Problem Set 2: Unconstrained and Equality-Constrained Maximization in $\mathbb{R}^N$

2. (i) Find all critical points of $f(x, y) = (x^2 - 4)^2 + y^2$ and show which are maxima and which are minima. (ii) Find all critical points of $f(x, y) = (y - x^2)^2 - x^2$ and show which are maxima and which are minima.

(i) The gradient at any critical point is
\[ \nabla f(x, y) = (4(x^2 - 4)x, 2y) = 0 \]
Thus $y$ must be zero. $x$ can be zero or $\pm \sqrt{2}$. This gives us three critical points. The Hessian is
\[ H(x, y) = \begin{bmatrix} 12x^2 - 16 & 0 \\ 0 & 2 \end{bmatrix} \]
The critical point $(0, 0)$ is neither a maximum nor a minimum, since the Hessian has eigenvalues 0 and 2. At $(\pm \sqrt{2}, 0)$, the Hessian has eigenvalues 2 and 32, so these critical points are minima.

(ii) The gradient at any critical point is
\[ \nabla f(x, y) = (-4(y - x^2)x - 2x, 2(y - x^2)) = 0 \]
The second equation gives $y = x^2$, and substituting this into the first equation gives
\[ -4(x^2 - x^2) - 2x = 0 \]
so $x = 0$, $y = 0$ is a critical point. The Hessian is
\[ H(x, y) = \begin{bmatrix} -4(y - x^2) + 12x^2 & -4x \\ -4x & 2 \end{bmatrix} \]
Evaluating at $(0, 0)$ yields
\[ H(x, y) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \]
which has eigenvalues $-2$ and 2, so $(0, 0)$ is neither a maximum nor a minimum.

3. Suppose a firm produces two goods, $q_1$ and $q_2$, whose prices are $p_1$ and $p_2$, respectively. The costs of production are $C(q_1, q_2)$. (i) Provide necessary and sufficient conditions for a production plan $(q_1^*, q_2^*)$ to be profit-maximizing and show how $q_1^*$ varies with $p_2$. If $C(q_1, q_2) = c_1(q_1) + c_2(q_2) + b q_1 q_2$, explain when a critical point is a local maximum of the profit function. (ii) If $C(q_1, q_2) = c_1(q_1) + c_2(q_2) + b q_1 q_2$, how does $q_1$ vary with $b$ and $p_1$? How do profits vary with $b$ and $p_1$?

(i) The firm’s profit-maximization problem is
\[ \max_{q_1, q_2} p_1 q_1 + p_2 q_2 - C(q_1, q_2) \]
The FONCs are
\[ p_1 - C_1(q_1^*, q_2^*) = 0 \]
\[ p_2 - C_2(q_1^*, q_2^*) = 0 \]
The SOSCs are that
\[
\begin{bmatrix}
C_{11}(q_1^*, q_2^*) & C_{12}(q_1^*, q_2^*) \\
C_{21}(q_1^*, q_2^*) & C_{22}(q_1^*, q_2^*)
\end{bmatrix}
\]
is negative definite. With the functional form \(C(q_1, q_2) = c_1(q_1) + c_2(q_2) + bq_1q_2\), we get a Hessian
\[
\begin{bmatrix}
-c''_1(q_1^*) & -b \\
-b & -c''_2(q_2^*)
\end{bmatrix}
\]
which must satisfy \(-c''_1(q_1^*) < 0, -c''_2(q_2^*) < 0,\) and \(c''_1(q_1^*)c''_2(q_2^*) - b^2 > 0\) to be negative definite. So \(b\) cannot be “too large” relative to the second derivatives of \(c_1(q_1)\) and \(c_2(q_2)\), or the “economies of scope” make the critical point characterized above a global minimum.

