

So if the agent is placing $\sigma_{ik}$ weight on strategy $k$, and $\sigma_{il}$ weight on strategy $l$, shifting it all to strategy $k$ gives a higher payoff:

$$(\sigma_{ik} + \sigma_{il}) \left( \sum_{s_{-i}} pr(s_{-i})u_i(s_{ik}, s_{-i}) \right) \geq \sigma_{ik} \left( \sum_{s_{-i}} pr(s_{-i})u_i(s_{ik}, s_{-i}) \right) + \sigma_{il} \left( \sum_{s_{-i}} pr(s_{-i})u_i(s_{il}, s_{-i}) \right)$$

So weakly dominated strategies should be eliminated from consideration.
Step 2: Make Your Opponents Indifferent Between Their Pure Strategies

Note that we can factor a player’s strategy, as follows:

\[ E[u_r(\sigma^*)] = \sigma^*_c h \sigma^*_r h (1) + \sigma^*_r t \sigma^*_c h (-1) + \sigma^*_r h \sigma^*_c t (-1) + \sigma^*_r t \sigma^*_c t (1) \]

\[ = \sigma^*_r h \left( \frac{\sigma^*_c h (1) + \sigma^*_c t (-1)}{\text{Exp Utility of Heads}} \right) + \sigma^*_r t \left( \frac{\sigma^*_c h (-1) + \sigma^*_c t (1)}{\text{Exp Utility of Tails}} \right) \]

Notice that the terms with underbraces are the expected utility the row player gets from *just playing heads* and *just playing tails*, respectively, where no randomization is involved. Also note that if one is strictly bigger than the other, the row player should just play the strategy with the better expected return.
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\mathbb{E}[u_r(\sigma^*)] = \sigma_{ch}^* \sigma_{rh}^* (1) + \sigma_{rt}^* \sigma_{ch}^* (-1) + \sigma_{rh}^* \sigma_{ct}^* (-1) + \sigma_{rt}^* \sigma_{ct}^* (1)
\]

\[
= \sigma_{rh}^* \left( \frac{\sigma_{ch}^* (1) + \sigma_{ct}^* (-1)}{\text{Exp Utility of Heads}} \right) + \sigma_{rt}^* \left( \frac{\sigma_{ch}^* (-1) + \sigma_{ct}^* (1)}{\text{Exp Utility of Tails}} \right)
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→ A player is only willing to play a mixed strategy if she is indifferent between the expected payoffs coming from her pure strategies.
Mixed Strategy Equilibrium in Battle of the Sexes

Recall

\[
\begin{array}{c|cc}
 & b & \\
\hline
l & 1,2 & 0,0 \\
a & 0,0 & 2,1 \\
\end{array}
\]

Does this game have a mixed-strategy Nash equilibrium?
Mixed Strategy Equilibrium in Matching Pennies

Recall

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Does this game have a mixed-strategy Nash equilibrium?
Example

\[
\begin{array}{ccc}
 & a & b & c \\
 u & 3,3 & 1,1 & 2,0 \\
 m & 1,7 & 5,0 & 1,2 \\
 d & 2,1 & 1,1 & 3,2 \\
\end{array}
\]

Solve for all Nash equilibria.
Existence of Nash Equilibrium

Theorem

In any finite game with a finite number of pure strategies, a (mixed-strategy) Nash equilibrium is guaranteed to exist.

So we’ve finally found a solution concept that works in any situation: Give me any finite game, and I know that a Nash equilibrium exists, unlike IDDS or pure-strategy Nash equilibrium which only works for special kinds of games.
Interpreting Mixed Nash Equilibrium

- **As Randomization**: The players simply attempt to play randomly, as in rock-paper-scissors.

- **Pure Strategies in Large Populations**: No one actually randomizes. There’s simply a large population of players who use pure strategies, and on average no player has an incentive to switch. For example, think of drivers using congested routes to work: Some proportion take one route and some another, and each driver does so deliberately, but on average they balance out to equilibrium proportions.

- **Purification**: Here, we imagine the players have a small amount of private information, and this causes what appears to be randomization, but it is really just uncertainty (this is a somewhat technical idea).
Bank Runs

Suppose there are two agents who put a dollar in a bank. On the news, there’s a story that there’s been some event that may adversely effect the economy. They might now withdraw their funds, fearing that the bank will fail.

- If they both leave the money in the bank, it gives a return $1 + R$. If they both withdraw, they keep their dollar.
- If one withdraws, that agent gets a dollar, but the bank folds and the other agent loses his dollar.

What are the Nash equilibria of the game?

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If $R$ goes up, is a bank run where both players withdraw more or less likely?
The Bystander Effect

Someone who can’t swim falls into a river at a park. There are $i = 1, 2, ..., N$ people at the park who witness the accident. All the people receive a positive payoff $v$ if drowning person is saved, and each person assess the risk of personal injury and other costs at $c$ if they act. What are some of the pure and mixed Nash equilibria of the game?
Wars of Attrition

There are two animals fighting over their territory. The value to each animal is $v$, and the cost of continuing the battle is $c$. At each moment in time, they decide whether to continue or stop. If they both continue, they both incur a cost of $c$. If they both stop, they get payoffs of zero. If one animal continues but the other stops in the $t$-th period, the first gets $v - (t - 1)c$, while the other gets $-(t - 1)c$. What is the (symmetric) mixed Nash equilibria of the game? What is the probability that the game reaches the $t$-th period? Is it possible for the animals to fight long enough for their costs to outweigh the value of the prize?
Wars of Attrition and “Swoopo”

From Wikipedia:

- Swoopo was a bidding fee auction site where purchased credits were used to make bids.
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- In order to participate in an auction, registered users had to first buy bids (called credits, and henceforth referred to as ”Bid-credits”) before entering into an auction. For the US version of the site, Bid-credits cost $0.60 apiece and were sold in lots (called BidPacks) of 40, 75, 150, 400, and 1,000. Each credit is good for one bid. Standard auctions begin with an opening price of $0.12 and every time someone bids the price increases by $0.12. Other auction types use different values, penny auctions use $0.01, 6 cent auctions $0.06, etc. The price of bids and the incremental values vary depending on the regional version of the site used.
Wars of Attrition and “Swoopo”

- The method of selling employed by Swoopo is controversial and has been criticized. The company, responding to claims that Swoopo is a type of gambling, stated that winning auctions involves skill and is not reliant upon chance.[3]
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Nevertheless, the argument about “skill game” is put down by MSN Money: “Chris Bauman [[director of Swoopo in the US]] told one blogger: ‘Winning takes two things: money and patience. Every person has a strategy.’ Indeed, he undoubtedly does. The problem is that, as with the gambling systems peddled by countless books, none of those strategies will actually work. Just remember that no matter how many times you bid, your chance of winning does not increase”.

Econ 400 (ND) Mixed Strategy Nash Equilibrium
Wars of Attrition and “Swoopo”

- Ian Ayres writing for New York Times blog called Swoopo a “scary website that seems to be exploiting the low-price allure of all-pay auctions”. 
Wars of Attrition and “Swoopo”

Ian Ayres writing for New York Times blog called Swoopo a “scary website that seems to be exploiting the low-price allure of all-pay auctions”. MSN Money has called Swoopo “The crack cocaine of online auction websites”, and stated that “in essence, what your 60 bidding fee gets you at Swoopo is a ticket to a lottery”. The New York Times has called the process “devilish.”

See quibids, zeekler