Strategic Timing and Network Formation

Terence R. Johnson*
University of Notre Dame

First Draft: 7/2012
This Draft: 11/2012

Abstract
Using a continuous time framework, this paper proves existence of Markov Perfect equilibria in a strategic network formation game when agents can bargain over whether or not to add links, as well as choose how they pursue opportunities to form relationships. This is accomplished using a normalization from the operations research literature that converts a continuous time payoff into a structure that is recursive in the number of network events. Agents are then allowed to make side payments, where their expectations about the future value of their relationships endogenously determine bargaining position. The level of network intensity is then endogenized, allowing agents to vary how aggressively they seek out relationships with other participants. As the network evolves, this allows calm periods where few changes to the network occur, as well as periods of frantic activity where agents seek out scarce opportunities. Network intensity is a strategic substitute for participants when payoffs in the network game are uniformly positive and higher effort crowds out the efforts of others in equilibrium. Finally, a class of games is characterized in which myopic behavior is optimal, and a repeated auction game is analyzed.

JEL Classification Codes: D85, C73, C78
Keywords: Networks, Network Formation, Markov Perfect Equilibrium, Bargaining

1 Introduction

Network theory provides an opportunity to explain not only how overlapping sets of relationships and obligations determine economic behavior within the network, but the evolution and timing of changes to the network structure itself. For example, agents who make risk-sharing agreements are not only exposed to the risks presented by their immediate partners, but the partners of their partners, and so on. Events that occur in the network can have cascading consequences: If a crucial agent should be eliminated, the remaining agents will likely want to act quickly to make the best of the ensuing chaos. Conversely, environments in which there is little change should feature relatively stable relationships and agreements. To capture these kinds of phenomena in a dynamic network

*Thanks to participants at the Fall 2012 Midwest Microeconomic Theory conference. 917 Flanner Hall, University of Notre Dame, tjohns20@nd.edu.
setting, this paper provides a model in which forward-looking agents make decisions about not only about which relationships to form, but how much to pay for them and when.

Using a continuous time framework, existence of Markov Perfect equilibria is shown in a baseline model of network formation, and then extended to include bargaining over link addition as well as the endogenous timing of offers. This creates an interesting model of timing in which the network evolves stochastically but deliberately, and the competition to alter the network is analytically well-behaved. A typical dynamic model would adopt a discrete time framework, and allow agents to move simultaneously each period to add or remove links. This leads to many — potentially mixed strategy — equilibria in the link adjustment game, leading to many dynamic equilibria and serious challenges in computing a given equilibrium or engaging in comparative statics analysis. Similarly, the static literature on network formation typically uses notions of stability to select an equilibrium network configuration. However, these static models cannot capture the features of the game that drive agents’ actions in a dynamic framework, and stable equilibria may not even exist.

To avoid these drawbacks, each agent is associated with an action process and a death process. When the action process experiences an arrival, the agent is allowed to approach other agents in the network to propose changes, possibly involving bargaining or side-payments. When the death process experiences an arrival, the agent is removed from the game. When these are Poisson processes, the model has an elegant structure in which all the processes superimpose, and we are leaving a single aggregate Poisson process that describes the aggregate level of network activity. Since the Poisson process has the feature that conditional on an arrival, the probability of a second arrival is zero, the network evolves in an orderly fashion.

While the action processes are initially driven by exogenous factors, Section 4 endogenizes them by allowing the agents who are already networked to pick the intensity of their action processes. This makes a study of the speed of network adjustment possible, providing a model that explains how the level of activity varies across states. In particular, when the flow payoffs are positive at all states, action intensity is a strategic substitute, so that when one agent pursues opportunity more aggressively, it crowds out the efforts of the other agents. When the flow payoffs are negative at all states, the opposite holds, and action intensity is a strategic complement. When payoffs take both positive and negative values, this feature of the model is ambiguous.

As a simple example, consider a network of extraction firms that supply raw materials to inter-
mediate goods producers, that then build the components that final goods producers assemble for sale to consumers. Each final goods producer requires a number of components, which may come from many different intermediate goods producers, who may, in turn, rely on flows of materials from many different extraction firms. Should a given firm be removed from the game — an oil well might go dry, an intermediate firm might become unprofitable, or a goods producer may be destroyed by a lawsuit concerning its product — a number of things might happen. New firms may enter in response, eager to replace the exiting firm. The existing firms might endogenously adjust their behavior to compensate. There might be a scramble of activity as firms renegotiate their existing contracts to adjust to the new world they find themselves in. Such activities, however, require coordination within and between economic agents, creating a chance element. The most dynamic agents will make the most of such opportunities (or, do the best job of limiting their losses), while less adaptive organizations might miss out on profitable opportunities. Central to this process is the network of overlapping connections which dictate the terms of trade among the agents.

Section 3 develops such a model showing how, if network events arrive according to an exogenous Poisson process that depends on the network topology, there exists a dynamic equilibrium which corresponds to a “discrete-time” game where the index corresponds to the number of market events that have occurred. This establishes existence of a continuous-time equilibrium, which is usually difficult or impossible to prove in other continuous-time frameworks (see Fudenberg and Tirole, Chapter 13). Using the same framework, a bargaining game is nested in the network adjustment phase, allowing agents to make side-payments. This greatly expands the possibilities, since many links may be pairwise efficient but not strictly profitable for each party. By adding this bargaining game in which the players’ payoffs explicitly depend on their expectation of future payoffs, much more realistic network evolution is possible. Section 4 then endogenizes the process by which the market evolves, by allowing each agent to control the intensity of his own Poisson process. Whenever the agent experiences a “success”, he is allowed to act deliberately in a manner that changes the structure of the network. For example, a firm might work aggressively on a new project so that it can present it to a new client, creating a new relationship in the network; after the collapse of a rival firm, the remaining firms might scramble to meet new clients and expand their business; politicians might spend more time campaigning in the hopes of meeting influential donors or allies. This kind of endogenous growth processes has a unique equilibrium in action intensities given that
the cost of increasing the intensity of their process is sufficiently convex, guaranteeing the existence of a dynamic equilibrium. Section 5 considers some classes of games in which theoretical results can be obtained by verifying that myopic behavior is dynamically optimal. Typically, this would be impossible, since the trade-off in many dynamic games is between a higher payoff immediately and a higher payoff in the long run. However, when considering relationships, there is often no gain to foregoing the best opportunities now, especially if the market is sufficiently flexible.

A number of papers study the strategic formation of networks. In particular, Bala and Goyal (2000) study a dynamic model of forward-looking agents. Charness and Jackson (2007) and Galeotti et al (2010), Galeotti et al (2005), Jackson and Wolinsky (1996) and Bloch and Jackson (2007) study network formation in various frameworks where players take strategic actions that determine the resulting network, sometimes including transfers. This is similar to Kranton and Minehart (2001), who study how goods are allocated in a model of overlapping auctions. Abreu and Manea (2011), Manea (2011a), and Manea (2011b) consider bargaining in networks where the structure is fixed, and focus on deriving stationary values when the agents bargain in a manner similar to Rubinstein-Stahl bargaining games. The goal of these papers is to explain how the network topology determines payoffs, rather than explain how the network forms.

There is a large literature on dynamic stochastic games, starting with Shapley (1953), and including many papers on existence and uniqueness, including Duffie et al (1994), Haller and Lagunoff (2000), and Duggan (2012). In particular, Lee and Fong (2010) consider a network formation framework with bargaining that fixes the number of agents. To generate variation over time, there are relationship-specific shocks that change the value of links. They adopt a discrete time formulation and allow only one agent to move at once. The current paper instead derives the single-proposal feature from the model, rather than assuming it, and has the added benefit of allowing the study of endogenous timing. A recent paper by Doraszelski and Judd (2012) exploits a continuous time framework to study dynamic games, but focuses on the computational benefits of this approach. Since networks are inherently intractable as a state variable, this is not the focus of the current paper, but some of their arguments for adopting a continuous time framework are similar.

Finally, a number of papers study the dynamics of markets with entry and exit. In particular, Shimer and Smith (2000) look at a matching market with search frictions where agents meet randomly and decide whether to match, and these temporary partnerships are destroyed according to
an exogenous process. The current paper is an initial attempt to extend a framework like Shimer and Smith (2000) to situations in which the match is a non-cooperative game among a number of players. For technical reasons, the distribution of agents is not endogenized in the paper, but the conclusion contains some discussion of how this might be accomplished. Other papers like Spulber (1996), McAfee (1992), Peters (1997), Satterthwaite and Sheneyov (2007), and Rust and Hall (2003) focus on models of price competition in dynamic frameworks where the relationships among agents are essentially bi-partite graphs of varying complexity. These kinds of relationships are naturally modelled as labelled networks that evolve over time in the current paper, providing a framework for building game theoretic models not just of price competition or competition in mechanisms, but other economic relationships.

2 Model

There are \( j = 1, 2, \ldots, J \) distinct observable types of agents; for example, buyers and sellers, firms and workers, and so on. Each type has a *representative entrant* who is prepared to begin participating in economic activity, but currently has no links to the existing network\(^1\).

A *labelled network* is a tuple of objects \( N = \{I, E, L\} \), where \( I \) is the set of *nodes* or *networked agents*; \( E \) is an *adjacency matrix* of *edges* or *links* whose entries \( e_{ii'} \) take the value 1 if \( i \) is connected to \( i' \), and take the value zero otherwise; and \( L \) is a *labelling* of the graph \( \{I, E\} \) assigning an observable type \( j = 1, 2, \ldots, J \) to each node.

Thus, a labelled network provides a “snapshot” of the extant relationships in the market between agents of potentially diverse types. For example, there might be three buyers and two sellers, connected as

---

\(^1\)See the second paragraph of the conclusion.
Considering the agents in the graph arranged clockwise starting from the first seller, with $I = \{1, 2, 3, 4, 5\}$,

\[
E = \begin{bmatrix}
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

and $L = \{S, S, B, B, B\}$.

Since the node $i$ maps to the label $\ell$ associated with that node as well as the set of relationships between $i$ and the other agents summarized in $E$, it is unambiguous refer only to an agent’s index $i$ and the network $N$, or $(i, N) = (i, \{I, E, L\})$.

More complicated relationships can be accommodated by allowing non-integer edges of varying intensity, or *weights*, taking values in some set $W = \{0, w_1, w_2, \ldots, \bar{w}\}$. For example, agents might have risk-sharing agreements or a firm might divide up its time among various agents, allowing the model to accommodate much more complicated decisions about how agents relate to one another. While the rest of the paper considers relationships in $E = \{0, 1\}$ for simplicity, the results generalize immediately to the case of a $W$ with a finite number of weights with minor modifications.