(ii) Profits are easy. The profit function is
\[
V(b, p_1) = [p_1q_1 + p_2q_2 - c_1(q_1) - c_2(q_2) - bq_1q_2]_{(q_1, q_2) = (q_1^*, q_2^*)}
\]
The envelope theorem implies
\[
\frac{\partial V}{\partial b} = -q_1^*q_2^* < 0
\]
\[
\frac{\partial V}{\partial p_1} = q_1^* > 0
\]
To compute the comparative statics, we totally differentiate the FONCs,
\[
p_1 - c'_1(q_1^*) - bq_2^* = 0
\]
\[
p_2 - c'_2(q_2^*) - bq_1^* = 0
\]
and re-write the result as a matrix equation,
\[
\begin{bmatrix}
-c''_1(q_1) & -b \\
-b & -c''_2(q_2)
\end{bmatrix}
\begin{bmatrix}
\partial q_1^*/\partial b \\
\partial q_2^*/\partial b
\end{bmatrix}
= \begin{bmatrix}
-q_2^* \\
-q_1^*
\end{bmatrix}
\]
Cramer’s rule then implies
\[
\frac{\partial q_1^*}{\partial b} = \frac{\det\begin{bmatrix}
-q_2^* & -b \\
-q_1^* & -c''_2(q_2^*)
\end{bmatrix}}{\det H} = \frac{q_2^*c''_2(q_2^*) - bq_1^*}{\det H}
\]
The denominator \(\det H\) is positive because we are at a maximum, and the numerator is positive if \(q_2^*c''_2(q_2^*) - bq_1^* > 0\). This is ambiguous, so we should expect changes in the spillover effect to have uncertain changes on firm behavior. Similarly,
\[
\begin{bmatrix}
-c''_1(q_1) & -b \\
-b & -c''_2(q_2)
\end{bmatrix}
\begin{bmatrix}
\partial q_1^*/\partial p_1 \\
\partial q_2^*/\partial p_2
\end{bmatrix}
= \begin{bmatrix}
-1 \\
0^*
\end{bmatrix}
\]
And using Cramer’s rule,
\[
\frac{\partial q_1^*}{\partial p_1} = \frac{\det\begin{bmatrix}
-1 & -b \\
0 & -c''_2(q_2^*)
\end{bmatrix}}{\det H} = \frac{c''_2(q_2^*)}{\det H} > 0
\]
This is unambiguously positive (actually, this is just the law of supply, right? We’ll see this always holds for price-taking firms).
4. A consumer with utility function \( u(q_1, q_2, m) = (q_1 - \gamma_1)q_2^\alpha + m \) and budget constraint \( w = p_1q_2 + p_2q_2 + m \) is trying to maximize utility. (i) Solve for the optimal bundle \((q_1^*, q_2^*, m^*)\) and check the second-order sufficient conditions. (ii) Show how \( q_1^* \) varies with \( p_2 \), and how \( q_2^* \) varies with \( p_1 \), both using the closed-form solutions and the implicit function theorem. How does the value function vary with \( \gamma_1 \)? Briefly provide an economic interpretation for the parameter \( \gamma_1 \).

(i) The maximization problem is

\[
\max_{q_1, q_2} (q_1 - \gamma_1)q_2^\alpha + w - p_1q_1 - p_2q_2
\]

with FONCs

\[
q_2^\alpha - p_1 = 0 \\
(q_1 - \gamma_1)\alpha q_2^{\alpha-1} - p_2 = 0
\]

The critical point then is

\[
q_2^* = p_1^{1/\alpha} \\
q_1^* = \frac{p_2}{\alpha p_1^{(\alpha-1)/\alpha}} + \gamma_1
\]

and \( m^* = w - p_1q_1^* - p_2q_2^* \).

The Hessian is

\[
H(q_1, q_2) = \begin{bmatrix}
0 & \alpha q_2^{\alpha-1} \\
\alpha q_2^{\alpha-1} & (q_1 - \gamma_1)\alpha(\alpha - 1)q_2^{\alpha-2}
\end{bmatrix}
\]

which has leading determinants 0 and \(- \left(\alpha q_2^{\alpha-1}\right)^2 < 0\). So this is, yikes, not negative definite. We cannot conclude this is a maximum (actually, we will be able to prove it later, since the objective function is “quasi-concave”).

(ii) As for comparative statics, we are in somewhat awkward territory because we are not sure that the proposed critical point is an optimum. We can still try to use the implicit function theorem, but we cannot assume that the Hessian has the alternating sign pattern associated with a negative definite matrix. Differentiating the closed form solution yields

\[
\frac{\partial q_1^*}{\partial p_2} = \frac{1}{\alpha p_1^{(\alpha-1)/\alpha}}
\]

To use the IFT, we totally differentiate the FONCs to get

\[
\alpha q_2^{\alpha-1}\frac{\partial q_2^*}{\partial p_2} = 0
\]

\[
\frac{\partial q_2^*}{\partial p_2}\alpha q_2^{\alpha-1} + (q_1^* - \gamma_1)\alpha(\alpha - 1)q_2^{\alpha-2}\frac{\partial q_2^*}{\partial p_2} - 1 = 0
\]

