1. Consider the indirect utility function

\[ v(p_1, p_2, w) = \frac{w}{p_1^\alpha p_2^\beta} \equiv \frac{w}{p_1^\alpha p_2^\beta} \]

where \( \alpha + \beta = 1 \).

i. Derive the expenditure function.

ii. Derive the Hicksian demand functions. Do these satisfy the law of demand?

iii. Derive the Walrasian demand functions. Do these satisfy the uncompensated law of demand?

iv. Suppose initial wealth is 10, the initial prices are \( p^0 = (1, 1) \) and the final prices are \( p^1 = (1 + t, 1) \). Compute the equivalent variation. If you interpret \( t \) as a tax on good 1, what is the deadweight loss of the tax?

\[ i. \]

\[ e(p, u) = \frac{w p_1^\alpha p_2^\beta}{\gamma} \]

\[ ii. \] Since \( \nabla_p e(p, u) = h(p, u) \), we have

\[ h_1(p, u) = \frac{u \alpha p_1^{\alpha-1} p_2^\beta}{\gamma} \]

\[ h_2(p, u) = \frac{u \beta p_1^\alpha p_2^{\beta-1}}{\gamma} \]

To show that these satisfy the law of demand, note that since \( h(p, u) = \arg \min \{x : u(x) \geq u\} \)
by definition, we have

\[ ph' \geq ph \]

\[ p'h \geq ph \]

so that \((p - p')(h - h') \leq 0\) and the law of demand holds.

iii. By Roy’s identity,

\[ x_i(p, w) = -\frac{\partial v(p, w)}{\partial p_i} \frac{\partial v(p, w)}{\partial w} \]

so that

\[ x_1(p, w) = \frac{\alpha w}{p_1} \]

\[ x_2(p, w) = \frac{\beta w}{p_2} \]

Note that \( x_1 \) does not vary in \( p_2 \) and vice versa. Thus, raising the price of \( p_1 \) drives down \( x_1 \) and has no effect on \( x_2 \); similarly, raising the price of \( p_2 \) drives down \( x_2 \) and has no effect on \( x_1 \). Therefore, \( x(p, w) \) satisfies the law of demand.

iv. Compute

\[ 2. \] A potential investor has wealth \( w_0 \) and a strictly increasing, concave Bernoulli utility function \( u() \) over final wealth. There is a risky investment opportunity, which has a net
return of $r$ per share with probability $p$, but 0 with probability $1-p$. There is a safe asset with a net return of 1. Thus, the agent selects the amount of the risky asset $z$ and the amount of the safe asset $s$ subject to $s \geq 0$, $z \geq 0$, $s + z \leq w_0$ to maximize his (vNM) expected utility.

i. Characterize the optimal level of investment in the risky asset for the investor. In particular, when does he purchase zero shares of the risky asset? When does he spend all of his wealth on the risky asset?

ii. How does the optimal level of investment respond to an increase in $r$ when $r > 1$?

iii. Suppose there is a second investor, who is “more risk averse” than the first. Provide two rigorous definitions of this concept. Which agent invests more in the risky asset?

iv. Suppose an agent’s wealth increases from $w_0$ to $w_1$. Does he invest more or less in the risky asset at $w_1$ than at $w_0$? If you need any extra assumptions to guarantee an unambiguous prediction, explain.

i. The constraints and increasing utility function imply $s = w_0 - z$ and $w_0 \geq z \geq 0$. Then the investor solves

$$\max_{w_0 \geq z \geq 0} p u (w_0 + (r - 1)z) + (1 - p)u (w_0 - z)$$

or

$$pu (w_0 + (r - 1)z) + (1 - p)u (w_0 - z) + \mu_0 z - \mu_1 (z - w_0)$$

with FONC

$$pu' (w_0 + (r - 1)z) (r - 1) - (1 - p)u' (w_0 - z) + \mu_0 - \mu_1 = 0$$

and SOSC at an interior solution

$$pu'' (w_0 + (r - 1)z) (r - 1)^2 + (1 - p)u'' (w_0 - z) < 0.$$  

He purchases zero shares if

$$pu' (w_0) (r - 1) - (1 - p)u' (w_0) + \mu_0 = 0$$

or $pr < 1$ so that the expected return is less than the return on the same asset. He purchases only risky shares if

$$pu' ((r - 1)w_0) (r - 1) - (1 - p)u' (0) - \mu_1 = 0$$

or

$$pu' ((r - 1)w_0) (r - 1) \geq (1 - p)u' (0)$$

which requires, in particular, that $u'(0)$ is finite.

