Continuous Games

Econ 400

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Another way of thinking about Nash equilibrium is by thinking about player $i$’s best-response function

$$B_i(\sigma_{-i}) = \{ \text{Player } i \text{’s best-response to } \sigma_{-i} \}$$

Then a strategy profile $\sigma^* = (\sigma^*_1, ..., \sigma^*_N)$ is a Nash equilibrium if, for each player $i$,

$$B_i(\sigma^*_{-i}) = \sigma^*_i$$

so that $\sigma^*$ is a mutual best response.
Example: Quantity Competition

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 4, 4 & 3, 6 & 2, 6 \\
2 & 6, 3 & 4, 4 & 2, 3 \\
3 & 6, 2 & 3, 2 & 0, 0 \\
\end{array}
\]

The best-response functions are the underlined entries. There are three “intersections”: \((1, 3)\), \((2, 2)\), \((3, 1)\). These are the spots where \(B_r(1, 3) = 1\) and \(B_c(3, 1) = 3\), or \(B_r(2, 2) = 2\) and \(B_c(2, 2) = 2\), and \(B_r(3, 1) = 3\) and \(B_c(1, 3) = 1\).
Example: Battle of the Sexes

The intersection of the best-response functions gives all of the Nash equilibria.
Nash Equilibrium

**Definition**

For each player $i = 1, 2, ..., n$, let $B_i(\sigma)$ be player $i$’s best-response function. Then a Nash equilibrium is a strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, ..., \sigma_n^*)$ where, for each player $i$,

$$B_i(\sigma_{-i}^*) = \sigma_i^*$$

This is how we’re going to think about Nash equilibria for our next class of games.
Nash Equilibrium

In particular, suppose there are two players, \( a \) and \( b \), with best-response functions

\[ B_a(\sigma_b) \]

and

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$$B_a(\sigma_b)$$

and

$$B_b(\sigma_a)$$

Then to find Nash equilibria, we substitute $b$’s best-response function into $a$’s best-response function:

$$\sigma^*_a = B_a(B_b(\sigma^*_a))$$
Nash Equilibrium

In particular, suppose there are two players, \( a \) and \( b \), with best-response functions

\[
B_a(\sigma_b)
\]

and

\[
B_b(\sigma_a)
\]

Then to find Nash equilibria, we substitute \( b \)’s best-response function into \( a \)’s best-response function:

\[
\sigma_a^* = B_a(B_b(\sigma_a^*))
\]

See how this eliminates \( \sigma_b \) from the problem? We find \( a \)’s best response to \( b \) best responding to \( a \), and we have an equilibrium.
Definition

A simultaneous-move game with complete information is *continuous* if players choose their strategies from intervals, such as \([a, b]\), \([0, 5]\), or \([-3, \infty)\).
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- Many quantities are essentially continuous: For example, pennies are such small denominations that money is essentially a continuous variable. If we’re considering how many fish to catch in a season, where the measurement is in millions of tons, the marginal fish is negligible.
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- For example, we might be interested in a strategy that tells an investor *when* to sell a particular asset: Time is a continuous variable, so we should let it take any value in \([0, \infty)\).
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- We’ve already seen some continuous games: Mixed strategies take a discrete choice (pure strategies) and smooth the situation out (mixed strategies). So choosing a probability \(p\) in \([0, 1]\) fits this class.
The Cournot Game

There are two firms, $a$ and $b$. They simultaneously choose any quantity $q_a, q_b \geq 0$. The price in the market is given by $p(q_a, q_b) = A - q_a - q_b$, and their total costs are $C(q) = cq$. 


There are two firms, \( a \) and \( b \). They simultaneously choose any price \( p_a, p_b \geq 0 \). If one firm chooses a strictly lower price than the other, it faces a market demand curve \( D(p) = A - p \) and the other firm gets no customers. If the two firms announce the same price, they split the market demand \( D(p) = (A - p)/2 \). Their total costs are \( C(q) = cq \).
The Hotelling Game

There are two firms, $a$ and $b$. They simultaneously choose any price $p_a, p_b \geq 0$. Firm $a$ is located at the left endpoint of the interval $[0, 1]$, and firm $b$ is located at the right endpoint. Consumers are uniformly distributed on $[0, 1]$, so for any $1 \geq b \geq a \geq 0$, there are $b - a$ consumers in $[a, b]$. The consumer living at address $0 \leq x \leq 1$ visiting firm $f$ gets a payoff

$$u(x, f) = \begin{cases} 
  v - p_a - tx & \text{, visit firm } f = a \\
  v - p_b - t(1 - x) & \text{, visit firm } f = b \\
  0 & \text{, make no purchase}
\end{cases}$$