Rewriting this as a matrix equation yields

\[
\begin{bmatrix}
0 & \alpha q_2^{\alpha-1} \\
\alpha q_2^{\alpha-1} & (q_1^* - \gamma_1)\alpha(\alpha - 1)q_2^{\alpha-2}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial q_1^*}{\partial p_2} \\
\frac{\partial q_2^*}{\partial p_2}
\end{bmatrix}
= \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

Using Cramer’s rule to solve for \( \frac{\partial q_1^*}{\partial p_2} \) yields

\[
\frac{\partial q_1^*}{\partial p_2} = \frac{\det \begin{bmatrix}
0 & \alpha q_2^{\alpha-1} \\
1 & (q_1^* - \gamma_1)\alpha(\alpha - 1)q_2^{\alpha-2}
\end{bmatrix}}{\det(H)}
\]
or
\[
\frac{\partial q_1^*}{\partial p_2} = -\alpha q_2^{*\alpha-1} = \frac{1}{\alpha q_2^{*\alpha-1}}
\]
Which gives the same result as differentiating the closed form solution.

The optimized value of the consumer’s utility is given by
\[
V(\gamma_1) = [(q_1 - \gamma_1)q_2^\alpha + w - p_1q_1 - p_2q_2]|_{(q_1, q_2) = (q^*_1, q^*_2)}
\]
and the Envelope Theorem implies
\[
V'(\gamma_1) = -q_2^{*\alpha} < 0
\]
so that an increase in \(\gamma_1\) makes the consumer worse off. The parameter \(\gamma_1\) is the minimum amount of \(q_1\) that the agent must consume to get positive utility from consuming any \(q_1\) or \(q_2\). Notice that if the consumer can’t afford enough \(\gamma_1\), he should just put all of his wealth into \(m\), which gives a constant marginal utility of 1. Let’s think about \(q_1\) as the quality of a “primary good” like a computer, electric guitar, or camera, and \(q_2\) as “secondary” or complementary goods like software, an amplifier, or lenses; anything that you also require to make \(q_1\) worth owning. Then the consumer needs to purchase a primary good of quality at least \(\gamma_1\), or the payoff is actually negative, and the consumer regrets making the purchase, no matter how good the secondary goods are that he also purchases.

5. Consider the maximization problem
\[
\max_{x_1, x_2} x_1 + x_2
\]
subject to
\[
x_1^2 + x_2^2 = 1
\]
Sketch the constraint set and contour lines of the objective function. Find all critical points of the Lagrangian. Verify whether each critical point is a local maximum or a local minimum. Find the global maximum.

You should sketch the constraint set and contour lines/indifference curves. The constraint is a circle of radius one, and the objective is a linear hyperplane with slope \(-1\).

The Lagrangian is
\[
\mathcal{L} = x_1 + x_2 - \lambda(x_1^2 + x_2^2 - 1)
\]
with FONCs
\[
1 - \lambda x_1^* = 0
\]
\[
1 - \lambda x_2^* = 0
\]
\[
-(x_1^*^2 + x_2^*^2 - 1) = 0
\]
The first two equations imply that \(x_1^* = x_2^*\). The third equation implies that \(x_1^{*2} + x_2^{*2} = 1\), or
\[
x_k^* = \pm \frac{1}{\sqrt{2}}
\]
So there are two potential candidates,
\[
\left(\frac{\sqrt{1}}{\sqrt{2}}, \frac{\sqrt{1}}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),
\]
The Bordered Hessian is
\[
\begin{bmatrix}
0 & -2x_1 & -2x_2 \\
-2x_1 & -2\lambda & 0 \\
-2x_2 & 0 & -2\lambda
\end{bmatrix}
\]
Since \(\lambda\) appears in it, we’ll need to compute it as well, which is different from many examples we saw in class. Also different is that the \(x_i\)'s appear along the top and left-hand borders. The determinant of the third principal minor of the Bordered Hessian is
\[
2x_1(4x_1\lambda) + 2x_2(4\lambda x_2) = 8\lambda(x_1^2 + x_2^2)
\]
So that the sign only depends on the multiplier, \(\lambda\). Since
\[
\lambda = \frac{1}{2x_1^*} = \frac{1}{2x_2^*}
\]
it will be positive if both terms are positive, and negative if both terms are negative. Consequently, the critical point
\[
\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)
\]
is a maximum while
\[
\left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)
\]
is actually a minimum.
Since the two critical points where \(x^*\) and \(y^*\) have the same sign are local maxima and yield the same value of the objective function, they are both global maxima.

