Midterm Exam

All parameters are strictly positive unless stated otherwise.

1. Let

\[ f(x_1, x_2) = [(x_1 + 1)^\rho + x_2^\rho]^{1/\rho}, \]

\[ \rho < 1, \] and \( g(x_1, x_2) = px_1 + x_2 - c. \) Consider the optimization problem

\[ \max_{x_1, x_2} f(x_1, x_2) \]

subject to \( g(x_1, x_2) = 0 \) and \( x_1, x_2 \geq 0. \)

i. Does this optimization problem have a solution? Is the solution unique?

ii. Provide first-order necessary conditions that characterize a maximum. Show that your proposed maxima are actually maxima, by checking second-order sufficient conditions or providing some other argument.

iii. Use the implicit function theorem to show how the optimal \( x_2 \) varies with \( p. \)

iv. How does the optimized value of the objective vary with \( c? \)

i. The set \( S = \{x_1, x_2 : g(x_1, x_2) = 0 \text{ and } x_1, x_2 \geq 0\} \) is compact since it is bounded by an open ball of radius \( 1 + c, \) and closed since for any point \( y \) in \( \mathbb{R}^2 \setminus S, \) we can draw an open ball around \( y \) that doesn’t intersect \( S, \) so \( \mathbb{R}^2 \setminus S \) is a union of open balls, therefore open. Therefore \( S \) is closed. Therefore \( S \) is compact. The objective is a composition of power functions, which are continuous. Therefore, a solution exists by the Weierstrass theorem.

The solution is unique since the objective is strictly quasi-concave. For if we raise \( f \) to the power \( \rho, \) the sum of the two inner terms is strictly concave. Therefore, \( f \) is strictly quasi-concave, and since \( S \) is a convex set, we must have a unique solution.

ii. As stated in part i, strict quasi-concavity of \( f \) and convexity of the feasible set implies the first-order conditions are sufficient for identifying maxima. The Lagrangean is

\[ \mathcal{L} = (x_1 + 1)^\rho + x_2^\rho - \lambda(px_1 + x_2 - c) + \mu_1x_1 + \mu_2x_2 \]

with FONC’s

\[ \rho(x_1 + 1)^{\rho - 1} - \lambda p + \mu_1 = 0 \]
\[ \rho(x_2)^{\rho - 1} - \lambda + \mu_2 = 0 \]
\[ -(px_1 + x_2 - c) = 0 \]

and complementary slackness conditions \( \mu_1x_1 = 0, \mu_2x_2 = 0, \mu_1, \mu_2 \geq 0. \)

- So if \( x_1, x_2 > 0, \) we have an interior solution characterized by

\[ \rho(x_1 + 1)^{\rho - 1} - \lambda p = 0 \]
\[ \rho(x_2)^{\rho - 1} - \lambda = 0 \]
\[ -(px_1 + x_2 - c) = 0 \]

- If \( x_1 = 0, x_2 > 0, \) we have

\[ \rho(x_1 + 1)^{\rho - 1} - \lambda p + \mu_1 = 0 \]
\[ \rho(x_2)^{\rho - 1} - \lambda = 0 \]
\[ -(x_2 - c) = 0 \]

From the first equation, we can see

\[ \frac{\rho(x_1 + 1)^{\rho - 1}}{p} \leq \lambda = \rho(x_2)^{\rho - 1} \]

or

\[ \frac{\rho}{p} \leq \rho(c)^{\rho - 1} \]

giving us concrete criteria to be at a corner in \( x_1. \)
• We can’t have the case $x_2 = 0$, $x_1 > 0$, since
\[
\rho (x_1 + 1)^{\rho - 1} - \lambda p = 0
\]
\[
\rho (0)^{\rho - 1} - \lambda + \mu_2 = 0
\]
\[- (px_1 + x_2 - c) = 0
\]
but $\rho < 1$ implies $\rho - 1 < 0$, so $0^{\rho - 1} = \infty$, and this case doesn’t provide a candidate solution.

iii. We have two candidate cases. In the corner case, $x_2 = c$ doesn’t vary with $p$. At the interior solution, use the implicit function theorem.

iv. By the envelope theorem,
\[
V'(c) = \lambda^*
\]
where $\lambda^*$ is the multiplier in each solution case, and equals the ratio of the marginal utility of each consumed good to its price.