The set of all networks with $K$ or fewer nodes is $\mathcal{N}(K)$, and the set of all bounded networks, $\bar{\mathcal{N}}$, are all those labelled networks with a finite number of nodes. It is necessary for some key results to obtain estimates of upper bounds on some kinds of activity, so the restriction to a finite number of nodes is sufficient to ensure that some aspects of the analysis are valid. In particular, as long as there is an upper bound on the number of network transitions per unit time for all networks of finite size, the analysis can be extended to $\bar{\mathcal{N}}$.

In some situations, links between some types might not be allowed. For example, in a two-sided matching model of marriage in a conservative state, only men and women can match, but not men to men or women to women, while in a more liberal state, men can match to men or women and vice versa. These kinds of restrictions can easily be incorporated by removing the disallowed networks from $\mathcal{N}$ or by suitably adjusting the player’s payoffs.

A *network game* is a tuple $\Gamma(N) = \{N, S(i, N, \theta), u(s, i, N, \theta); \Theta(N), p_\theta(\theta, N)\}$, where $\theta$ is a vector of *transitory private information* taking values in the set $\Theta$ with common prior $p_\theta(\theta, N)$;
$S(i, N, \theta)$ is the strategy set of player $i$ given the network and realization of private information; and $u(s, i, N, \theta)$ is the player to $i$ at strategy profile $s$ given the realized types and the network.

A strategy profile $s^* : (\Theta, N) \rightarrow \times_i S(i, N, \theta)$ is a (Bayesian Nash) equilibrium if, for every player-type $\theta_i$ and any alternative strategy $s'$,

$$\mathbb{E}_{\theta_{-i}} [u(s^*(\theta_i), s^*_{-i}(\theta_{-i}), i, N, \theta)|\theta_i] \geq \mathbb{E}_{\theta_{-i}} [u(s', s^*_{-i}(\theta_{-i}), i, N, \theta)|\theta_i]$$

Assume that the payoff functions are bounded, so that

$$\sup_{s,i,N,\theta} |u(s, i, N, \theta)| \leq \bar{u} < \infty$$

Let $\Psi(\Gamma(N), \Theta(N))$ be the set of equilibria for every network-information pair, and assume this set is non-empty for all $N$ and $\theta$. Then an equilibrium selection rule is a mapping from every network-information pair $(N, \theta)$ into a probability distribution over equilibria, $p_{\psi}[\psi|N, \theta]$. Then the ex ante expected utility of an agent at $(i, N)$ is given by

$$U(i, N) = \mathbb{E}_\theta [\mathbb{E}_{\psi(\theta)} \{u(s(\theta_i, \psi), s_{-i}(\theta_{-i}, \psi); i, N, \theta)|\theta_i\}]$$

By design, the players use equilibrium strategies against one another, but the equilibrium selected at each moment is realized as part of the set of common knowledge at the time that play occurs.

**Network Events**

The model is set in continuous time, and changes to the network will be determined by a set of Poisson processes that correspond to each agent. For example, imagine a financial network in which agents are linked through options contracts that allow them to hedge. It is unlikely that any firm will be literally negotiating simultaneously with multiple potential partners in the sense that simultaneous-move games imply. It is more likely that they split attention between the demands of immediate activity in the network game and monitoring the evolution of the network, and serious negotiations over link addition take place quickly in a one-on-one fashion.

Each networked agent $i = 1, 2, ..., |I|$ is associated with two Poisson processes. The first is an action process which has intensity $\eta(i, N)$, and the second is a death process which has intensity $\eta$. 
Figure 1: Event Timing

If agent $i$’s action process arrives at time $t$, the agent can unilaterally remove one of his links or approach another agent in the network to propose forming a new link between them. If agent $i$’s death process arrives at time $t$, that agent is removed from the network.

Each representative entrant $j = 1, ..., J$ is associated with a single arrival process, $\lambda(j, N)$, whose arrival allows the $j$-representative entrant to enter without forming any links, or approach some networked agent about forming a link and thereby becoming a networked agent.

The structural changes — addition or removal of links, deaths of networked agents, entry of new agents — are network events. Since Poisson processes superimpose, let

$$\lambda(N) = \sum_{i=1}^{\mid I \mid} \eta(i, N) + \mid I \mid \eta + \sum_{j=1}^{\mid J \mid} \eta(j, N)$$

be the level of aggregate activity at $(i, N)$. This gives the unambiguous probability of a network event occurring, and the conditional probabilities of actions and deaths are easily defined as in Figure 1. Therefore, with probability $\lambda(N)dt + o(dt)$, a network event occurs, and with complementary probability nothing happens.

The set of network events is then equivalent to selecting an agent to act or be removed, of which there are $\mid I \mid$ networked agents and $\mid J \mid$ representative entrants. Let $\Omega(N)$ be the set of $2\mid I \mid + J$ possible events at network $N$, with generic element $\omega$. If $\omega = i$, agent $i$ has been selected to make
a proposal, remove a link, or start a market at \( \omega \); if \( \omega = -i \), agent \( i \) has been selected for removal from the network; if \( \omega = j \), representative entrant \( j \) has been selected to make a proposal to a networked agent or enter into the market with no links.

Let \( A(i, N, \omega) \) be the set of actions available to agent \( i \) in network \( N \) at event \( \omega \). For a networked agent \( i \) selected to move, \( A(i, N, i) \) includes removing any existing link between \( i \) and \( i' \); approaching some agent \( i' \in N \) and proposing that they form a link, or nothing. For a representative entrant, \( A(j, N, j) \) includes approaching some agent \( i' \in N \) and proposing that they form a link; entering the network alone; or nothing. For a networked agent \( i' \) when \( i \) is selected to move, \( A(i', N, i) \) only includes accepting a proposal from agent \( i \) or rejecting it. The timing is such that the proposal is made first, and then the other party decides whether to accept or reject it.

If networked agent \( i \) proposes adding a link to agent \( i' \) and \( i' \) accepts, then the new connectivity matrix \( E' \) is given by \( E' = E \oplus e_{ii'} \), where the \((i, i')\) entry of the connectivity matrix becomes a one. If agent \( i \) removes a link to agent \( i' \), then the new network \( E' \) is given by \( E' = E \ominus e_{ii'} \), and the \((i, i')\) entry of the connectivity matrix becomes a zero. If a representative entrant of type \( j \) joins the network, the agent is added to the set \( I \) as \(|I|+1\), which coincides with a new row and column in the adjacency matrix \( E' \),

\[
N' = \{ I \cup \{|I|+1\}, E \oplus e_{|I|+1,i'}, L \cup \{ j \} \}.
\]

**Definition** A (Markov) network strategy for agent \( i \) at network \( N \) and event \( \omega \), \( \sigma_i(N, \omega) \), is a mapping \( \sigma_i : (N, \omega) \rightarrow A(i, N, \omega) \) for all states \((i, N, \omega)\). A (Markov) network strategy profile is a mapping \( \sigma \) giving network strategies for all networked agents \( i \) at all networks \( N \) and events \( \omega \).

One of the following assumptions will be required to establish the main results:

- (B1) The network process takes values in the set of bounded networks, \( \mathcal{N} = \bar{N} \), and

\[
\sup_N \lambda(N) = \bar{\lambda} < \infty
\]

and to ensure the process remains bounded as \( t \to \infty \),

\[
\eta \geq \sup_N \frac{\sum_j \eta(j, N)}{|I|}
\]
so that the probability of a death is weakly more likely than a new entry at every network state \( N \in \mathcal{N} \).

• (B2) The network process takes values in the set of networks with \( K \) or fewer vertices, \( \mathcal{N} = \mathcal{N}(K) \), and

\[
\max_{N \leq K} \xi(K) = \bar{\lambda} < \infty
\]

Whenever the number of networked agents reaches or exceeds \( K \), \( \lambda(j, N) = 0 \) for all \( j \), so that entry ceases.

These assumptions are required because without an upper bound on the level of network activity that is independent of the size of the network, some calculations in the paper would fail. In Section 4, the levels of aggregate activity will be endogenized by allowing each player to control his own individual activity process.

**Payoffs**

All agents seek to maximize their discounted expected utility

\[
V(i_0, N_0) = \lim_{T \to \infty} \mathbb{E} \left[ \int_0^T e^{-\rho t} U(i_t, N_t) dt \right | \sigma
\]

where \( i_0 \) is the agent’s initial position in the network, \( N_0 = \{I_0, E_0, L_0\} \) is the initial network structure, and \( \sigma \) is a fixed dynamic strategy profile.

Note that the value function is defined conditional on being the agent in position \( i_t \) in network \( N_t \) with label \( \ell_t \). Therefore, the “identity” of an agent is somewhat fluid, similar to how behavioral strategies in finite games of complete information organize an agent’s decision problem with respect to the node or how strategies in games of incomplete organize an agent’s decision around the type of private information he receives. Here, network strategies will be a Markovian mapping from network position into the agent’s strategy set, and this is reflected in the payoffs.

3 Dynamic Payoffs

Since the game is set in continuous time and the interval between network transitions is random, computing the agents’ payoffs presents a challenge. Games in continuous time are frequently poorly
behaved, but the framework developed here allows use of a procedure from the operations research literature to avoid these difficulties.

Fix a network strategy profile $\sigma$. This generates a law of motion over $\mathcal{N}$ from the perspective of node $i$, $\mu_{\sigma}[i_{t+s} = i', N_{t+s} = N'|N_t = N, i_t = i]$, which is the probability that agent $i$ in network $N$ at time $t$ transitions to agent $i'$ in network $N'$ at time $t + s$, given that all agents have adopted network strategies $\sigma$.

Through a normalization called uniformization in the operations research literature, (Ross, 1970; Lippman, 1975; Serfozo, 1979; Yin and Zhang 1998), the continuous process described by a transition law like $\mu_{\sigma}[i_{t+s} = i', N_{t+s} = N'|N_t = N, i_t = i]$ can be converted into one that recovers a “discrete-index” framework:

**Proposition 3.1** An agents’ discounted payoff can be written

$$
E \left[ \int_0^\infty e^{-\rho t} U(i_t, N_t) dt \right] = \sum_{k=0}^{\infty} \left( \frac{\bar{\lambda}}{\lambda + \rho} \right)^k E \left[ \frac{U(i_k, N_k)}{\rho + \lambda} \right] \left| \sigma \right.
$$

And the Bellman equation for an agent $i \in N$ can be written

$$
V(i, N) = \frac{U(i, N)}{\rho + \lambda} + \beta E_{i', N', \omega}[\max_{\sigma(i, N, \omega)} \{ V(i', N') \}] \mid \sigma
$$

where

$$
\beta = \frac{\bar{\lambda}}{\rho + \lambda}
$$

and for a representative entrant $j$,

$$
V(j, N) = \beta E_{i', N', \omega}[\max_{\sigma(j, N, \omega)} \{ V(i', N') \}] \mid \sigma
$$

The details are in the proof of Proposition 3.1, but the intuition is relatively straightforward.