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6. Consider the maximization problem

\[
\max_{x,y} \quad xy
\]
subject to
\[
\frac{a}{2}x^2 + \frac{b}{2}y^2 = r
\]
Sketch the constraint set and contour lines of the objective function. Find all critical points of the Lagrangian. Verify whether each critical point is a local maximum or a local minimum. Find the global maximum. How does the value of the objective vary with \(r\)? \(a\)? How does \(x^*\) respond to a change in \(r\); show this using the closed-form solutions and the IFT.

Sketch. The constraint set is an ellipse and the objective function is “Cobb-Douglas”. The Lagrangian is
\[
L(x, \lambda) = xy - \lambda \left(\frac{a}{2}x^2 + \frac{b}{2}y^2 - r\right)
\]
with FONCs
\[
y - \lambda ax = 0 \\
x - \lambda by = 0 \\
-(\frac{a}{2}x^2 + \frac{b}{2}y^2 - r) = 0
\]
The first two equations imply that \( y/x = ax/(by) \), or \( by^2 = ax^2 \). Using the third equation, we get
\[
ax^2 = r
\]
or
\[
x^* = \pm \sqrt{\frac{r}{a}}
\]
implying that
\[
y^* = \pm \sqrt{\frac{r}{b}}
\]
so we have four critical points,
\[
\left( \sqrt{\frac{r}{a}}, \sqrt{\frac{r}{b}} \right), \left( \sqrt{\frac{r}{a}}, -\sqrt{\frac{r}{b}} \right), \left( -\sqrt{\frac{r}{a}}, \sqrt{\frac{r}{b}} \right), \left( -\sqrt{\frac{r}{a}}, -\sqrt{\frac{r}{b}} \right)
\]
The constraint is non-vanishing if the radius must be 1, and the gradient of the objective is non-vanishing on the constraint set, so there are no non-differentiabilities or violations of the qualification constraint.

Now that we have a candidate list, we can check the SOSCs using the bordered Hessian:
\[
\begin{bmatrix}
0 & -ax & -by \\
-ax & -a\lambda & 1 \\
-by & 1 & -b\lambda
\end{bmatrix}
\]
The determinant of the third principal leading minor is
\[
ax(axb\lambda + by) + by(ax + a\lambda by)
\]
which equals
\[
ab\lambda(x^2 + y^2) + 2axby = ab\lambda r + 2axby = ab(\lambda r + 2xy)
\]
For the solutions with both \( x^* \) and \( y^* \) strictly positive or strictly negative, this is positive, and we have a local maximum, since
\[
\lambda = \frac{y}{ax}
\]
so that the multiplier has the same sign as \( y/x \).

For the solutions with one of \( x^* \) and \( y^* \) positive and one of \( x^* \) and \( y^* \) negative, we have a negative outcome, which means it is a local minimum.

7. Solve the following two dimensional maximization problems subject to the linear constraint \( w = p_1 x_1 + p_2 x_2 \), \( p_1, p_2 > 0 \). Sketch contour lines of the objective function and the constraint set. Then compute the change in \( x_1 \) with respect to \( p_2 \), and a change in \( x_1 \) with respect to \( w \). Assume \( \alpha_1 > \alpha_2 > 0 \). For \( i \) and \( ii \), when do the SOSCs hold?

i. Cobb-Douglas

The Lagrangian is
\[
\mathcal{L}(x_1, x_2, \lambda) = x_1^{\alpha_1} x_2^{\alpha_2} - \lambda(p_1 x_1 + p_2 x_2 - w)
\]
The FONCs are
\[
\alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} - \lambda p_1 = 0
\]
\[ x_1^{\alpha_1} x_2^{\alpha_2-1} - \lambda p_2 = 0 \]
\[ -(p_1 x_1 + p_2 x_2 - w) = 0 \]

The first two equations imply that
\[ \frac{\alpha_1 x_2}{\alpha_2 x_1} = \frac{p_1}{p_2} \]

Then
\[ x_2 = \frac{p_1 \alpha_2}{p_2 \alpha_1} x_1 \]

Substituting this into the constraint yields
\[ p_1 x_1 + \frac{p_1 \alpha_2}{\alpha_1} x_1 = w \]

so that
\[ x_1^* = \frac{\alpha_1}{p_1 (\alpha_2 + \alpha_1)} w \]

and
\[ x_2^* = \frac{\alpha_2}{p_2 (\alpha_2 + \alpha_1)} w \]

Using the closed form solution above,
\[ \frac{\partial x_1^*}{\partial w} = \frac{\alpha_1}{p_1 (\alpha_2 + \alpha_1)} \]

and
\[ \frac{\partial x_1^*}{\partial p_2} = 0 \]

I highly recommend doing these comparative statics using the implicit function theorem for practice.