2. There are two risky assets available: The mean return on each risky asset is $\mu_i$, each has a variance of $\sigma_i^2$. Suppose $\mu_1 < \mu_2$ but $\sigma_1^2 < \sigma_2^2$, and the covariance between the two assets is $\rho < 0$. As a fund manager, your boss has asked you to choose a portfolio $(z_1, z_2)$ to maximize the following mean-minus-variance objective function:
\[
\mu_1 z_1 + \mu_2 z_2 - \frac{1}{2} \sigma_1^2 z_1^2 - \frac{1}{2} \sigma_2^2 z_2^2 - \rho z_1 z_2
\]
where $z_1 + p z_2 \leq E$. Assume that $\sigma_1^2 \sigma_2^2 > \rho^2$.

i. If you can short an asset, or buy a negative amount of it, what is the solution to the optimal portfolio problem? (You don’t have to solve the system of first-order conditions explicitly for closed form solutions, but explain whether there is a maximum or not and whether it’s unique). Provide a brief sketch of the geometry of the solution for each case.

ii. If you cannot short assets, so $z_1, z_2 \geq 0$, what is the solution to the optimal portfolio problem? Under what conditions do you buy one asset but not the other? Do you ever buy neither asset? (You don’t have to solve the system of first-order conditions explicitly for a closed form solutions, but explain whether there is a maximum or not and whether it’s unique). Provide a brief sketch of the geometry of the solution for each case.

iii. For part ii, in the case where the non-negativity constraints are slack but the constraint $z_1 + p z_2 \leq E$ binds with an equality, how does $z_1$ vary with $\rho$ and $p$? Provide an economic interpretation of your result.

iv. How does the optimized value of the portfolio vary with $\rho$? Provide an economic interpretation of your result. (“Hedge” might be a useful word to use in your answer.)

Note that the objective is a quadratic form, which Hessian
\[
\begin{bmatrix}
-\sigma_1^2 & -\rho \\
-\rho & -\sigma_2^2
\end{bmatrix}
\]
which is negative definite from the condition that $\sigma_1^2 \sigma_2^2 > \rho^2$. Consequently, the objective function is strictly concave. As long as there is a critical point, it is the unique solution. In particular, we don’t have to worry about unbounded arbitrage, where we can make infinite profits by infinitely shorting one of the assets. More mathematically, the unconstrained maximization problem has a unique, bounded solution, so that if we impose additional constraints, we’ll simply pick the feasible point closest to the unconstrained maximum.

i. In this case, we don’t have non-negativity constraints, so the Lagrangean is (I’ll use $\phi$’s as multipliers since $\mu$ already shows up as the mean)
\[
\mathcal{L} = \mu_1 z_1 + \mu_2 z_2 - \frac{1}{2} \sigma_1^2 z_1^2 - \frac{1}{2} \sigma_2^2 z_2^2 - \rho z_1 z_2 - \phi (z_1 + p z_2 - E)
\]
with FONC’s
\[ \mu_1 - \sigma_1^2 z_1 - \rho z_2 - \phi = 0 \]
\[ \mu_2 - \sigma_2^2 z_2 - \rho z_1 - \phi p = 0 \]
and complementary slackness condition \( \phi \geq 0 \) and
\[ \phi(z_1 + p z_2 - E) = 0 \]
Note that even in this case, we might not spend all of the money. There are two cases:

- \( \phi = 0 \) so that we don’t spend all our money, and
  \[ \mu_1 - \sigma_1^2 z_1 - \rho z_2 = 0 \]
  \[ \mu_2 - \sigma_2^2 z_2 - \rho z_1 = 0 \]
- \( \phi \geq 0 \) and the budget constraint is binding,
  \[ \mu_1 - \sigma_1^2 z_1 - \rho z_2 - \phi = 0 \]
  \[ \mu_2 - \sigma_2^2 z_2 - \rho z_1 - \phi p = 0 \]
  \[ z_1 + p z_2 - E = 0 \]