The dynamics described by

$$
\mu_{\sigma}[i_{t+s} = i', N_{t+s} = N'|N_t = N, i_t = i]
$$

can be broken into two parts: An arrival process describing how long it takes a network event to arrive at each state, and a Markov transition density which gives the probability of moving from
each state to each other state, conditional on the current network. Since the size and complexity of the network varies with the number of agents and links, the time between network events is not homogeneous across all possible network states. However, by “speeding up” the slow states by adding fictional transitions from those state to themselves, the network event arrival process can be made uniform, so that the probability of $k$ network events occurring by time $t$ is given by

$$pr[k \text{ network events by time } t] = \frac{(\bar{\lambda}t)^k e^{-\bar{\lambda}t}}{k!}$$

and the holding times are distributed exponentially with parameter $\bar{\lambda}$ across all states. Similarly, the transitions between states conditional on a market event occurring are given by a Markov process that depends only on the strategies, network event, and current network state,

$$pr[(i,N) \text{ transitions to } (i',N')] \text{ at network event } \omega|\sigma] = \tau_\sigma[i',N'|i,N,\omega]$$

Consequently, the entire framework fits together with little or no awkward regularity conditions that are required by studying limits of discrete-time optimization problems that converge to a continuous-time problem, and the accompanying arguments that the discrete-time strategies converge to the continuous-time strategies.

4 Existence of Dynamic Equilibrium

Given the uniformization result, the agents’ payoffs can be analyzed in a semi-discrete framework, allowing a precise definition of Markov perfect equilibrium in continuous time.

**Definition** A network strategy profile $\sigma^*$ is a *Markov perfect network equilibrium* if, for all $i$, $N$ and $\omega$,

$$\sigma^*_i(N,\omega) \in \text{Argmax}_{\sigma'_i} \frac{U(i,N)}{\rho + \lambda} + \beta \mathbb{E}_{i',N'} \{ V(i',N') | \omega, \sigma'_i, \sigma^*_{-i} \}$$

(1)

Consider the following operator: For each $(i,N)$

$$V'(i,N) = \max_{\sigma'_i} \mathbb{E}_\omega \left[ \frac{U(i,N)}{\rho + \lambda} + \beta \mathbb{E}_{i',N'} \{ V(i',N') | \omega, \sigma'_i, \sigma^*_{-i} \} \right]$$
and as a mapping from value functions to value functions,

\[ V' = T_\sigma V \]

Then a value function \( V^* \) supports \( \sigma^* \) as a Markov perfect equilibrium if

\[ V^* = T_\sigma V^* \]

By showing that fixed points of \( T_\sigma \) exist, it follows that some selection from the set of maximizers of \( V^* \) is a Markov perfect equilibrium, establishing existence of equilibrium.

**Proposition 4.1** Under assumption B1 or B2, a Markov perfect network equilibrium \( \sigma^* \) exists.

In games where more links and entry uniformly improve the payoffs of all players, the operator \( T_\sigma \) will be monotone since any adjustment will raise all agents’ payoffs. In this case, there will be a unique equilibrium, since \( T_\sigma \) will be a contraction. However, this is unlikely to be the case in general, since relationships often entail trade-offs. For example, by approaching the customers of its competitors, a firm might raise its payoff and those of the customers, but harm its opponents, so that some components of \( V' \) increase relative to \( V \) under \( T_\sigma \), while other components decrease. These non-monotonicities create the possibility of multiple equilibria or the failure of \( T_\sigma \) to be a contraction mapping.

### 4.1 Bargaining over link addition

In many situations, agents pay one another the first time a relationships or link is established. For example, a client may pay a retainer to a law firm, or there might be a large, initial fee to subscribe to a service like Monster.com or Match.com, after which a new entrant can begin building relationships. This section considers how such payments can be used to add socially valuable links to a network where one party unambiguously loses in discounted value terms, but the other gains a compensating amount. This expands the kinds of dynamics that can arise in networks, since it is no longer necessary that both parties have positive discounted value for the link. However, agents’ willingness to pay for links is determined by their expectations about future transfers and network opportunities, so that bargaining position is endogenously determined. This section establishes existence of equilibrium under two bargaining models of interest.
Adding a link is pairwise efficient for \(i\) and \(i'\) if

\[ \beta V(i, N \oplus e_{ii'}) + \beta V(i', N \oplus e_{ii'}) \geq \beta V(i, N) + \beta V(i', N) \]

And, for payment \(y_{ii'}\), it is individually rational for \(i\) to add link \(e_{ii'}\) if

\[ \beta V(i, N \oplus e_{ii'}) + y_{ii'} \geq \beta V(i, N) \]

and individually rational for \(i'\) to add link \(e_{ii'}\) if

\[ \beta V(i', N \oplus e_{ii'}) - y_{ii'} \geq \beta V(i', N) \]

Note that if it is pairwise efficient to add a link \(e_{ii'}\), there is a compact, convex set of payments \(y_{ii'}\) that are individually rational for both parties and depend on the agents’ discounted expected values. Call this set \(Y_{ii'}(V)\), the set of transfers that implement pairwise efficient link addition:

\[ Y_{ii'}(V) = \left[ \beta \{ V(i, N) - V(i, N \oplus e_{ii'}) \} \land \beta \{ V(i', N \oplus e_{ii'}) - V(j', N) \} \right] \]

\[ \beta \{ V(i, N) - V(i, N \oplus e_{ii'}) \} \lor \beta \{ V(i', N \oplus e_{ii'}) - V(i', N) \} \]

A bargaining agreement is a function \(\phi_{ii'} : Y \to Y\) that maps the set of pairwise efficient and individually rational payments into a single payment, with the convention that \(\phi_{ii'} : \emptyset \to \emptyset\) in the case that trade is pairwise inefficient. Note that by construction, \(\phi_{ii'}(N, Y) = -\phi_{i'i}(N, Y)\).

This approach is a reduced form way of summarizing bargaining games where the surplus is known to both players. For example, one bargaining agreement might be generated by a game where (i) whenever a networked agent approaches a networked agent to add a link, the proposer makes a single take-it-or-leave-it offer which the other agent either accepts or rejects, and (ii) when a representative entrant approaches a networked agent, the networked agent gets to make a take-it-or-leave-it offer to the representative entrant. The important features of the bargaining agreement is that its image is in the set of mutually individually rational transfers \(Y_{ii'}\), so that a payment is always selected that is acceptable for both parties, and the bargaining agreements does not depend on \(N\) or \(V\) directly, but only on the gains from trade available to the agents by adding the link.
Definition A network strategy profile \( (\sigma^*, \phi^*) \) is a Markov perfect network equilibrium with bargaining if, for all \( i \) and \( N \), and \( \omega \),

\[
\sigma^*_i(i, N, \omega) \in \text{Argmax}_{\sigma'_i} \frac{U(i, N)}{\rho + \lambda} + \beta \mathbb{E}_{i', N'} \{ V(i', N') + \phi^*_{ii'}(Y_{ii'}(V(i, N))) \mid \omega, \sigma'_i, \sigma^*_{-i} \}
\]

The definition requires that the network strategies \( \sigma^* \) still be optimal when the payments \( \phi^*_{ii'} \) are made whenever the network transitions. For a similar argument to that of Proposition 4.1 to hold, it must be the case that the bargaining agreements is sufficiently continuous, so that the operator \( T_\phi V = V' \) defined by

\[
V'(i, N) = \max_{\sigma'_i} \frac{U(i, N)}{\rho + \lambda} + \beta \mathbb{E}_{i', N'} \{ V(i', N') + \phi^*_{ii'}(Y_{ii'}(V(i, N))) \mid \omega, \sigma'_i, \sigma^*_{-i} \}
\]

is a continuous operator on the space of value functions. Fortunately, existence of sufficiently continuous bargaining agreements can be shown by Browder’s selection theorem.

Proposition 4.2 Under assumption B1 or B2, a Markov perfect network equilibrium with bargaining agreement \( \phi^*_{ii'}(Y) \) and network strategies \( \sigma^* \) exists. A Markov perfect network equilibrium with bargaining exists for every bargaining agreement \( \phi^*_{ii'}(Y) \) that is a continuous selection from \( Y_{ii'}(V) \).

This result allows agents to compensate one another for adding links to the network that benefit one party, but weakly harm the other. For example, by introducing a weak rival into a market, a firm might suffer a small loss of business, but gain through a payment by the rival as well as blockading entry by future opponents. The introduction of pairwise transfers allows these deals to be made, where previously, links would only be added if both parties gained dynamically.

The previous proposition would fail if \( \phi^*_{ii'}(Y) \) was sufficiently discontinuous that there were ubiquitous “oscillations” in the network strategies. For example, imagine proceeding by backwards induction. One iteration might see agent \( i \) approach agent \( i' \), but as soon as this is incorporated into the value functions, it becomes attractive for \( i \) to approach agent \( i'' \). If cycles arise where agent \( i \) wants to go back and forth, the process never converges, and it stands to reason that a fixed point might not exist at all, either. By restricting attention to transfers that are continuous selections, this problem can be avoided.

The downside of this extension is that for a given set of network strategies — how agents add...
links to the network — there can be uncountably many bargaining agreements that can implement those strategies.

**Proposition 4.3** Consider a Markov perfect network equilibrium with bargaining in which agent $i$ and $i'$ exchange a payment at $N$, $Y_{ii'}(V)$ is a correspondence and not a function, and both agents strictly prefer to add the link, given the bargaining agreements $\phi_{ii'}(N,Y_{ii'}(V))$. Then there are uncountably many bargaining agreements for which the network strategies are still a Markov perfect equilibrium with bargaining.

This highlights the benefit of picking a particular non-cooperative bargaining game to determine the side-payments between the players. As long as the bargaining game has a unique equilibrium this multiplicity will vanish, making analysis or computation more straightforward.

### 4.2 Networks with endogenous intensity

The intensity of activity in the market in previous sections was exogenous, determined by $\lambda(N)$ and $\eta(k,N)$. This section makes the intensity of the market an endogenous phenomenon that arises from the competition between agents for opportunities.

