Bayes’ Rule and Adverse Selection

Econ 400

University of Notre Dame
Information and Probability

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Suppose you and some friends are competing to see who can hit the bullseye blindfolded (never do this). Suppose that hits are uniformly distributed across the dartboard, so each spot is equally likely.

What is the probability of hitting the bullseye? Suppose that after your throw, your friends say that you hit either the inner ring or the bullseye. What is the probability of hitting the bullseye?
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$$pr[A|B] = \frac{pr[A \text{ and } B]}{pr[B]}$$
Bayes’ Rule

Suppose the probability that event $B$ occurs is not zero. Then the probability of event $A$ given (that event) $B$ (has occurred) is

$$pr[A|B] = \frac{pr[A \cap B]}{pr[B]}$$

This is called Bayes’ Rule.
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\[
pr[A|B] = \frac{pr[A \cap B]}{pr[B]}
\]

This is called \textit{Bayes’ Rule}. Suppose that \( B_1, B_2, ..., B_K \) are disjoint events (none can occur at the same time) and \( B = B_1 \cup B_2 \cup ... \cup B_K \). Then

\[
pr[A|B] = \sum_{k} pr[A|B_k]pr[B_k]
\]

This is called \textit{the Total Law of Probability}. (This is just a way of “breaking up” Bayes’ Rule across “sub-events”)

Example: Defective Parts

Given a malfunction, what is the probability the faulty part came from factory A, versus factory B? Or what is

$$\Pr[(\text{Came from factory}) \ A | \text{Defective}]$$
Example: Defective Parts

\[
\begin{align*}
Pr[A] &= \frac{2}{3} \\
Pr[B] &= \frac{1}{3} \\
Pr[A \cap D] &= \frac{2}{9} \\
Pr[D|A] &= \frac{1}{3} \\
Pr[OK|A] &= \frac{2}{3} \\
Pr[A \cap OK] &= \frac{4}{9} \\
Pr[B \cap D] &= \frac{2}{9} \\
Pr[D|B] &= \frac{2}{3} \\
Pr[OK|B] &= \frac{1}{3} \\
Pr[B \cap OK] &= \frac{1}{9}
\end{align*}
\]
Example: Defective Parts

Then

\[ pr[A|D] = \frac{pr[A \cap D]}{pr[D]} \]

or

\[ pr[A|D] = \frac{pr[A \cap D]}{pr[D|A]pr[A] + pr[D|B]pr[B]} = \frac{2/9}{(1/3)(2/3) + (2/3)(1/3)} = \frac{1}{2} \]
Suppose you go to the hospital for a routine physical. The doctor comes back and says that, using a test that is 90 percent accurate, you are positive for a disease that occurs with probability $i$ in the population. He then recommends that you undergo an expensive and painful treatment.
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Let $H$ be the event that you are healthy and $S$ be the event that you are sick. Let $+$ be the event that the test was positive, and $-$ be the event that the test was negative. Then

$$pr[S|+] = \frac{pr[S \cap +]}{pr[+]}$$
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\[
pr[S|+] = \frac{pr[S \cap +]}{pr[+]} = \frac{pr[S \cap +]}{pr[+|S]pr[S] + pr[+|H]pr[H]}
\]

so

\[
pr[S|+] = \frac{.9i}{.9i + .1(1 - i)}
\]
# Medical Testing and False Positives

| $i$   | $pr[S|+]$ |
|-------|-----------|
| .1    | .5        |
| .01   | .083      |
| .001  | .0089     |
| .0001 | .00089    |
Important Distinction

Unless the events $A$ and $B$ are independent,

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- The probability a person is shy, given they are a librarian, is $3/4$; the probability that a person is shy, given that they are a businessperson, is $1/4$. 
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- The probability a person is shy, given they are a librarian, is $3/4$; the probability that a person is shy, given that they are a businessperson, is $1/4$.
- Suppose you’ve met a shy person at a party attended by librarians and businesspeople and are trying to decide what to talk about (books or the stock market). What’s the probability they are a librarian?
Suppose we have a random variable $X$ that takes values $x_1, x_2, ..., x_K$. Whenever we have conditional probability distribution $pr[x|B]$, we can write out the conditional expectation of $X$ given $B$,