ii. Now we have non-negativity constraints, so the Lagrangean is
\[ L = \mu_1 z_1 + \mu_2 z_2 - \frac{1}{2} \sigma_1^2 z_1^2 - \frac{1}{2} \sigma_2^2 z_2^2 - \rho z_1 z_2 - \phi(z_1 + p z_2 - E) + \phi_1 z_1 + \phi_2 z_2 \]
with FONC’s
\[ \mu_1 - \sigma_1^2 z_1 - \rho z_2 - \phi + \phi_1 = 0 \]
\[ \mu_2 - \sigma_2^2 z_2 - \rho z_1 - \phi p + \phi_2 = 0 \]
and complementary slackness conditions
\[ \phi(z_1 + p z_2 - E) = 0, \phi_1 z_1 = 0, \phi_2 z_2 = 0 \]
with \( \phi, \phi_1, \phi_2 \geq 0 \).

So in this world, we really do have \( 2^3 - 1 \) cases, since the global maximum of the unconstrained problem can be anywhere, except \( z_1, z_2 < 0 \), since \( \mu_1, \mu_2 > 0 \).

- Do we ever choose \( z_1, z_2 = 0 \)? This implies \( \phi = 0, \phi_1, \phi_2 > 0 \), requiring
  \[ \mu_1 + \phi_1 = 0 \]
  \[ \mu_2 + \phi_2 = 0 \]
  which yields a contradiction, since \( \mu_1, \mu_2 > 0 \). So we always buy at least some of one of the assets.
  (one case)
- Do we ever buy \( z_1 \) but not \( z_2 \)? This requires \( \phi_1 = 0 \) but \( \phi_2 \geq 0 \), and we’ll leave \( \phi \) in to cover two cases at once. Then
  \[ \mu_1 - \sigma_1^2 z_1 - \phi = 0 \]
  \[ \mu_2 - \rho z_1 - \phi p + \phi_2 = 0 \]
  \[ \phi(z_1 - E) = 0, \phi \geq 0 \]
  So we fail to buy any \( z_2 \) if it’s the case that
  \[ \mu_2 \leq \rho z_1^* + \phi p \]
  Since \( \rho < 0 \) and \( z_1^* > 0 \), this can only occur if the budget constraint binds: Basically, we can have a corner solution only if \( z_1 \) is such a good deal that adding any \( z_2 \) loses us money. But if we aren’t spending all of our capital on \( z_1 \), then we definitely want to buy some \( z_2 \). (two cases)
• In the case that we buy $z_2$ but not $z_1$, it looks like the above case with the roles reversed. (two cases)
• At a solution with $z_1, z_2 > 0$, we have the system

\[
\begin{align*}
\mu_1 - \sigma_1^2 z_1 - \rho z_2 - \phi &= 0 \\
\mu_2 - \sigma_2^2 z_2 - \rho z_1 - \phi p &= 0 \\
\phi(z_1 + pz_2 - E) &= 0, \phi \geq 0
\end{align*}
\]

The budget constraint fails to bind if

\[
\begin{align*}
\mu_1 - \sigma_1^2 z_1 - \rho z_2 &\leq 0 \\
\mu_2 - \sigma_2^2 z_2 - \rho z_1 &\leq 0
\end{align*}
\]

at the optimum. (two cases)

So that was 7 cases, and we are done.

iii. Use the implicit function theorem.

iv. The comparative static is

\[
V'(\rho) = -z_1 z_2 < 0
\]

so that profits are decreasing in $\rho$. Basically, $\rho < 0$ means that the assets are negatively correlated, so that when one does well the other does poorly in expectation. This allows us to use one to hedge the risks associated with the other. As $\rho$ increases, this hedging feature vanishes, increasing the portfolio’s variance and reducing profits.

3. There is a single representative consumer with utility function $u(q, m) = \phi(q) + m$ over bundles of quantity of the market good, $q$, and money $m$. Assume $\phi'(q) > 0$, $\phi''(q) < 0$, and

\[
\lim_{q \to \infty} \phi'(q) = 0
\]

The consumer maximizes utility subject to a budget constraints $w = pq + m$. There are $j = 1, 2, ..., J$ firms who each have a cost function $c(q_j, \theta_j)$ which satisfy

\[
c(0, \theta_j) = 0, \quad \lim_{q_j \to \infty} c(q_j, \theta_j) = \infty, \quad \frac{\partial c(q_j, \theta_j)}{\partial q_j} > 0, \quad \frac{\partial^2 c(q_j, \theta_j)}{\partial q_j^2} > 0, \quad \frac{\partial c(q_j, \theta_j)}{\partial \theta_j} < 0, \quad \text{and} \quad \frac{\partial^2 c(q_j, \theta_j)}{\partial \theta_j \partial q_j} < 0
\]