The choice of intensity might most naturally be modelled as a menu cost: If an agent pays a cost $C$ at date $t$, he is allowed to approach another agent to discuss adding a link. While natural, there are a number of problems. First, if many agents want to act at the same time with probability one, the network will potentially undergo an uncountable number of changes in a short time. A “hysterical” period of uncountable changes and counter-changes is both unrealistic and technically challenging to model. Second, many appealing strategies — such as, “As soon as agent $k$ moves, make an offer to agent $k''$” — may be undefined, since there is no “first moment after time $t$”, leading to technical, measurability issues. By using the Poisson control approach, these issues are avoided. Behaviorally, the Poisson control model approximates a situation in which a certain amount of preparation is required to make an offer, and completing this work takes a stochastic amount of time. Since the agent must divide his attention between the stage game and preparing an offer, these offers take the form of a stochastic process.

More formally, consider a model where every networked agent $i$ controls a Poisson process with intensity $\eta(i,N)$, and incurs a cost $c(\eta)$ per unit time where $c()$ is an increasing, convex function with $c(0) =$. Then at each state $(i,N)$ agents choose $\eta(i,N)$ to maximize their discounted expected
utility at that state. Since Poisson processes superimpose, this yields an aggregate Poisson process with intensity

\[
\lambda(N) = \sum_{i=1}^{|I|} \eta(i, N) + \sum_{j=1}^{|J|} \eta(j, N) + |I|\eta
\]

which replaces the exogenous \(\lambda(N)\) function of the previous sections.

The timing of agents’ decision-making requires some consideration. Say that a stochastic process \(X_t\) has finite time scale if there is a bijection between the number of arrivals of \(X_t\) and a finite set of numbers \(\{1, 2, ..., K\}\); countable time scale if there is a bijection between the number of arrivals of \(X_t\) and \(1, 2, ..., \); and uncountable time scale if there is a bijection between the number of arrivals of \(X_t\) and \([0, 1]\). The arrivals of moves that alter the structure of the network exists on a countable time scale, since it corresponds to the arrivals of a Poisson process, which is a countable set with probability one. Decisions about action intensity, however, exist on an uncountable time scale since this must be chosen for every instant of the game. Consequently, it seems natural that the most frequently occurring decision offers the least commitment, and should be decided subject to the less-frequently occurring decisions. This implies that agents should choose \(\eta(i, N)\) treating the value of each state \(V(i, N)\) as given, and then maximize over their network strategies \(\sigma\) anticipating how intensities will be decided at each state. This creates a natural composition that reflects the level of commitment of each decision.

**Definition** A *Markov perfect network equilibrium with endogenous intensity* consists of a network strategy profile \(\sigma^*\) and a profile of action intensities at each state \(\eta^*(i, N)\) so that no agent \(i\) in network \(N\) at state \(\omega\) has a profitable deviation from \(\sigma^*_i(N, \omega)\) and no agent \(i\) in network \(N\) has a profitable deviation from the intensity strategy \(\eta(i, N)\).

To establish existence of an equilibrium, the optimal intensity strategies are first characterized in terms of the value function. Fix the value function \(V(i, N)\) for all states. In equilibrium, the networked agents each control their own Poisson process that determines the arrival of opportunities
to make changes to the network. Consider a short period of time $dt$ in state $(i, N)$ for agent $i$:

$$V(i, N) = dt \{ U(i, N) - c(\eta(i, N)) \}$$

$$+ e^{-\rho t} \left\{ \sum_{k \in I} \eta(k, N) dt \left( \prod_{k' \neq k} (1 - \eta(k', N)) dt \right) E[V(i', N') | i, N, \sigma, \omega = k] \right.$$  

$$+ \left( 1 - \sum_{k \in I} \eta(k, N) dt \left( \prod_{k' \neq k} (1 - \eta(k', N)) dt \right) \right) V(i, N) \right\}$$

This implies that whenever two or more agents “succeed” at generating an opportunity, nothing happens, but this is simply to minimize the amount of notation and is without loss of generality. The probability that two or more events occur is an $o(dt^k)$ event, $k \geq 2$, which will vanish in the limit as $dt \to 0$.

Re-arranging and dividing by $dt$ yields

$$V(i, N) \left\{ \frac{1 - e^{-\rho dt}}{dt} + \sum_{k \in I} \eta(k, N) \left( \prod_{k' \neq k} (1 - \eta(k', N)) dt \right) \right\} =$$

$$U(i, N) - c(\eta(i, N)) + e^{-\rho t} \left\{ \sum_{k \in I} \eta(k, N) \left( \prod_{k' \neq k} (1 - \eta(k', N)) dt \right) \right\} E[V(i', N') | i, N, \sigma, \omega = k]$$

And taking the limit as $dt \downarrow 0$ yields

$$V(i, N) = \frac{U(i, N) - c(\eta(i, N)) + \sum_{k \in I} \eta(k, N) E[V(i', N') | i, N, \sigma, \omega = k]}{\rho + \sum_{k \in I} \eta(k, N)}$$

Then agents choose their intensity parameter $\eta(i, N)$ to maximize

$$g_i(\eta(i, N)) = \frac{U(i, N) - c(\eta(i, N)) + \sum_{k \in I} \eta(k, N) E[V(i', N') | i, N, \sigma, \omega = k]}{\rho + \sum_{k \in I} \eta(k, N)} \quad (3)$$

This step usually appears as a precursor to deriving the Hamilton-Jacobi-Bellman equation, but there are no continuous time state variables that depend on the agents’ choice of control.

A necessary condition for equilibrium is that agents are maximizing their payoffs with respect to $\eta(i, N)$. The challenge here is that if the equilibrium fails to be unique at some states, the composition of the Bellman operator with the equilibrium intensity operator that maps value functions to profiles of intensity strategies will then become a correspondence, and a particular intensity parameter...
equilibrium will have to be selected. However, this selection will have to be made consistent and continuous across all states to prove existence of a fixed point, leading to a much more complicated problem. The following proposition shows that these technicalities can be avoided:

**Proposition 4.4** The best-reply correspondence for each agent is a function. If $c(\lambda)$ is sufficiently convex, there is a unique equilibrium in action intensities. The partial effect of $k$’s action intensity on $i$’s action intensity is

$$\frac{\partial \eta(i,N)}{\partial \eta(k,N)} = \frac{3V(i,N) + E[V(i',N')|i,N,\sigma,\omega = k]}{g''_i(\eta(i,N))(\rho + \sum_k \eta(k,N))^2}$$

where $g''_i(\eta(i,N))$ is the second derivative of Eq. (2), agent $i$’s maximization problem at $(i,N)$. If $U(i,N) \geq 0$ for all $(i,N)$, action intensities are strategic substitutes; if $U(i,N) < 0$ for all $(i,N)$, action intensities are strategic complements.

In the strategic substitutes case, as more agents are added to the environment, there is a phenomenon similar to a “bystander effect” that lowers overall intensity of effort. However, this also complicates the process of establishing uniqueness to some extent, since even for well-behaved games like Cournot quantity competition, the general results on uniqueness are quite restrictive. Without the sufficient condition given in the proof of Proposition 4.1, there is an equilibrium selection issue that could prove disruptive to the existence of dynamic equilibria in general.

Consider the operator $T_\eta$ given by

$$V'(i,N) = \max_{\eta(i,N)} \frac{U(i,N) - c(\eta(i,N)) + \sum_{k \in I} \eta(k,N)E[V(i',N')|i,N,\sigma,\omega = k]}{\rho + \sum_{k \in I} \eta(k,N)}$$

with

$$V' = T_\eta V$$

For a given value function $V$, this computes the best intensity strategy, and then maps it to a new payoff $V$.

**Proposition 4.5** Assume B2. Let $T_\sigma$ be the operator defined in Proposition 4.2. Then the composition $T = T_\sigma \circ T_\lambda$ is a continuous mapping from a closed, compact set $\mathcal{V}$ to itself, so it has a fixed point $V^* = TV^*$ in $\mathcal{V}$. Therefore, a Markov Perfect equilibrium exists with bargaining and endogenous activity.
This establishes existence of a Markov perfect equilibrium where the activity intensities are endogenous, extending the analysis of Section 4 to allow a much richer model of market timing and network evolution.

5 Regret-Free Games

Consider a game with two observable types of agents, where each observable type only wants to link to the opposite type (e.g., firms and workers in a labor market, buyers and sellers, heterosexual men and women, and so on). The situation is then one of bi-partite matching, and can be represented by a single connectivity matrix where each row corresponds to the networked agents of one of the observable types, and each column corresponds to the networked agents of the other observable type. For two of each kind of agent there are 18 such networks, for three of each kind there are 532, and for four of each kind there are 66,066. In general, there will be

\[ \sum_{k=1}^{K} \sum_{\{j_1 \leq j_2 \leq \ldots \leq j_J \leq k\}} 2^{j_1j_2\ldots j_J} \]

potential network states. In more general, non-bi-partite games, the growth will be even faster.

This curse of dimensionality seems particularly severe, and identifying classes of games where the complexities are not overwhelming is desirable. Four obvious options are:

- Approximate a large network by projecting the long-run payoffs onto an agent’s local network, and ignore the rest of the network structure.

- Find games where the players’ payoffs can be reduced to sufficient statistics computed from the network topology.

- Instead of all the pairwise relationships that can be considered, focus on models where agents arrange themselves in markets within which they are all connected.

- Characterize classes of games where myopically optimal behavior is also dynamically optimal.

The first option is essentially a computational approach, and is not pursued here. The second reduces the network problem to a regular dynamic stochastic game in which the state variables
summarize the network. Since these are either beyond the scope of this paper or focus on eliminating
the network from the problem, I focus on the last option.

Studying myopic play is a dead-end for many games. For example, a myopic agent faced with a
classic capital accumulation problem will consume all capital in the first period, save nothing, and
end with a very low payoff. However, network games involve cooperation among agents, and it is
not immediately clear that myopically optimal behavior is not also dynamically optimal. As long
as acting myopically always leads to a “good” distribution over future states for all players at all
states, the consequence is that it will be a Markov perfect equilibrium.

**Definition** A network strategy profile $\sigma^*$ is a myopic equilibrium if, for all $(i, N)$,

$$
\mathbb{E}_{i', N'}[U(i', N')|\sigma^*_i, \sigma^*_{-i}, i, N] \geq \mathbb{E}_{i', N'}[U(i', N')|\sigma'_i, \sigma^*_{-i}, i, N]
$$

A strategy profile is a myopic equilibrium if it maximizes a player’s next-period payoff, given
that all the other agents have adopted their parts of the strategy profile.