ii. If $r$ goes up, $z$ weakly increases, and using the IFT, the sign of $\partial z^*/\partial r$ is

$$pu' (w_0 + (r - 1)z) + pu'' (w_0 + (r - 1)z) z (r - 1)$$

which is ambiguous: There is an income and substitution effect, so that the agent doesn’t have to risk as much to get the same expected consumption in the risky state of the world. This is positive if

$$1 \geq r_A (w_0 + (r - 1)z^*) z^* (r - 1)$$
so that if the agent isn’t too risk averse or the return \( r \) isn’t too high, investment goes up.

iii. Agent 2 is more risk averse than agent 1 if his coefficient of absolute risk aversion is higher, or if \( u_2(x) = \psi(u_1(x)) \) where \( \psi() \) is increasing and concave. Agent 1 invests more in the risky asset, because 2’s interior FONC is

\[
p u_2' \left( w_0 + (r - 1)z_2^* \right) (r - 1) - (1 - p) u_2' \left( w_0 - z_2^* \right) = 0
\]

\[
p \psi'(u_2^H) u_1' \left( w_0 + (r - 1)z_2^* \right) (r - 1) - (1 - p) \psi'(u_2^L) u_1' \left( w_0 - z_2^* \right) = 0
\]

\[
p u_1' \left( w_0 + (r - 1)z_2^* \right) (r - 1) - (1 - p) \frac{\psi'(u_2^L)}{\psi'(u_2^H)} u_1' \left( w_0 - z_2^* \right) = 0
\]

and since \( \psi \) is concave \( \psi' \) is decreasing so that \( \psi'(u_2^H) < \psi'(u_2^L) \), and

\[
p u_1' \left( w_0 + (r - 1)z_2^* \right) (r - 1) - (1 - p) u_1' \left( w_0 - z_2^* \right) > 0.
\]

This is 1’s FONC evaluated at 2’s solution, and it is strictly positive and decreasing in \( z \). Therefore, we must increase \( z \) to get to a zero, so that \( z_1^* > z_2^* \).

iv. Assume decreasing absolute risk aversion. Then an decrease in wealth is a concave transformation. By iii, it follows that the wealthier agent is less risk averse, and invests more.

### 3. There are firms \( j = 1, 2, \ldots, J \) who all hire inputs \( z_j = (h_j, k_j) \) to produce a single output \( y_j \), where \( h_j \) is labor for firm \( j \) and \( k_j \) is capital for firm \( j \). Let \( y_j = f_j(z_j) \) be the production function and \( y_j - f_j(z_j) = F_j(y_j, z_j) \leq 0 \) the transformation frontier. Let the wage rate of labor be \( w \) and the wage rate of capital be \( r \), and the price of the output be \( p \).

Suppose the firms maximize profits subject to their technological constraints and \( h_j \geq 0, k_j \geq 0, y_j \geq 0 \). Assume that (a) if \( h_j = 0 \) or \( k_j = 0 \), then \( y_j = 0 \) and (b) \( f_j(z_j) \) is strictly concave.

i. Prove that \( Y_j = \{(y_j, z_j) : F_j(y_j, z_j) \leq 0 \} \) is a convex set.

ii. Prove that a representative firm exists. In particular, show that the aggregate profit function

\[
\pi(y) = \max_{\{y_1 \in Y_1, y_2 \in Y_2, \ldots, y_J \in Y_J\}} \sum_{j=1}^{J} \{py_j - rk_j - wh_j\}
\]

is the same as the sum of the individual firms’ profits, and the optimal solution to the aggregate profit maximization problem is the same as the solution to the individual firms’ profit-maximization problems.

iii. Now assume that capital is fixed in the short run, so that the firms have a fixed amount \( k_j \) which cannot be traded among the firms. Provide conditions under which the representative firm that can reallocate capital achieves a better outcome. Write a sentence or two about the implications for models of economies where only aggregate capital is used to model a representative firm, but there is reason to believe that capital markets are incomplete or inefficient.

iv. Assume that firms share information, so that the production function of each firm \( j \),

\[
y_j = f_j(z_j; y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_J)
\]
depends on the production of the \( J - 1 \) other firms. For example, labor in the tech industry is highly mobile, and the ideas and solutions developed at one firm transfer easily to another. Does the representative firm that solves

\[
\pi(y) = \max_{\{y_1 \in Y_1, y_2 \in Y_2, \ldots, y_J \in Y_J\}} \sum_{j=1}^{J} \{py_j - rk_j - wh_j\}
\]

subject to \( y_j = f_j(z_j; y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_J) \) have the same solution as the dis-aggregated firms solving their independent maximization problems, taking the choices of the other firms as given? Explain the difference. Write a sentence or two about the implications for models of economies where only aggregate capital is used to model a representative firm, but there is reason to believe that there are non-trivial within-industry gains to learning-by-doing between firms.