Their total costs are $C(q) = cq$. 
Our Problem:

Since the players in these games have strategy sets $s_i > 0$, we can't use a strategic form with rows and columns to solve the game. We're going to have to approach these as maximization problems, and use calculus.
General Approach:

- **Step 1**: Write down the players' payoff functions. Are they differentiable? (If not, we can’t use calculus.)
- **Step 2**: Maximize each player’s payoff with respect to his own strategy, taking the behavior of the other players as given. Solving this problem gives that player’s best-response function.
- **Step 3**: Now that we have a best-response function for each player, we can look for an intersection, which is a Nash equilibrium. In particular, this is the exact spot where we use the idea

\[ \sigma^*_i = B_i(B_{-i}(\sigma^*_i)) \]
Example: The Cournot Game, Step 1

In the Cournot game, the players’ payoff functions are

\[ \pi_a(q_a, q_b) = (A - q_a - q_b)q_a - cq_a \]

\[ \pi_b(q_b, q_a) = (A - q_a - q_b)q_b - cq_b \]

(These payoff functions are differentiable, so we are free to use calculus.)
Example: The Cournot Game, Step 2

Maximize with respect to \( q_a \),

\[
\frac{\partial \pi_a(q_a, q_b)}{\partial q_a} = A - q_a - q_b - q_a - c = 0
\]

and solve to get

\[
q_a = \frac{A - c - q_b}{2}
\]

This is firm \( a \)'s best-response function. Doing the same work for firm \( b \) gives

\[
q_b = \frac{A - c - q_a}{2}
\]
Example: The Cournot Game, Step 3

Now we’ve got two best-response functions:

\[ q_a = \frac{A - c - q_b}{2}, \quad q_b = \frac{A - c - q_a}{2} \]

We need to solve these simultaneously. Meaning, we have two equations in two unknowns, and the entire system determines their values, not just one equation at a time.
Example: The Cournot Game, Equilibrium

Then the Nash equilibrium is:

\[ q_a^* = \frac{A - c}{3}, \quad q_b^* = \frac{A - c}{3} \]
Consider the two functions

\[ q_a(q_b) = \frac{A - c - q_b}{2} ,\quad q_b(q_a) = \frac{A - c - q_a}{2} \]

We might want to get a better sense of what competition “looks like” in these games.
Best-Response Functions in the Cournot Game

\[
q_a(0) = \frac{A-c}{2}
\]

\[
q_b(A-c) = 0
\]

\[
(q_a^*, q_b^*)
\]

\[
q_a(A-c) = 0 \quad q_b(0) = A-c
\]
The Bertrand Game, Step 1

The firm’s payoffs in the Bertrand game are

\[ \pi_a(p_a, p_b) = \begin{cases} 
(A - p_a)(p_a - c), & p_a < p_b \\
\frac{A - p_a}{2}(p_a - c), & p_a = p_b \\
0, & p_a > p_b 
\end{cases} \]

Is this function continuous? Differentiable?
For $c \leq p_b \leq p_m$, our profit function is, in general, not differentiable. So firm a’s best-response function doesn’t exist.
It is a Nash equilibrium of the Bertrand Game for both firms to choose $p^*_a = p^*_b = c$. 
The Hotelling Game

There are two firms, \(a\) and \(b\). They simultaneously choose any price \(p_a, p_b \geq 0\). Firm \(a\) is located at the left endpoint of the interval \([0, 1]\), and firm \(b\) is located at the right endpoint. Consumers are uniformly distributed on \([0, 1]\), so for any \(1 \geq b \geq a \geq 0\), there are \(b - a\) consumers in \([a, b]\). The consumer living at address \(0 \leq x \leq 1\) visiting firm \(f\) gets a payoff

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    u(x, f) = \begin{cases} 
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    0 & \text{, make no purchase}
    \end{cases}
\]

Their total costs are \(C(q) = cq\).
What are the firms’ payoffs?
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\[ \pi_a(p_a, p_b) = \left( \frac{1}{2} + \frac{p_b - p_a}{2t} \right) \ast (p_a - c) \]

and

\[ \pi_b(p_b, p_a) = \left( \frac{1}{2} + \frac{p_a - p_b}{2t} \right) \ast (p_b - c) \]
We maximize each player's payoff with respect to the strategy that player controls:
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\[
\frac{\partial \pi_a(p_a, p_b)}{\partial p_a} = \frac{1}{2} + \frac{p_b - p_a}{2t} - \frac{1}{2t}(p_a - c) = 0
\]

\[
\frac{\partial \pi_b(p_b, p_a)}{\partial p_b} = \frac{1}{2} + \frac{p_a - p_b}{2t} - \frac{1}{2t}(p_b - c) = 0
\]
Solving each player’s first-order condition in terms of that player’s strategy gives a best-response function:
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\[ p_a(p_b) = \frac{t + p_b + c}{2} \]

and

\[ p_b(p_a) = \frac{t + p_a + c}{2} \]
Now we substitute one best-response function into the other to solve them simultaneously, giving the Nash equilibrium prices:
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\[ p_a^* = t + c \]

and

\[ p_b^* = t + c \]