**Definition** Let state $(i, N) \succeq (i', N')$ if and only if $U(i, N) \geq U(i', N')$. Then $\sigma'_i$ dynamically
dominates $\sigma''_i$ relative to $\sigma^*$ if, for all $(i^o, N^o)$,

$$
\sum_{(i'', N'') \preceq (i^o, N^o)} \sum_{(i', N')} \mu_{\sigma^*}[i'', N''|i', N'] \mu_{(\sigma'_i, \sigma^*_{-i})}[i', N'|i, N] \\
\geq \sum_{(i'', N'') \preceq (i^o, N^o)} \sum_{(i', N')} \mu_{\sigma^*}[i'', N''|i', N', \sigma^*] \mu_{(\sigma'_i, \sigma^*_{-i})}[i', N'|i, N]
$$

This condition involves a number of ideas. First, a partial ordering is fixed over the state space,
so that $(i, N) \succeq (i', N')$ if and only if the flow payoff at $(i, N)$ is greater than the flow payoff at
$(i', N')$. Second, some network strategy profile $\sigma^*$ is fixed. Finally, an action $\sigma'_i$ is said to dynamically
dominate $\sigma''_i$ with respect to the continuation profile $\sigma^*$ if $\sigma'_i$ gives a better transition density next
period and $\sigma''_i$. In short, the agent looks ahead one period and — assuming that all players use
$\sigma^*$ — asks whether playing $\sigma'_i$ today will give a better transition density than $\sigma''_i$ tomorrow in the
sense of first-order stochastic dominance.

If a network strategy profile is constructed such that play is myopic for all $(i, N)$ and $\sigma^*$ stochas-
tically dominates any other action for all agents and all network states, then myopic play both
maximizes the agents’ payoffs next period and gives them the best transition density from next period’s state. Together, this implies that myopic play would generally be optimal.

**Definition** A network strategy profile $\sigma^*$ is regret-free if, for all $i$ and $N$, $\sigma^*$ forms a myopic equilibrium and stochastically dominates any other action the players could choose.

There are a number of reasons why a game might not be regret-free. First, waiting for entry of new agents might be better than forming relationships with existing agents in the current configuration, so that networked agents want to “hoard” links. Second, the addition of links that result in “unintended consequences” for other agents in the network might raise a networked agent’s payoff today but lower it in the future; for example, by adding a link, one agent might reduce his or her value to a number of partners, results in the loss of existing relationships in the next period.

By combining the myopic equilibrium with the stochastic dominance condition on transition densities, however, it follows that, basically, no one-shot deviations are profitable, and hence that the strategies form a Markov perfect network equilibrium.

**Proposition 5.1** Any regret-free network strategy profile $\sigma^*$ is a Markov perfect network equilibrium.

This construction is convenient because it is relatively easy to find myopic network strategy profiles that form an equilibrium of a one-stage game. Once a network strategy profile is provided, checking whether it is a Markov perfect equilibrium then reduces to verifying whether the transition density satisfies stochastic dominance. Since these conditions are so close to the “primitives” of the model and calculation of a value function is not required, it is possible to verify whether or not a particular network strategy profile is an equilibrium without resorting to a computational approach.

**Example: Auction Hopping**

Let the set of types $J$ include buyers and sellers, so $J = \{B, S\}$. Every moment $dt$, the seller has the capacity to produce one unit of a good. Buyers each independently draw a privately known value from an identical, log-concave distribution $F(v)$ with strictly positive support on $[0, \bar{v}]$. Each moment, the seller designs and implements a profit-maximizing mechanism, taking the buyers to which he is matched as given. For concreteness, let the sellers all select the second-price auction
with optimal reserve price \( r^* \), given by

\[
r^* = \frac{1 - F(r^*)}{f(r^*)}
\]

A network \( N \) is a many-to-one matching from buyers to sellers. Each seller can be matched to as many buyers as desired, but each buyer can be matched to only a single seller at any given moment. Let \( \alpha(i) \) be the set of agents to whom agent \( i \) is matched, and \( \alpha(\alpha(i)) = \alpha^2(i) \) the set of agents to whom agent \( i \)'s partners are matched. Then buyers receive an ex ante expected payoff

\[
U(i, N) = (1 - F(r^*)) \sum_{k=0}^{\lfloor \alpha^2(i) \rfloor - 1} \frac{(|\alpha^2(i)| - 1)!}{(|\alpha^2(i)| - 1 - k)!k!} (1 - F(r^*))^k (F(r^*))^{|\alpha^2(i)| - 1 - k} \int_{r^*}^{\theta} F(x)^k dx
\]

and sellers receive an ex ante expected payoff

\[
U(i, N) = \sum_{k=0}^{\lfloor \alpha(i) \rfloor} \frac{|\alpha(i)|!}{(|\alpha(i)| - k)!k!} (1 - F(r^*))^k (F(r^*))^{|\alpha(i)| - 1 - k} \int_{r^*}^{\theta} \left( x - \frac{1 - F(x)}{f(x)} \right) f_{1:|\alpha(i)|}(x) dx
\]

where \( f_{1:|\alpha(i)|}(x) \) is the distribution of the first order statistic of \( |\alpha(i)| \) draws from a density \( f(x) \).

This game fits the framework developed in previous sections, where the market actions include:

- The death of a networked agent
- A representative entrant approaches a networked agent to propose adding a link
- A buyer approaches a seller and requests to join that seller’s market, severing ties to any previous sellers if the proposal is accepted
- A seller approaches a buyer to “poach” them from another seller, on the condition that the seller sever previous ties to all other sellers

The only difference is that upon joining a new seller, buyers sever ties to their previous matches.

Define the most attractive seller as the one with the fewest number of buyers, and the least attractive seller as the one with the largest number of buyers.

Consider the following network strategies for each buyer:

- If selected to act, approach one of the most attractive sellers at random. If approached by a seller, join only if the new seller is strictly more attractive than the current match after joining.
and the network strategies for each seller:

- Randomly approach a buyer matched to a strictly less attractive seller or an unmatched buyer.

These network strategies drive the markets towards an equilibrium where there are a roughly equal number of buyers matched to each seller. However, since entry and exit keep the market in a constant state of churning. In particular, when a seller is removed from the market there is a sudden glut of unmatched buyers. If bargaining and endogenous activity are allowed, this will lead to a flurry of activity as buyers’ bargaining power has fallen substantially. On the other hand, the death of a single buyer has little effect on the market when there are many players, since the biggest impact it can have for the remaining agents is changing the identity of the most attractive seller.

**Proposition 5.2** Suppose B1 or B2 hold. The network strategies $\sigma^*$ are regret-free, so they are a Markov-perfect network equilibrium of the auction-hopping game.

This implies that no matter what, buyers are better off flowing the most attractive seller. If that seller is removed from the network, the ensuing scramble will be less intense, since that seller had the fewest number of clients. If further entry occurs on that seller, the gains to buyers are competed away, but having such gains for a small time is better than having no gains at all. Once that seller’s client list is equated with the next most attractive seller, entry ceases, and only resumes when future shocks again make that seller attractive to buyers. On the seller side, the argument is essentially the same. Clients come and go, but poaching them is always an option, since they are better off going over to a strictly more attractive seller.

**6 Conclusion**

By endogenizing network formation, bargaining, and timing, the framework developed here provides a variety of opportunities for better understanding social and economic interaction. The network evolves in a manageable way, presenting an opportunity for the study of more complicated network games that incorporate more realistic features of such markets.

It is difficult to construct a “large” version of the game, in which the distribution of agents searching for trading opportunities is endogenous. In papers like Shimer and Smith (2000) this is accomplished by matching two continuua of agents, where some measure go unmatched, and each
agent is measure zero from an economy-wide perspective, but has positive measure within each partnership. In the framework adopted in the current paper, a continuum could be assigned to each type of the \( J \) types of agents, but then we would require a continuum of markets in which to match them, leading to measurability problems. Similarly, if we assign a countably infinite number of agents to each of the \( J \) types and match them into a countably infinite number of simultaneously existing networks, finitely additive measures will be required rather than countably additive ones. Solving these problems, while technical, would allow a more general construction of dynamic network games with an endogenous search distribution.

The flexibility of the model to accommodate variation in the timing of changes might provide deeper insight into the scrambles and frenzies that accompany important changes to market structure or participants, and can provide useful welfare comparisons for policy purposes. For example, high frequency trading is a financial services practice that is increasingly drawing criticism due to how quickly it can cause damage to markets, and models like the one presented here could be used to investigate whether policies that limit the intensity of agents’ network activity can make all participants better off. By imposing constraints on the agents to reduce or encourage a higher level of activity, welfare gains might be achieved relative to a decentralized equilibrium.

Since only a single agent is moving at any given time, an agent’s strategy set can be made significantly more complicated. For instance, he could approach any number of agents to add new links, or remove any number of existing links. An interesting extension is to consider situations in which network agents can make many moves at once, but the other networked agents can only observe the moves that pertain to them. Consequently, agents may realize ex post that what looked like a good agreement is actually a bad one, leading to inefficiencies that arise due to the problem of inferring another agent’s intentions merely from the proposal.

Proposition 4.2 also opens the door to incorporating private information into the framework using a mechanism design approach. Rather than using bargaining agreements, the type of a representative entrant might be unobservable. Consequently, a networked agent will be forced to design some kind of mechanism to solicit information from any potential partner. This opens a number of interesting avenues for research, since the very fact that the representative entrant is approaching a particular networked agent reveals some information, and the networked agents are essentially competing in mechanisms. Exploring these games might provide useful models of
screening in small industries or wherever the outcome of initial relationships can be observed later on.

Lastly, the network game could be modelled as meeting periodically, rather than occurring continuously. This maintains the continuous time structure of the network dynamics while achieving a more realistic notion of repeated play in the network game. Adding this feature would complicate the analysis, making value functions and stationary distributions “cyclical” and dependent on the calendar date between market meetings.
References


[33] M. Satterthwaite and A. Shneyerov. Dynamic matching, two-sided incomplete information,

[34] R. Serfozo. An equivalence between continuous and discrete time markov decision problems.


2000.

580, 1996.


7 Appendix: Proofs

Proof of Proposition 3.1

Proof Let \( \bar{\lambda} = \sup_N \lambda(N) \) and fix a network strategy profile \( \sigma \). The proof strategy is to renormalize the stochastic process in a way that the probability that an event occurs is the same across all states, but the payoffs and stochastic process are exactly the same as the original chain. Recall that \( \mu_{\sigma}[i_{t+s} = i', N_{t+s} = N'|N_t = N, i_t = i] \) is the probability that agent \( i \) in network \( N \) at time \( t \) transitions to \( i' \) in network \( N' \) at time \( t + s \).