The bordered Hessian is
\[
\begin{bmatrix}
0 & -p_1 & -p_2 \\
-p_1 & \alpha_1 (\alpha_1 - 1) x_1^{\alpha_1-2} x_2^{\alpha_2} & \alpha_1 \alpha_2 x_1^{\alpha_1-1} x_2^{\alpha_2-1} \\
-p_2 & \alpha_1 \alpha_2 x_1^{\alpha_1-1} x_2^{\alpha_2-1} & \alpha_2 (\alpha_2 - 1) x_1^{\alpha_1} x_2^{\alpha_2-2}
\end{bmatrix}
\]

The determinant of the third principal leading minor is
\[ p_1 (-p_1 \alpha_2 (\alpha_2 - 1) x_1^{\alpha_1} x_2^{\alpha_2-2} + \alpha_1 \alpha_2 x_1^{\alpha_1-1} x_2^{\alpha_2-1} - p_2) - p_2 (-p_1 \alpha_1 \alpha_2 x_1^{\alpha_1-1} x_2^{\alpha_2-1} + p_2 \alpha_1 (\alpha_1 - 1) x_1^{\alpha_1-2} x_2^{\alpha_2}) \]

Note that \( x_1^{\alpha_1} x_2^{\alpha_2} \) appears in every term. Divide these out to get
\[ p_1 (-p_1 \alpha_2 (\alpha_2 - 1) x_2^{-2} + \alpha_1 \alpha_2 x_1^{-1} x_2^{\alpha_2-1} - p_2) + p_2 (p_1 \alpha_1 \alpha_2 x_1^{-1} x_2^{-1} - p_2 \alpha_1 (\alpha_1 - 1) x_1^{-2}) \]

This will be positive (the SOSCs will be satisfied) if
\[ p_1 \alpha_1 \alpha_2 x_1^{-1} x_2^{-1} p_2 + p_2 \alpha_1 \alpha_2 x_1^{-1} x_2^{-1} > p_1^2 \alpha_2 (\alpha_2 - 1) x_2^{-2} + p_2^2 \alpha_1 (\alpha_1 - 1) x_1^{-2} \]

or
\[ 2p_1 p_2 x_1^{-1} x_2^{-1} \alpha_1 \alpha_2 > p_1^2 \alpha_2 (\alpha_2 - 1) x_2^{-2} + p_2^2 \alpha_1 (\alpha_1 - 1) x_1^{-2} \quad (1) \]

So a set of sufficient conditions to ensure this are that \( 0 < \alpha_1 < 1 \) and \( 0 < \alpha_2 < 1 \), so that the left-hand side is positive but the right-hand side is negative.
ii. Stone-Geary

\[ f(x) = (x - \gamma_1)^{\alpha_1}(x - \gamma_2)^{\alpha_2} \]

The Lagrangian is

\[ \mathcal{L}(x, \lambda) = (x - \gamma_1)^{\alpha_1}(x - \gamma_2)^{\alpha_2} - \lambda(p_1 x_1 + p_2 x_2 - w) \]

The FONCs are

\[
\begin{align*}
\alpha_1(x - \gamma_1)^{\alpha_1-1}(x - \gamma_2)^{\alpha_2} - \lambda p_1 &= 0 \\
\alpha_2(x - \gamma_1)^{\alpha_1}(x - \gamma_2)^{\alpha_2-1} - \lambda p_2 &= 0 \\
-(p_1 x_1 + p_2 x_2 - w) &= 0
\end{align*}
\]

The first two equations imply

\[ \frac{\alpha_1(x_2 - \gamma_2)}{\alpha_2(x_1 - \gamma_1)} = \frac{p_1}{p_2} \]

Solving for \( x_2 \) yields

\[ x_2 = \frac{p_1}{\alpha_1 p_2} \alpha_2(x_1 - \gamma_1) + \gamma_2 \]

Substituting this into the constraint yields

\[ p_1 x_1 + \frac{p_1}{\alpha_1} \alpha_2(x_1 - \gamma_1) + p_2 \gamma_2 = w \]