Interpret $\theta_j$ as the quality of firm $j$’s technology parameter.

i. What is the solution to the consumer’s utility problem? What is the solution to the firm’s profit maximization problem? Provide some proof that your solution is a global optimum.

ii. Define a perfectly competitive equilibrium and characterize it for this model. Does an equilibrium always exist? How does the equilibrium quantity traded vary in a given firm’s technology parameter, $\theta_j$? How does a firm’s profits vary in its technology parameter, $\theta_j$?

iii. Now define a research and development equilibrium in $(\theta_1, \theta_2, ..., \theta_J)$ as follows:

1. All firms take the technology parameters of the other firms as given, and choose their own research parameter to maximize

\[
\max_{\theta_i} \pi(\theta_1, \theta_2, ..., \theta_J) - k\theta_j
\]

where $\pi(\theta_j)$ is the profit function from part i.

2. Given the technology parameters selected by the other firms, no firm wishes to change its technology parameter.
Provide a system of equations that characterize a research and development equilibrium.

iv. If the firms all acted together in selecting their research and development parameters in part iii, rather than doing it separately, would they achieve the same result? If you are done with the rest of the exam: Do the individual firms over- or under-invest in their technology parameters relative to when they act together?

i. The consumer solves
\[ \max_q \phi(q) - pq + w \]
with FONC
\[ \phi'(q^*) = p \]
and SOS\(\text{C}\)
\[ \phi''(q^*) < 0 \]
which is automatically satisfied since \(\phi(q)\) is concave, so there is a unique solution. This defines the demand curve.

Seller \(j\) solves
\[ \max_{q_j} pq_j - c(q_j, \theta_j) + \mu_j q_j \]
with FONC
\[ p - c_q(q^*_j, \theta_j) + \mu_j = 0 \]
so that \(p = c_q(q^*_j, \theta_j)\) if \(q^*_j \geq 0\), and \(p < c_q(0, \theta_j)\) if \(q^*_j = 0\), and at an interior solution, the SOS\(\text{C}\) is
\[ -c_{qq}(q^*_j, \theta_j) < 0 \]
which is automatically satisfied since \(c(q_j, \theta_j)\) is convex in \(q_j\). This defines firm \(j\)’s supply curve.

ii. A price-taking equilibrium is an allocation \((q^*; q^*_1, ..., q^*_J)\) and price \(p^*\) so that (i) the consumer’s utility is maximized at \(q^*\), taking the price \(p^*\) as given, (ii) each firm \(j\) maximizes profits by selecting quantity \(q^*_j\) taking the price \(p^*\) as given, (iii) markets clear, so that \(q^* = \sum_{j=1}^J q^*_j\).

To show equilibrium existence, define the excess demand function
\[ E(p) = q^*(p) - \sum_{j=1}^J q^*_j(p) \]
Note that \(q^*(p)\) is decreasing in \(p\) while each \(q^*_j(p)\) is increasing in \(p\), by the implicit function theorem applied to the FONC’s for each agent. Therefore, \(E(p)\) is a continuous, decreasing function. Also, \(q^*(p) \to 0\) as \(p \to \infty\) since \(\lim_{q \to \infty} \phi'(q) = 0\), but \(q^*_j(p)\) is strictly increasing, so that we know that there is some \(\bar{p}\) for which \(E(\bar{p}) < 0\). So as long as \(E(0) > 0\), continuity of \(E(p)\) and the intermediate value theorem imply there exists a \(p^*\) for which \(E(p^*) = 0\), and the market clears. So we need
\[ E(0) = q^*(0) - \sum_{j=1}^J q^*_j(0) > 0 \]
as the sufficient condition for existence. An easy assumption would be \(\lim_{q \to 0} \phi'(0) = \infty\), or \(\phi'(0) > \min_j c_q(0, \theta_j)\).