Definition The transition probabilities \( \mu_{\sigma}[i_{t+s} = i', N_{t+s} = N'|N_t = N, i_t = i] \) are homogeneous if

\[
\mu_{\sigma}[i_{t+s} = i', N_{t+s} = N'|N_t = N, i_t = i] = \mu_{\sigma}[i_s = i', N_s = N'|N_0 = N, i_0 = i]
\]

for all \( N, N', i, i', t \) and \( s \). Call \( P_{\sigma}(t) \) the matrix whose \( ((i', N'), (i, N)) \) entry is \( \mu_{\sigma}[i_t = i', N_t = N'|N_0 = N, i_0 = i] \). Then \( P_{\sigma}(t) \) is a stochastic semigroup if

1. \( P_0 \) is the identity matrix.
2. Chapman-Kolmogorov Equations: \( P_{t+t'} = P_t P_{t'} \) for all \( t, t' \geq 0 \).
3. \( P_t \) is stochastic for all \( t \): All entries are positive, and the row sums are equal to 1.

A semigroup is standard if \( P_t \to I \) as \( t \to 0 \).

Note that \( \mu_{\sigma}[] \) is a standard stochastic semi-group. We now characterize the derivative of the semigroup with respect to time, \( D_t P_t = P_t Q \), and then characterize the dynamics in terms of the matrix \( Q \) (see Ethier and Kurz [7]).

Consider the transition matrix \( P_t \) whose \( ((i, N), (i', N')) \)-entry is the probability that the system transitions from \( (i, N) \) to \( (i', N') \), given \( t \) moments of time in state \( N \). Then

\[
Q = \lim_{h \to 0} \frac{1}{h} (P_h - I)
\]

is the infinitesimal generator or \( Q \)-matrix of the process. Let \( \tau_{\sigma}[i', N'|i, N, \omega] \) be the probability of a transition to \( (i', N') \), given that the network is at state \( (i, N) \), and event \( \omega \) has occurred. Note
that this transition probability is unambiguously determined by the network strategies \( \sigma \), the state \((i, N)\), and \( \omega \).

Since the Poisson process has, for small \( h \), probability approximately \( \lambda(N)h \) of an event, the \( Q \)-matrix has entries, for \((i, N) \neq (i', N')\),

\[
\lim_{h \downarrow 0} \frac{\lambda(N)h \sum_{\omega} \tau_\sigma[i', N'|i, N, \omega]p[\omega|N]}{h} = \lambda(N) \sum_{\omega} \tau_\sigma[i', N'|i, N, \omega]p[\omega|N]
\]

and for \((i', N') = (i, N)\),

\[
\lim_{h \downarrow 0} \frac{\lambda(N)h \sum_{\omega} \tau_\sigma[i, N|i, N, \omega]p[\omega|N] + (1 - \lambda(N)h) - 1}{h} = \lambda(N) \sum_{\omega} \tau_\sigma[i, N|i, N, \omega]p[\omega|N] - \lambda(N) - 1
\]

Then the law of motion for the system is

\[
\frac{d}{dt} P_t = P_t Q
\]

from which the dynamics can be solved, with the boundary condition \( P_0 = I \),

\[
P_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} Q^k = e^{Qt}
\]

This is the fundamental equation for describing the evolution of the system. However, the probability of remaining in network configuration \( N \) is not constant across \( N \): \( \lambda(N) \) is allowed to vary in the size of the network. Consequently, the holding times across various states influence the discounted expected value of being at the state, complicating the task of deriving a recursive representation of the agents’ payoffs. To remedy this, a transformation is made to the \( Q \)-matrix so that (i) the instantaneous dynamics are the same, and (ii) the expected holding times of the transformed process are constant across all states. Define

\[
Q' = \frac{Q}{\lambda} + I
\]

First, note that \( Q' \) has the same instantaneous dynamics as \( Q \):

\[
P_t = e^{Qt} = e^{\tilde{\lambda}(Q'-I)t} = e^{-\tilde{\lambda}t} e^{-\lambda Q'} = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(Q'\lambda t)^k}{k!}
\]
Rearranging yields
\[ P_t = \sum_{k=0}^{\infty} e^{-\lambda t}(\bar{\lambda}t)^k k! (Q')^k \]

Then differentiating \( P_t \) in \( t \) yields
\[ \frac{d}{dt} P_t = \sum_{k=0}^{\infty} \bar{\lambda}k (\bar{\lambda}t)^{k-1} k! (Q')^k = \left\{ -I\bar{\lambda} + \bar{\lambda}Q' e^{-\bar{\lambda}t} - \bar{\lambda}tQ' e^{-\bar{\lambda}t} + ... \right\} \]

And taking the limit as \( t \downarrow 0 \)
\[ \lim_{t \downarrow 0} \frac{d}{dt} P_t = \lim_{t \downarrow 0} \left\{ -I\bar{\lambda} + \bar{\lambda}Q' e^{-\bar{\lambda}t} - \bar{\lambda}tQ' e^{-\bar{\lambda}t} + ... \right\} = -I\bar{\lambda} + \bar{\lambda}Q' = Q \]

So the dynamics of the two processes are the same, since \( D_tP_t = P_tQ = P_tQ' \). The transformation \( Q' = Q/\bar{\lambda} + I \) simply adjusts the holding times across all states so that they are uniform, yielding a homogeneous Poisson process with constant intensity \( \bar{\lambda} \) at each state.

Let \( t_k \) be the time at which the \( k \)-th transition time occurs, and \( \xi_k = t_k - t_{k-1} \) be the time increment between \( t_{k-1} \) and \( t_k \). Note that since the process is Poisson and holding times are uniform, the increments \( \xi_k \) are independently and identically exponentially distributed random variables.

Now consider the agents’ payoffs:
\[ \lim_{K \to \infty} E \left[ \int_0^{t_K} e^{-\rho t} U(i_k, N_k) dt \left| \sigma \right. \right] \]

Note that the payoffs \( U(i_k, N_k) \) depend on the agent’s current position in the network and the network structure, but not the time spent in that state or the history, due to the transformed \( Q' \) probabilities; events now arrive uniformly at all states \( N \) with probability \( \bar{\lambda} \). Then this can be written
\[ \lim_{K \to \infty} \sum_{k=0}^{K} \left[ \int_{t_k}^{t_{k+1}} e^{-\rho t} U(i_k(t), N_k(t)) dt \right] \left[ \sigma \right] = \lim_{K \to \infty} \sum_{k=0}^{K} \left[ \int_{t_k}^{t_{k+1}} e^{-\rho t} dt \right] \left[ \sigma \right] E[U(i_k, N_k)|\sigma] \]

This is because the system state at time \( t \) is independent of the time it has spent in each of the previous states, since the uniformization procedure has equated holding times across all states.
Then the term

\[
\mathbb{E} \left[ \int_{t_k}^{t_{k+1}} e^{-\rho t} dt \right] = \frac{1}{\rho} \mathbb{E} \left[ e^{-\rho t_k} - e^{-\rho t_{k+1}} \right] = \frac{1}{\rho} \mathbb{E} \left[ e^{-\rho \sum_{i=0}^{k} \xi_i} \left( 1 - e^{-\rho \xi_{k+1}} \right) \right]
\]

The increments \( \xi_k \) are independent and exponentially distributed with intensity \( \bar{\lambda} \), so that this becomes

\[
\mathbb{E} \left[ (e^{-\rho \xi})^k (1 - e^{-\rho \xi}) \right] = \mathbb{E} \left[ e^{-\rho \xi} \right]^k \left( 1 - \mathbb{E} \left[ e^{-\rho \xi} \right] \right) = \frac{1}{\rho} \left( \frac{\bar{\lambda}}{\rho + \lambda} \right)^k \left( 1 - \frac{\bar{\lambda}}{\rho + \lambda} \right) = \left( \frac{\bar{\lambda}}{\rho + \lambda} \right)^k \left( \frac{1}{\rho + \lambda} \right)
\]

Yielding an equation for the payoffs

\[
\lim_{K \to \infty} \sum_{k=0}^{K} \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} e^{-\rho t} U(i_k, N_k) dt \right] = \lim_{K \to \infty} \sum_{k=0}^{K} \left( \frac{\bar{\lambda}}{\lambda + \rho} \right)^k \frac{1}{\rho + \lambda} \mathbb{E} \left[ U(i_k, N_k) \right]
\]

Then the Bellman equation can be written as

\[
V(i, N) = \frac{U(i, N)}{\rho + \lambda} + \frac{\bar{\lambda}}{\rho + \lambda} \left( 1 - \frac{\lambda(N)}{\lambda} \right) V(i, N) + \frac{\bar{\lambda}}{\rho + \lambda} \sum_{i', N', \omega} \max_{\sigma_i(\omega)} \frac{\lambda(N)}{\lambda} \tau(\sigma_i(\omega), \sigma_{i-1}) [i', N', i, N, \omega] p[\omega | N] V(i', N')
\]

The recursive structure is more obvious when written

\[
V(i, N) = \frac{U(i, N)}{\rho + \lambda} + \frac{\bar{\lambda}}{\rho + \lambda} \left( 1 - \frac{\lambda(N)}{\lambda} \right) V(i, N) + \frac{\lambda(N)}{\lambda} \sum_{\omega} \max_{\sigma_i(N, \omega)} p[\omega | N] \tau(\sigma_i(\omega), \sigma_{i-1}) [i', N', i, N, \omega] V(i', N')
\]

\[
V(i, N) = \frac{U(i, N)}{\rho + \lambda} + \frac{\bar{\lambda}}{\rho + \lambda} \mathbb{E}_{i', N', \omega} \left[ \max_{\sigma_i} V(i', N') \right] \sigma
\]

and letting \( \beta = \frac{\bar{\lambda}}{\rho + \bar{\lambda}} \),

\[
V(i, N) = \frac{U(i, N)}{\rho + \lambda} + \beta \mathbb{E}_{i', N', \omega} \left[ \max_{\sigma_i} V(i', N') \right] \sigma
\]

\[\square\]

**Proof of Proposition 4.1**
Proof Let the set $\mathcal{V} = \times_1^K [-\bar{v}, \bar{v}]$, where

$$\bar{v} = \frac{1}{1 - \beta} \left\{ \sup_{i,N} \left| \frac{U(i, N)}{\rho + \lambda} \right| + \frac{1}{1 - \beta} \sup_{i,N} \left| \frac{U(i, N)}{\rho + \lambda} \right| \right\}$$

If $N(K)$ is under consideration, $K$ is finite, but if $\tilde{N}$ is under consideration, this is a subset of the space of real-valued sequences, $\ell_\infty$. Note that this is equal to the best payoff any agent can receive in any state, plus the discounted value of the best payoff any agent can receive at any state, discounted as an infinite sum. In other words, this is the payoff of an agent lucky enough to receive the supremum over all payoffs in all states as well as the maximal feasible transfer from a new entrant every period forever. Since this payoff is not attainable, the operators considered throughout this proof and the rest of the paper will map $\mathcal{V}$ into itself.