Solving for \( x_1 \) yields

\[
\begin{align*}
x_1^* &= \frac{\alpha_1 w - p_2 \alpha_1 \gamma_2 + p_1 \alpha_2 \gamma_1}{p_1 (\alpha_1 + \alpha_2)} \\
x_2^* &= \frac{\alpha_2 w - p_1 \alpha_2 \gamma_1 + p_2 \alpha_1 \gamma_2}{p_2 (\alpha_2 + \alpha_1)}
\end{align*}
\]

and the comparative statics are

\[
\begin{align*}
\frac{\partial x_1^*}{\partial w} &= \frac{-\alpha_1 \gamma_2}{p_1 (\alpha_1 + \alpha_2)} < 0 \\
\frac{\partial x_1^*}{\partial p_2} &= \frac{-\alpha_2 \gamma_2}{p_1 (\alpha_1 + \alpha_2)} < 0
\end{align*}
\]

The bordered Hessian is

\[
\begin{bmatrix}
0 & -p_1 \\
-p_1 & \alpha_1(\alpha_1 - 1)(x_1 - \gamma_1)^{\alpha_1-2}(x_2 - \gamma_2)^g + \alpha_2(\gamma_1)^{\alpha_1-1}(x_2 - \gamma_2)^{\alpha_2-1} \\
-p_2 & -\alpha_1 \alpha_2(x_1 - \gamma_1)^{\alpha_1-1}(x_2 - \gamma_2)^{\alpha_2-1} + \alpha_2(\alpha_2 - 1)(x_1 - \gamma_1)^{\alpha_1-2}(x_2 - \gamma_2)^g
\end{bmatrix}
\]

which has determinant

\[
\begin{align*}
p_1(-p_1 \alpha_2(\alpha_2 - 1)(x_1 - \gamma_1)^{\alpha_1}(x_2 - \gamma_2)^{\alpha_2-2} + p_2 \alpha_1 \alpha_2(x_1 - \gamma_1)^{\alpha_1-1}(x_2 - \gamma_2)^{\alpha_2-1}) \\
&p_2(-p_1 \alpha_1 \alpha_2(x_1 - \gamma_1)^{\alpha_1-1}(x_2 - \gamma_2)^{\alpha_2-1} + p_2 \alpha_1(\alpha_2 - 1)(x_1 - \gamma_1)^{\alpha_1-2}(x_2 - \gamma_2)^g)
\end{align*}
\]

which equals

\[
\begin{align*}
p_1(-p_1 \alpha_2(\alpha_2 - 1)(x_2 - \gamma_2)^{-2} + p_2 \alpha_1 \alpha_2(x_1 - \gamma_1)^{-1}(x_2 - \gamma_2)^{-1}) \\
&p_2(-p_1 \alpha_1 \alpha_2(x_1 - \gamma_1)^{-1}(x_2 - \gamma_2)^{\alpha_2-1} + p_2 \alpha_1(\alpha_2 - 1)(x_1 - \gamma_1)^{-2})
\end{align*}
\]

If we set \( z_1 = (x_1 - \gamma_1) \) and \( z_2 = (x_2 - \gamma_2) \), this reduces to

\[
\begin{align*}
p_1(-p_1 \alpha_2(\alpha_2 - 1)z_2^{-2} + p_2 \alpha_1 \alpha_2 z_1^{-1} z_2^{-1}) \\
&p_2(-p_1 \alpha_1 \alpha_2 x_1^{\alpha_1-1} z_2^{-1} + p_2 \alpha_1(\alpha_2 - 1)z_1^{-2})
\end{align*}
\]

This is then equivalent to Eq. (1) above, so that this is basically just the Cobb-Douglas case, as long as \( z_1, z_2 > 0 \). So the requirements will be the same.
iii. Constant Elasticity of Substitution

\[
\left(\alpha_1 x_1^{1/\rho} + \alpha_2 x_2^{1/\rho}\right)^\rho
\]

I am going to take the transformation \(y^{1/\rho}\) of the objective, so the Lagrangian is

\[
\mathcal{L}(x, \lambda) = \alpha_1 x_1^{1/\rho} + \alpha_2 x_2^{1/\rho} - \lambda (p_1 x_1 + p_2 x_2 - w)
\]

This has FONCs

\[
\begin{align*}
\alpha_1 \rho x_1^{1/\rho - 1} - \lambda p_1 &= 0 \\
\alpha_2 \rho x_2^{1/\rho - 1} - \lambda p_2 &= 0 \\
-(p_1 x_1 + p_2 x_2 - w) &= 0
\end{align*}
\]