$$
\mathbb{E}[X|B] = \sum_k x_k pr[x_k|B]
$$

or if an agent has utility $u(x_k)$ for outcome $k$, the expected utility of $X$ given $B$,

$$
\mathbb{E}[u(X)|B] = \sum_k u(x_k) pr[x_k|B]
$$
Suppose there is an asset whose time zero price is 1. At time 1, the asset appreciates by a factor of \( u > 1 \) with probability \( p \) or depreciates by a factor of \( 0 < d < 1 \) with probability \( 1 - p \). At time 2, the asset appreciates by a factor of \( u > 1 \) with probability \( q \) or depreciates by a factor of \( 0 < d < 1 \) with probability \( 1 - q \). What is the “fair” price of the asset at time zero and time 1?
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An option is a contract purchased at date zero at a strike price $K$, allowing the bearer of the option to purchase an option at date $t$ for a fixed price, $p_t$. At some point, the contract expires and trade is no longer possible.
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Pricing a (Call) Option

What is the “fair” strike price at time zero? What is the expected value of the option at date 1, given $u$ or given $d$? If the contract expires at date 2, for what prices and events should you exercise the option?

\[
\begin{align*}
S_0 &= 1 \\
& \quad \downarrow \quad \downarrow \\
u, p & \quad u, q \\
d, 1 - q & \quad u^2 S_0 \quad \text{Value if exercised: } (u^2 - p) - K \\
d, 1 - q & \quad u d S_0 \quad \text{Value if exercised: } (u d - p) - K \\
d, 1 - q & \quad d^2 S_0 \quad \text{Value if exercised: } (d^2 - p) - K \\
d, 1 - p & \quad d S_0 \\
\end{align*}
\]

The strike price is paid if the contract is purchased. The option is exercised or not. The final value of the asset is realized.
Adverse Selection

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There are *buyers* and *sellers* in a market. There are two kinds of sellers:

- **Peaches**: The value of the good is $v_h$, and these kinds of sellers occur with probability $1 - p$
- **Lemons**: The value of the good is $v_l$, and these kinds of sellers occur with probability $p$
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Assume $v_h > v_l$. Buyers value the good at $v$, while sellers value their own unit of the good at $\alpha v$, with $\alpha < 1$. Assume, however, that the seller’s type cannot be observed until after the transaction takes place.
Then the buyer’s expected payoff, giving the trading price, \( t \), is

\[
\mathbb{E}[v|p] = \begin{cases} 
(1 - p)v_h + pv_l - t, & \text{trade at price } t, \text{ lemons and peaches} \\
v_l - t, & \text{trade at price } t, \text{ only lemons} \\
0, & \text{no trade}
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A peach will be willing to trade at price $t$ if

$$
t \geq \alpha v_h
$$

and for a lemon to want to trade if

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For simplicity, set $t^* = \mathbb{E}[v|t^*]$, so we give all the gains from trade to the seller.
Adverse Selection

Suppose peaches and lemons are in the market at $t^*$. Then $t^* = \mathbb{E}[v|t^*] = pv_h + (1 - p)v_l$. Peaches will want to trade if:

$$ (1 - p)v_h + pv_l \geq \alpha v_h $$

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- If only lemons are in the market, then
  
  $$t^* = \mathbb{E}[v|t^*] = v_l.$$
  
  Peaches all withdraw from trade, and only the lemons remain (and everybody knows they are lemons, *because the price is too low for the peaches to want to trade*).
Adverse Selection

Graphically,

\[ t^* = \mathbb{E}[v | t^*] \]

\[ v_h \]

\[ \alpha q_h \]

\[ v_l \]
Information Economics

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- If prices equate where supply equals demand, why can’t some people get loans, even when they are willing to pay higher interest rates than are charged in equilibrium?

- Why does bargaining break down between two parties? (If the project/trade isn’t profitable, they shouldn’t waste their time, right? But if the project/trade is profitable, our bargaining and repeated games models say they should be able to cooperate?)
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- Why do online stores charge different prices based on the browser, operating system, and computer hardware you are using?
Why are health care markets such a mess?
• Why are health care markets such a mess?
• Why did the last financial crisis happen, and why did it seem like no one could do anything about it?