Since there are a countable number of network configurations for $N(K)$ and $\tilde{N}$, the state space $\tilde{N}$ is countable.

Let $\sigma_i(N, \omega)$ be the network strategy for agent $i$ in network $N$ at event $\omega$. Then Proposition 3.1 characterizes the agent’s Bellman equation at state $(i, N)$:

$$V(i, N) = \frac{U(i, N)}{\rho + \lambda} + \beta \left\{ \left( 1 - \frac{\lambda(N)}{\lambda} \right) V(i, N) + \frac{\lambda(N)}{\lambda} \sum_{i', N', \omega} \max_{\sigma_i^{'}} p[\omega | N] \tau(\sigma_i^{'}, \sigma_{-i}) [i', N' | i, N, \omega] V(i', N') \right\}$$

Consider the operator

$$V'(i, N) = \frac{U(i, N)}{\rho + \lambda} + \beta \left\{ \left( 1 - \frac{\lambda(N)}{\lambda} \right) V(i, N) + \frac{\lambda(N)}{\lambda} \sum_{i', N', \omega} \max_{\sigma_i^{'}} p[\omega | N] \tau(\sigma_i^{'}, \sigma_{-i}) [i', N' | i, N, \omega] V(i', N') \right\}$$

$$V' = T_\rho V$$

34
Then

\[ \|T_\sigma V(i, N) - T_\sigma V'(i, n)\| = \sup_{i, N} \frac{\lambda(N)}{\lambda} \sum_{i', N', \omega} p[\omega|N] \left\{ \max_{\sigma_i'} \tau(\sigma_i', \sigma_{i-1}) | i', N'| i, N, \omega V(i', N') \right\} - \]

\[ \sum_{i', N', \omega} p[\omega|N] \left\{ \max_{\sigma_i'} \tau(\sigma_i', \sigma_{i-1}) | i', N'| i, N, \omega V'(i', N') \right\} \]

Let \( \varepsilon > 0 \) be given, and let \( \|V - V'\| < \delta \). Suppose that

\[ \sum_{i', N', \omega} p[\omega|N] \left\{ \max_{\sigma_i'} \tau(\sigma_i', \sigma_{i-1}) | i', N'| i, N, \omega V(i', N') \right\} \]

\[ \geq \sum_{i', N', \omega} p[\omega|N] \left\{ \max_{\sigma_i'} \tau(\sigma_i', \sigma_{i-1}) | i', N'| i, N, \omega V'(i', N') \right\} \]

(If the opposite is true, then reversing the roles of \( V \) and \( V' \) in what follows yields the same result.)

Since \( \|V - V'\| < \delta \), for all \((i, N), -\delta < V(i, N) - V'(i, n) < \delta \), implying \( V(i, N) < V'(i, N) + \delta \), so that

\[ \|T_\sigma V(i, N) - T_\sigma V'(i, n)\| \leq \beta \frac{\lambda(N)}{\lambda} \sup_{i, N} \sum_{i', N'} p[\omega|N] \left\{ \max_{\sigma_i'} \tau(\sigma_i', \sigma_{i-1}) | i', N'| i, N, \omega V'(i', N') + \delta \right\} \]

\[ - \sum_{i', N'} p[\omega|N] \left\{ \max_{\sigma_i'} \tau(\sigma_i', \sigma_{i-1}) | i', N'| i, N, \omega V'(i', N') \right\} \]

\[ \leq \beta \frac{\lambda(N)}{\lambda} \delta \]

So for \( \|V - V'\| < \delta = \frac{\lambda}{\beta \lambda(N)} \varepsilon \), \( \|T_\sigma V - T_\sigma V'\| < \varepsilon \), and \( T \) is a continuous mapping.

The operator \( T \) maps \( \mathcal{V} \) into itself, since \(-\bar{v} \) and \( \bar{v} \) are not achievable from any state \((i, N)\), so that a mapping with discounting on \( \mathcal{V} \) necessarily maps vectors \( V \) into values between \(-\bar{v} \) and \( \bar{v} \).

Then since \( \|T_\sigma V\| \leq \bar{v} \) for any \( V \in \mathcal{V} \), \( T_\sigma \) maps \( \mathcal{V} \) into a subset of \( \mathcal{V} \).

The following analysis is written in terms of \( \mathcal{N} \), but all the essential details hold for \( \mathcal{N}(K) \) as well, using Brouwer’s fixed point theorem rather than Schauder’s:

Consider the set \( \mathcal{V} = \times_1^\infty [-\bar{v}, \bar{v}] \), where the index runs of the set of all bounded networks. Since
it is a countable product of compact sets, by Tychonoff’s theorem it is compact. Since it is a
subset of $\ell_\infty$, it satisfies the Hausdorff property, since $\ell_\infty$ is a Banach space, so that convex sets
can be separated by hyperplanes. The set is also convex, since convex combinations of elements lie
in $\mathcal{V}$. It is a topological space with the topology induced by the supnorm. Therefore, it is a convex
set in a locally convex linear topological Hausdorff space. If $T_\sigma$ is a continuous map from $\mathcal{V}$ to $\mathcal{V}$,
then continuity and compactness that it maps compact sets to compact sets.

Consider the following version of Schauder’s fixed-point theorem:

**Theorem 7.1 (Cheney, p. 337)** Let $D$ be a convex set in a locally convex linear topological Haus-
dorff space. If $f$ maps $D$ continuously into a compact subset of $D$, then $f$ has a fixed point.

Then since $T_\sigma$ and $\mathcal{V}$ satisfy the hypotheses of the theorem, there is then a fixed point $V^*$
satisfying $V^* = T_\sigma V^*$. □

**Proof of Proposition 4.2**

Proof Note that the set of mutually individually rational payoffs $Y_{ii'}(V)$ is a correspondence map-
ing a closed, convex subset of $\mathbb{R}^2$ into $\mathbb{R}^2$ that is convex- and closed-valued. Then, by Browder’s
Selection Theorem, there exists a continuous selection $\phi_{ii'}()$ from $Y_{ii'}(V)$ for each pair of networked
agents and representative entrants. So trade can be arranged in a fashion that is (i) pairwise
efficient, (ii) individually rational, and (iii) the transfers are continuous in the value function.

Consider the space $\mathcal{V}$ of value functions defined in the proof of the previous proposition. The
agents’ dynamic values can now be written as

$$V(i, N) = \frac{U(i, N)}{\rho + \lambda} + \mathbb{E}_{i', N', \omega} \left[ \max_{\sigma_i(N, \omega)} \left\{ \phi_{\omega \sigma_\omega}(Y_{\omega \sigma_\omega}(V)) + \beta V(i', N') \right\} \right] \left| \sigma \right|,$$

This notation is consistent since, if event $\omega = i$ occurs, then $\phi_{\omega \sigma_\omega}(Y_{\omega \sigma_\omega}(V)) = \phi_{i \sigma_i}(Y_{\omega \sigma_\omega}(V))$, where $\sigma_i$ is the identity of the networked agent that $i$ approaches, or $\phi_{ii'}$. Since $\phi_{ii'}(Y_{ii'}(V)) = -\phi_{i' \sigma_i}(Y_{ii'}(V))$, this notation is consistent for tracking the change in the payoff of the accepter, as well.

Once it is shown that $T_\sigma$ is a continuous map from $\mathcal{V}$ to $\mathcal{V}$, the rest of the proof is identical to
Proposition 4.1. Let $\varepsilon > 0$ be given and let $\|V - V'\| < \delta$. 

36
Note that

\[ Y_{ii'}(V' + \delta) = \beta \{ V'(i, N) + \delta - V'(i, N + e_{ii'}) - \delta \} \land \beta \{ V'(i', N + e_{ii'}) + \delta - V'(i', N) - \delta \} \]

\[ \beta \{ V'(i, N) + \delta - V'(i, N + e_{ii'}) - \delta \} \lor \beta \{ V'(i', N + e_{ii'}) + \delta - V'(i', N) - \delta \} = Y_{ii'}(V') \]

Then the transfers satisfy \( \phi_{ii'}(Y_{ii'}(V' + \delta)) = \phi_{ii'}(Y_{ii'}(V')) \). Since \( \| V - V' \| < \delta \), for all \( (i, N), \) \( V(i, N) \leq V'(i, N) + \delta \), and (suppressing the arguments of \( \phi_{ii'}(Y_{ii'}(V) = \phi_{ii'} \) and \( \phi_{ii'}(Y_{ii'}(V') = \phi_{ii'} \))

\[ \| T_\sigma V - T_\sigma V' \| = \sup_{i,N} \| \mathbb{E}_{i',N',\omega} \left[ \max_{\sigma_i(N,\omega)} \{ \phi_{i\omega\sigma} + \beta V(i', N') \} \right] \| - \mathbb{E}_{i',N',\omega} \left[ \max_{\sigma_i(N,\omega)} \{ \phi_{i\omega\sigma} + \beta V(i', N') \} \right] \| \]

\[ \leq \sup_{i,N} \| \mathbb{E}_{i',N',\omega} \left[ \max_{\sigma_i(N,\omega)} \{ \phi_{i\omega\sigma} + \beta V(i', N') + \delta \} \right] \| - \mathbb{E}_{i',N',\omega} \left[ \max_{\sigma_i(N,\omega)} \{ \phi_{i\omega\sigma} + \beta V(i', N') \} \right] \| \]

Now, adding and subtracting \( \phi'_{i\omega\sigma} \) yields

\[ \| T_\sigma V - T_\sigma V' \| \leq \sup_{i,N} \| \mathbb{E}_{i',N',\omega} \left[ \max_{\sigma_i(N,\omega)} \{ \phi_{i\omega\sigma} + \phi'_{i\omega\sigma} - \phi_{i\omega\sigma} + \beta V(i', N') + \delta \} \right] \| \]

\[ - \mathbb{E}_{i',N',\omega} \left[ \max_{\sigma_i(N,\omega)} \{ \phi_{i\omega\sigma} + \beta V'(i', N') \} \right] \|
\]

Breaking up the first \( \max\) operator yields

\[ \| T_\sigma V - T_\sigma V' \| \leq \sup_{i,N} \| \mathbb{E}_{i',N',\omega} \left[ \max_{\sigma_i(N,\omega)} \{ \phi_{i\omega\sigma} - \phi'_{i\omega\sigma} + \beta \delta \} \right] \| \]

and, by the triangle inequality,

\[ \| T_\sigma V - T_\sigma V' \| \leq \| \mathbb{E}_{i',N',\omega} \left[ \max_{\sigma_i(N,\omega)} \{ \phi_{i\omega\sigma} - \phi'_{i\omega\sigma} \} \right] \| + \beta \delta \]

and, since the supremum distance between \( \phi_{i\sigma_i} \) and \( \phi'_{i\sigma_i} \) is greater than any expected difference between the two,

\[ \| T_\sigma V - T_\sigma V' \| \leq \| \phi_{i\omega\sigma} - \phi'_{i\omega\sigma} \| + \beta \delta \]