The first two equations imply that

\[
\frac{\alpha_1 x_1^{1/\rho - 1}}{\alpha_2 x_2^{1/\rho - 1}} = \frac{p_1}{p_2}
\]

Let’s raise this to the \(1/\rho - 1\) power to get

\[
x_1 \over x_2 = \left(\frac{p_1 \alpha_2}{p_2 \alpha_1}\right)^{1/\rho - 1} = \gamma(p_1, p_2)
\]

Substitute this into the constraint to get

\[
p_1 x_1 + p_2 \frac{x_1}{\gamma(p_1, p_2)} = w
\]

or

\[
x_1^* = \frac{\gamma(p_1, p_2)w}{p_1 \gamma(p_1, p_2) + p_2}
\]

and

\[
x_2^* = \frac{\gamma(p_1, p_2)w}{p_2 \gamma(p_1, p_2) + p_1}
\]

The comparative statics are

\[
\frac{\partial x_1^*}{\partial p_2} = \frac{\gamma_2(p_1 \gamma(p_1, p_2) + p_2) - (p_1 \gamma_2 + 1) \gamma w}{(p_1 \gamma(p_1, p_2) + p_2)^2}
\]

and

\[
\frac{\partial x_1^*}{\partial w} = \frac{\gamma(p_1, p_2)}{p_1 \gamma(p_1, p_2) + p_2} > 0
\]

iv. Leontief

\[
\min\{\alpha_1 x_1, \alpha_2 x_2\}
\]

Well, the Lagrangian is

\[
\mathcal{L}(x, \lambda) = \min\{\alpha_1 x_1, \alpha_2 x_2\} - \lambda (p_1 x_1 + p_2 x_2 - w)
\]

The objective is discontinuous whenever \(\alpha_1 x_1 = \alpha_2 x_2\), so that we must add all of those points to the candidate list. Otherwise, the gradient of the objective is \((\alpha_1, 0)\) whenever \(\alpha_1 x_1 < \alpha_2 x_2\),
and \((\alpha_2, 0)\) whenever \(\alpha_2 x_2 > \alpha_1 x_1\). None of these points can satisfy the constraint qualification, however, since the gradient of the constraint is \((p_1, p_2)\), but the gradient of the objective \((\alpha_1, 0)\) or \((0, \alpha_2)\) vanishes for one of the components; for example, the system of equations
\[
\alpha_1 - \lambda p_1 = 0 \\
0 - \lambda p_2 = 0 \\
-(p_1 x_1 + p_2 x_2 - w)
\]
cannot be solved, since Eq 1 implies that \(\lambda = \alpha_1 / p_1\) but Eq 2 implies that \(\lambda = 0\). Therefore, we must add all of these points to the candidate list. That implies that... everything in \(\mathbb{R}^2_+\) is on the candidate list.

Now, we must be a bit more clever. Suppose the condition \(\alpha_1 x_1 = \alpha_2 x_2\) fails. In particular, assume that \(x_1 > \alpha_2 x_2 / \alpha_1\). Then if we take \(\epsilon\) away from \(x_1\) and re-allocate it to \(x_2\), the value of the objective will increase to \(\alpha_2 (x_2 + p_1 \epsilon / p_2) > \alpha_2 x_2\). So at any solution, we must have \(\alpha_1 x_1 = \alpha_2 x_2\).

Given that condition must hold, we can substitute the condition \(\alpha_1 x_1 = \alpha_2 x_2\) into the constraint to get
\[
p_1 x_1 + p_2 \frac{\alpha_1 x_1}{\alpha_2} = w
\]
and solving yields
\[
x_1^* = \frac{\alpha_2}{p_1 \alpha_2 + p_2 \alpha_1} w
\]
and
\[
x_2^* = \frac{\alpha_1}{p_2 \alpha_1 + p_1 \alpha_2} w
\]
Since the function is non-differentiable at the optimum, we cannot check SOSC’s.

The comparative statics are
\[
\frac{\partial x_1^*}{\partial w} = \frac{\alpha_2}{p_1 \alpha_2 + p_2 \alpha_1} \\
\frac{\partial x_1^*}{\partial p_2} = \frac{-\alpha_2 \alpha_1}{(p_1 \alpha_2 + p_2 \alpha_1)^2} w < 0
\]

8. Suppose we take a strictly increasing transformation of the objective function and leave the constraints unchanged. Is a solution of the transformed problem a solution of the original problem? Suppose we have constraints \(g(x) = c\) and take a strictly increasing transformation of both sides. Is a solution of the transformed problem a solution of the original problem?