There’s a bunch of ways to do comparative statics here. This is the fastest way I found. The system is
\[ \phi'(q^*) - p^* = 0 \]
and letting \(A\) be the set of all firms \(j\) such that \(q^*_j \geq 0\),
\[ j \in A : p^* - c_q(q^*_j, \theta_j) = 0 \]
and
\[ q^* - \sum_{j \in A} q^*_j = 0 \]
We have a ton of endogenous variables though. Let’s eliminate some by substitution:

\[ j \in A : \phi'(q^*) - c_{q_j}(q^*_j, \theta_j) = 0 \]

Now, let’s perturb firm 1’s technology, holding all others constant, assuming \( q^*_1 > 0 \):

\[
\phi''(q^*) \frac{dq^*}{d\theta_1} - c_{qq}(q^*_1, \theta_1) \frac{\partial q^*_1}{\partial \theta_1} - c_{q\theta}(q^*_1, \theta_1) = 0
\]

\[
\phi''(q^*) \frac{dq^*}{d\theta_1} - c_{q\theta}(q^*_j, \theta_j) \frac{\partial q^*_j}{\partial \theta_1} = 0
\]

Note that we can rewrite the above two equations to isolate \( \frac{\partial q^*_j}{\partial \theta_1} \) as

\[
\frac{\partial q^*_1}{\partial \theta_1} = \frac{\phi''(q^*) \frac{dq^*}{d\theta_1} - c_{q\theta}(q^*_1, \theta_1)}{c_{qq}(q^*_1, \theta_1)}
\]

\[
\frac{\partial q^*_j}{\partial \theta_1} = \frac{\phi''(q^*) \frac{dq^*}{d\theta_1}}{c_{qq}(q^*_j, \theta_j)}
\]

Summing over \( j \in A \) yields

\[
\frac{dq^*}{d\theta_1} = \sum_{j \in A} \frac{\phi''(q^*) \frac{dq^*}{d\theta_1} - c_{q\theta}(q^*_j, \theta_j)}{c_{qq}(q^*_j, \theta_j)}
\]

and solving yields

\[
\frac{dq^*}{d\theta_1} = \frac{-c_{q\theta}(q^*_1, \theta_1)/c_{qq}(q^*_1, \theta_1)}{1 - \sum_{j \in A} \frac{\phi''(q^*)}{c_{qq}(q^*_j, \theta_j)}} > 0
\]

So when one firm’s technology improves, the model predicts that there will be an aggregate increase in quantity provided.

By the envelope theorem,

\[
\frac{\partial \pi(\theta_j)}{\partial \theta_j} = -c_{q_j}(q^*_j, \theta_j) > 0
\]

iii. In an R&D equilibrium, we have the maximization problem for each firm

\[
\max_{\theta_j} \pi(q^*_j(\theta_1, \ldots, \theta_j), \theta_j) - k \theta_j
\]

where \( \pi(q^*_j, \theta_j) \) is the payoff from the previous parts. Then the envelope theorem implies

\[
-c_{\theta_j}(q^*_j, \theta_j) - k + \mu_j = 0
\]

where \( \mu_j \geq 0 \) and \( \mu_j \theta_j = 0 \). If the above FONC holds for each firm given the technology selections of the other firms, it is an R&D equilibrium.

A social planner would solve

\[
\max_{\theta_1, \ldots, \theta_j} \sum_j \pi(q^*_j(\theta_1, \ldots, \theta_j), \theta_j) - k \theta_j
\]

with FONC for \( \theta_j \)

\[
\sum_{i \neq j} \frac{\partial \pi(q^*_i, \theta_j^*)}{\partial \theta_j} - c_{\theta_j}(q^*_j, \theta_j) - k + \mu_j = 0
\]

Note that while firm \( j \) doesn’t take into account the impact of its investment on the other firms in the product market, the social planner does. Consequently, there are a bunch of negative terms added to
the above FONC missing from firm $j$’s FONC. Therefore, firm $j$ is over-investing relative to the social planner, since

$$\sum_{i \neq j} \frac{\partial \pi(q^*_i, \theta^*_i)}{\partial \theta_j} - c_{\theta_j}(q^*_j, \theta_j) - k < -c_{\theta_j}(q^*_j, \theta_j) - k$$

and the right-hand side of the inequality is decreasing in $\theta_j$. This can be understood as firm $j$ acting to intimidate other firms by becoming more efficient, leading to losses of profits for all of the firms together.