So our Nash equilibrium is that both players charge \( t + c \), which is the marginal cost \( c \) plus a term that depends on product differentiation, \( t \).
Graphing Best-Response Functions

Consider the two functions

\[ p_a(p_b) = \frac{t + p_b + c}{2}, \quad p_b(p_a) = \frac{t + p_a + c}{2} \]

Again, we might want to get a better sense of what competition “looks like” in these games.
Best-Response Functions in the Hotelling Game

\[ p_a(0) = \frac{t}{2} \]

\[ p_b(0) = \frac{t}{2} \]
Strategic Complements and Strategic Substitutes

Definition

A game exhibits *strategic substitutes* if the players’ best-response functions are downward sloping in their opponents’ strategies (as in Cournot competition), and a game exhibits *strategic complements* if the players’ best-response functions are upward sloping in their opponents’ strategies (as in Hotelling competition).
Strategic Substitutes and Strategic Complements

![Graph showing strategic substitutes and strategic complements with BR1 and BR2 lines on s1(s2) and s2(s1) axes.](image)
Qualitative Analysis of Equilibria
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In games of strategic complements, if your opponent increases his strategy, you want to *increase* yours.
Qualitative Analysis of Equilibria

- In games of strategic complements, if your opponent increases his strategy, you want to *increase* yours.
- In games of strategic substitutes, if your opponent increases his strategy, you want to *decrease* yours.
An Important Trick: Symmetry

<table>
<thead>
<tr>
<th>Definition</th>
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<tbody>
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<td>A game is <em>symmetric</em> if any player’s payoff function $u_i(s_i, s_j, s_{-i,j})$ can be converted into any other player’s payoff function $u_j(s_j, s_i, s_{-j,i})$ simply by re-arranging the player’s “names” in the payoff functions.</td>
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An Important Trick: Symmetry

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A game is *symmetric* if any player’s payoff function $u_i(s_i, s_j, s_{-i,j})$ can be converted into any other player’s payoff function $u_j(s_j, s_i, s_{-j,i})$ simply by re-arranging the player’s “names” in the payoff functions.

**Theorem**

*Any symmetric game has a symmetric equilibrium, where every player uses the same strategy.*
A Simple Partnership Model

Suppose there are two (engineers, lawyers, doctors, etc.) who start a firm. They each exert effort $e_1, e_2 > 0$, and equally split the profits of the firm, $\pi(e_1, e_2) = se_1e_2, s > 0$. The cost of effort is $\frac{c}{2}e^2$ for both agents, where $c > 0$. If they work separately and do not monitor each other, what is the Nash equilibrium effort level if $s < c$?
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$$u_1(e_1, e_2) = se_1e_2 + e_1 - \frac{c}{2}e_1^2$$

and agent 2 gets

$$u_2(e_2, e_1) = se_2e_1 + e_2 - \frac{c}{2}e_2^2$$
There are three farmers in a market for beef. The cows all graze on a common pasture. The more cows that are put on the pasture, the lower the value of each cow, since the cows then compete more aggressively for food, for a price $p(q_1, q_2, q_3) = A - q_1 - q_2 - q_3$. The cows are otherwise costless.
There are three farmers in a market for beef. The cows all graze on a common pasture. The more cows that are put on the pasture, the lower the value of each cow, since the cows then compete more aggressively for food, for a price \( p(q_1, q_2, q_3) = A - q_1 - q_2 - q_3 \). The cows are otherwise costless.

- What is the Nash equilibrium of the game where the farmers simultaneously and non-cooperatively decide how many cows to graze?
- If the town placed a tax \( t \) per cow grazing on the common, could the efficient number of cows be achieved? What would the tax be?
$N$-player Cournot

Suppose there are $I = 1, 2, \ldots, N$ players in a Cournot market, where price $p(q_1, q_2, \ldots, q_N) = A - q_1 - q_2 - \ldots - q_N$ and $C(q) = cq$ for all firms.
Suppose there are $I = 1, 2, \ldots, N$ players in a Cournot market, where price $p(q_1, q_2, \ldots, q_N) = A - q_1 - q_2 - \ldots - q_N$ and $C(q) = cq$ for all firms.

- What is the equilibrium for $I$ players, in general?
- What happens to the equilibrium price and quantity as $I$ grows?