If we take a strictly increasing transformation \(h() : \mathbb{R} \rightarrow \mathbb{R}\) of \(f(x)\), we get the Lagrangian
\[
\mathcal{L}(x, \lambda) = h(f(x)) - \lambda g(x)
\]
The FONCs are
\[
h'(f(x^*)) \nabla f(x^*) - \lambda^* \nabla g(x^*) = 0 \\
-g(x^*) = 0
\]
So that if \((x^*, \lambda^*)\) is a critical point of the above equations, we can rewrite them as
\[
\nabla f(x^*) - \frac{\lambda^*}{h'(f(x^*))} \nabla g(x^*) = 0
\]
\[-g(x^*) = 0\]

and as
\[
\nabla f(x^*) - \lambda^* \nabla g(x^*) = 0
\]
\[-g(x^*) = 0\]

where \(\lambda^* = \frac{\lambda^*}{h'(f(x^*))}\), so that \((x^*, \lambda^*)\) is a critical point of the original Lagrangian,
\[
\mathcal{L}(x, \lambda) = f(x) - \lambda g(x)
\]

The trick is that \(h(x)\) maps all of \(\mathbb{R}\) into \(\mathbb{R}\) — If the objective \(f(x)\) takes negative values, for example, \(\log(x)\) or \(\sqrt{x}\) will not be able to handle those values, and the result is that any critical points that occur where \(f(x)\) is negative will be “dropped”. So you really need a transformation \(h(x)\) that maps the range of \(f(x)\) into \(\mathbb{R}\).

Let \(h(x)\) be a strictly increasing function that maps \(\mathbb{R}\) into \(\mathbb{R}\). Now, consider the transformed problem
\[
\mathcal{L}(x, \lambda) = f(x) - \lambda (h(g(x)) - h(c))
\]

The FONCs are
\[
\nabla f(x^*) - \lambda^* h'(g(x^*)) \nabla g(x^*) = 0
\]
\[-(h(g(x^*)) - h(c)) = 0\]

Note that the second equation, \(-(h(g(x^*)) - h(c)) = 0,\) implies that \(g(x^*) = c,\) so that the original constraint must be satisfied. If we let \(\lambda^* = \lambda^* h'(g(x^*))\), then \((x^*, \lambda^*)\) is a critical point of the Lagrangian
\[
\mathcal{L}(x, \lambda) = f(x) - \lambda (g(x) - c)
\]
so that a critical point of the transformed problem is a critical point of the original problem.

Again, if you apply a transformation \(h()\) to the constraint that doesn’t cover the entire range of \(g(x)\), then you will potentially destroy some critical points.

9. Suppose you have a maximization problem
\[
\max_x f(x)
\]
subject to \(g(x, t) = 0\). Show how to use the implicit function theorem to derive comparative statics with respect to \(t\). Explain briefly what the bordered Hessian looks like, and how it differs from the examples in the notes.

The Lagrangean is
\[
\mathcal{L} = f(x) - \lambda g(x, t)
\]
with FONC’s
\[
\nabla f(x^*) - \lambda^* \nabla_x g(x^*, t) = 0
\]
\[-g(x^*, t) = 0\]

Differentiating with respect to \(t\) yields
\[-\nabla_x g(x^*, t) \nabla_t x^* - g_t(x^*, t) = 0\]
\[
\n\nabla_x^2 f(x^*) \nabla_t x^* - \frac{\partial \lambda^*}{\partial t} \nabla_x g(x^*, t) - \lambda^* \nabla^2_x g(x^*, t) \nabla_t x^* - \lambda^* \nabla x g_t(x^*, t) = 0
\]

Rewriting this as a matrix equation,

\[
\begin{bmatrix}
0 & -\nabla_x g(x^*, t)'

-\nabla_x g(x^*, t) & \nabla^2_x f(x^*) - \lambda^* \nabla^2_x g(x^*, t)
\end{bmatrix}
\begin{bmatrix}
\partial \lambda^* / \partial t

\nabla_t x^*
\end{bmatrix} =
\begin{bmatrix}
g_t(x^*, t)

\lambda^* \nabla x g_t(x^*, t)
\end{bmatrix}
\]

So derivatives of \( g(x, t) \) appear everywhere, making the problem quite complicated. In particular, the bordered Hessian no longer has the constraint gradients on the borders and only the Hessian of the objective in the lower right-hand corner, but now also has the constraint there. Our previous examples have had a linear constraint, so the cross partials vanish, making the form of the bordered Hessian much simpler.