i. Since \( f_j(h_j, k_j) \) is concave, \(-f_j(h_j, k_j)\) is convex. Thus \( F_j(y_j, z_j) = y_j - f_j(z_j) \) is a convex function in \((y, h, k)\). Therefore its lower contour sets are convex sets. Therefore \( Y_j \) is convex, as a lower contour set of 0.

ii. Note that profit maximization for each \( j \) implies

\[
\max_{y_j \in Y_j} p \cdot y_j \geq p \cdot y'_j, y'_j \in Y_j
\]

so that

\[
\sum_{j=1}^{J} \max_{y_j \in Y_j} p \cdot y_j \geq \sum_{j=1}^{J} p \cdot y'_j, y'_1 \in Y_1, \ldots, y'_J \in Y_J
\]

and since the maximization for each firm is independent,

\[
\max_{\{y_j \in Y_j\}_{j=1}^{J}} \sum_{j=1}^{J} p \cdot y_j \geq \sum_{j=1}^{J} p \cdot y'_j, y'_1 \in Y_1, \ldots, y'_J \in Y_J
\]

so that a representative firm can’t produce higher profits than the aggregation of each of the individual firms’ plans, and the aggregate profit function equals the sum of individual firm profits.

iii. Now the aggregate firm solves, ignoring corner solutions,

\[
\mathcal{L} = \sum_{j=1}^{J} (pf_j(h_j, k_j) - wh_j) - \lambda \left( \sum_{j=1}^{J} k_j - \sum_{j=1}^{J} \bar{k}_j \right)
\]

yielding FONC’s for each \( j \)

\[
p\frac{f_j(h_j, k_j)}{\partial h_j} - w = 0
\]

\[
p\frac{f_j(h_j, k_j)}{\partial k_j} - \lambda = 0
\]

\[- \left( \sum_{j=1}^{J} k_j - \sum_{j=1}^{J} \bar{k}_j \right) = 0.
\]
Thus, the initial allocation of capital is optimal only if the marginal product of capital is equated across all of the firms from the start. If there is any misallocation, then, a representative firm ignores the distribution and assumes that \( \sum_{j=1}^{J} k_j \) is a sufficient statistic for the entire industry.

iv. Now the aggregate firm solves, ignoring corner solutions,

\[
L = \sum_{j=1}^{J} (py_j - wh_j - rk_j)
\]

\[- \sum_{j=1}^{J} \lambda_j \{ y_j - f_j[h_j, k_j; f_1(h_1, k_1), f_2(h_2, k_2), ..., f_{j-1}(h_{j-1}, k_{j-1}), f_{j+1}(h_{j+1}, k_{j+1}), ..., f_J(h_J, k_J)] \}
\]

yielding FONC’s

\[
p - \lambda_j = 0
\]

\[- w + \lambda_j \frac{\partial f_j[h_j, k_j, y_j]}{\partial h_j} + \sum_{\ell \neq j} \lambda_{\ell} \frac{\partial f_{\ell}[h_{\ell}, k_{\ell}, y_{-\ell}]}{\partial y_j} \frac{\partial f_j[h_j, k_j, y_j]}{\partial h_j} = 0
\]

\[- r + \lambda_j \frac{\partial f_j[h_j, k_j, y_j]}{\partial k_j} + \sum_{\ell \neq j} \lambda_{\ell} \frac{\partial f_{\ell}[h_{\ell}, k_{\ell}, y_{-\ell}]}{\partial y_j} \frac{\partial f_j[h_j, k_j, y_j]}{\partial k_j} = 0
\]

\[- (y_j - f_j[h_j, k_j, y_{-j}]) = 0
\]

Since \( p = \lambda_j \) for all \( j \), we get

\[
p \left( \frac{\partial y_j}{\partial h_j} + \sum_{\ell \neq j} \frac{\partial y_{\ell}}{\partial y_j} \frac{\partial y_j}{\partial h_j} \right) = w
\]

\[
p \left( \frac{\partial y_j}{\partial k_j} + \sum_{\ell \neq j} \frac{\partial y_{\ell}}{\partial y_j} \frac{\partial y_j}{\partial k_j} \right) = r
\]

so that the marginal products are now at the industry level, since the second term in each (...) represents the externalities of firm \( j \)'s production on the \( J - 1 \) other firms. If a model treats a representative firm as selecting production in the presence of externalities, it will over-state marginal products because it will attribute them to capital and labor, and not to learning-by-doing.