Then by continuity of \( \phi_{ii'}(Y_{ii'}(V)) \), by taking \( V \) and \( V' \) close enough, the first term can be made arbitrarily small since \( \phi_{ii'}(Y_{ii'}(V)) \) is a continuous selection, and the second term bounds \( \beta \| V - V' \| \).
Therefore, for \( \|V - V'\| < \delta \), where \( \delta \) is sufficiently small,

\[
\|T_{\sigma}V - T_{\sigma}V'\| \leq \varepsilon
\]

so that \( T_{\sigma} \) is a continuous mapping. \( \square \)

**Proof of Proposition 4.3**

**Proof** Since the two players strictly prefer to add the link, there is an open set of transfers around \( \phi_{i'i'}(Y_{i'i'}(V)) \) for which the players still strictly prefer to add the link. Choosing any of these sufficiently close to \( \phi_{i'i'}(Y_{i'i'}(V)) \) will have no impact on the network strategies, so that the adjusted bargaining agreements will generate value functions that support the original network strategies. Since there are an uncountable number of these adjustments, there are an uncountable number of bargaining agreements that lead to the same equilibrium network strategies. \( \square \)

**Proof of Proposition 4.4**

**Proof** The first-order necessary condition is

\[
\frac{\mathbb{E}[V(i', N')|i, N, \sigma, \omega = i] - c'(\eta(i, N))}{\rho + \sum_k \eta(k, N)} - \frac{U(i, N) - c(\eta(i, N)) + \sum_k \eta(k, N) \mathbb{E}[V(i', N')|i, N, \sigma, \omega = k]}{(\rho + \sum_k \eta(k, N))^2} = 0
\]

and the second-order sufficient condition is

\[-\frac{1}{\rho + \sum_k \eta(k, N)} c''(\eta(i, N)) < 0\]

so each player’s maximization problem is own-concave.

To ensure uniqueness of equilibrium, we find a condition that implies that the Jacobian of the best-reply mappings has a dominant diagonal. This is equivalent to, for all \( (i, N) \),

\[
c''(\eta(i, N)) \geq \sum_{k \neq i} \mathbb{E}[V(i', N')|i, N, \sigma, \omega = k]
\]

This can be ensured by assuming that for all \( \eta(i, N) \),

\[
c''(\eta(i, N)) \geq (K - 1) \frac{1}{1 - \beta} \sup_{i, N} \left| \frac{U(i, N)}{\rho + \lambda} \right| = (K - 1) \left\lfloor \frac{1}{\rho} \sup_{i, N} |U(i, N)| \right\rfloor
\]

38
Proof of Proposition 4.5

Proof Consider the operator $T_\lambda$ given by

$$V' = T_\lambda V = \max_{\eta(i,N)} \frac{U(i,N) - c(\eta(i,N)) + \sum_k \eta(k,N)\mathbb{E}[V(i',N')|i,N,\sigma,\omega = k]}{\rho + \sum_k \eta(k,N)}$$

This is a continuous mapping, since the action intensities vary continuously in $V$; in fact, it is differentiable in $V$ since the intensity strategies themselves are, by the Implicit Function Theorem.

To ensure the uniformization approach of Section 2 applies, an upper bound on the activity level at each state must be provided independent of $V$. Note that in equilibrium, the intensity strategies of the agents satisfy

$$c'(\eta(i,N)) = \mathbb{E}[V(i',N')|i,N,\sigma,\omega = i] - V(i,N)$$

so that an upper bound on $\eta(i,N)$ is

$$\bar{\eta}_k = (c')^{-1} \left\{ \frac{1}{1 - \beta} \sup_{i,N} \left| \frac{U(i,N)}{\rho + \lambda} \right| \right\} = (c')^{-1} \left\{ \frac{1}{\rho} \sup_{i,N} |U(i,N)| \right\}$$

The far right-hand side is independent of $\bar{\lambda}$ and the current value function $V(i,N)$ under consideration. Then adding up the upper bounds on each process yields a bound on the aggregate process,

$$\bar{\lambda} = \sup_N \left\{ \bar{\eta}_1 + ... + \bar{\eta}_J \right\} + \sum_{j=1}^{|J|} \eta_j + |I|\eta$$

is an upper bound on the transition rates across all states. As long as this value is finite for a given network set $\mathcal{N}$, the approach adopted in the paper can be applied. If this becomes infinite, the uniformization approach fails, and other tools must be used. Since this bound is linear in the number of agents in the network, it will only work when a fixed bound on the sized of the network is given. Depending on the application at hand, an upper bound on the size of the network might be available, allowing the analysis to be extended to the entire space of bounded networks. □

Proof of Proposition 5.1

Proof Consider a myopic network strategy profile $\sigma^*$ with the dynamic dominance property. By
the dynamic dominance property,

\[
\frac{U(i, N \oplus \sigma^*_i)}{\rho + \lambda} + \beta \sum_{i', N', \omega} p[\omega | N] \tau_{\sigma^*}[i', N' | i, N, \omega] U(i', N') \geq \frac{U(i, N \oplus \sigma^*_i)}{\rho + \lambda} + \beta \sum_{i', N', \omega} \tau_{(\sigma^*_i, \sigma^*_i)}[i', N' | i, N, \omega] U(i', N')
\]

So that \(\sigma^*\) is optimal if there is a network event left in the game.

Now, suppose that \(\sigma^*\) is \(K\)-times optimal, so that

\[
\frac{U(i, N \oplus \sigma^*_i)}{\rho + \lambda} + \mathbb{E} \left[ \sum_{k=1}^{K} \beta^k \sum_{i', N', \omega} \tau_{\sigma^*}[i', N' | i, N, \omega] U(i', N') \right] \sigma^* 
\]

\[
\geq \frac{U(i, N \oplus \sigma^*_i)}{\rho + \lambda} + \mathbb{E} \left[ \sum_{k=1}^{K} \beta^k \sum_{i', N', \omega} \tau_{\sigma^*}[i', N' | i, N, \omega] U(i', N') \right] \sigma^* \sigma^*_i, \sigma^*_i
\]

Now, taking the expectation on the left-hand side with respect to \(\sigma^*_i\) and on the left-hand side with respect to any alternative \(\sigma^*_i\) maintains the inequality by dynamic dominance, and adding \(U(i, N \oplus \sigma^*)/(\rho + \lambda)\) on the left-hand side and \(U(i, N \oplus (\sigma^*_i, \sigma^*_i))/ (\rho + \lambda)\) yields

\[
\frac{U(i, N \oplus \sigma^*_i)}{\rho + \lambda} + \mathbb{E} \left[ \sum_{k=1}^{K+1} \beta^k \sum_{i', N', \omega} \tau_{\sigma^*}[i', N' | i, N, \omega] U(i', N') \right] \sigma^* 
\]

\[
\geq \frac{U(i, N \oplus \sigma^*_i)}{\rho + \lambda} + \mathbb{E} \left[ \sum_{k=1}^{K+1} \beta^k \sum_{i', N', \omega} \tau_{\sigma^*}[i', N' | i, N, \omega] U(i', N') \right] \sigma^* \sigma^*_i, \sigma^*_i
\]

so that if \(\sigma^*\) is optimal with \(K\) network events remaining, it is optimal with \(K+1\) network events remaining. Consequently, by induction, it is optimal for any \(K\). Intuitively, the ordering \(\succeq\) generates an extended set of preferences over sequences of networks, where stochastic dominance implies that acting myopically places more probability weight on the better sequences, as well as maximizes an agent’s immediate payoff.

Consider the series

\[
V_K(i, N) = \lim_{K \to \infty} \left\{ \frac{U(i, N \oplus \sigma^*)}{\rho + \lambda} + \mathbb{E} \left[ \sum_{k=1}^{K} \beta^k \sum_{i', N', \omega} \tau_{\sigma^*}[i', N' | i, N, \omega] U(i', N') \right] \sigma^* \right\}
\]

By the dominated convergence theorem, the limit and integral can be interchanged, and the limit
of the series exists. Call this limiting function $V^*$. Applying the operator

$$V'(i, N) = \frac{U(i, N)}{\rho + \lambda} + \frac{\bar{\lambda}}{\rho + \lambda} \left(1 - \frac{\lambda(N)}{\lambda}\right) V(i, N)$$

$\rho + \lambda \sum_{i', N', \omega} \max_{\sigma_i(N, \omega)} \frac{\lambda(N)}{\lambda} \tau_{i', N', i, N, \omega} \sigma_{i, \omega} | i', N' | i, N, \omega | V(i', N')$

yields $T_p V^* = V^*$ for all $(i, N)$, so $\sigma^*$ is a Markov perfect equilibrium. □

**Proof of Proposition 5.2**

**Proof** First, note that for buyers, $U(i, N)$ is decreasing in $\alpha^2(i)$ and for sellers, $U(i, N)$ is increasing in $\alpha(i)$. If there is a single network event left, the proposed strategies clearly maximize a player’s payoff.

To show the stochastic dominance property holds, we fix $\succeq$ and argue that using the myopic strategies provides a better transition distribution with respect to $\succeq$ than any alternative. This in turn establishes the regret-freeness property.

For matched buyers, the worst network event is being exogenously removed from the game, which is worse than a network event in which a partner seller is removed from the game (which is increasing in $\alpha^2(i)$), which is worse than a network event in which the buyer’s next period seller partner is matched to $\alpha^2(i)$ buyers.

We can partition the set of possible buyer states by $\succeq$: Exiting, being unmatched with $K$ other buyers, being matched to a seller along with $K$ other buyers. Note that since sellers poach by selecting a buyer randomly from the set of all buyers who are at a less attractive seller or are unmatched, there is no gain to being unmatched relative to being matched to an unattractive seller, and that there is no mortality effect from being matched or unmatched. Therefore, the lottery over transitions is best when an agent is at the most attractive seller possible: Other buyers may become matched to that seller, but the payoff cannot become worse than any of the other sellers. Likewise, if that seller is removed, it is better to become unmatched with the fewest rivals for limited attention from sellers. Therefore, the transitions are always better when matched to a more attractive seller, compared to a less attractive seller or being unmatched. Therefore, the transition density for the buyers under the proposed network strategies is regret-free.

For sellers, the worst network event is being exogenously removed from the game, which is worse than being matched to $\alpha(i)$ buyers, where more buyers is always better. Similarly, the number of
buyers has no effect on the likelihood of being exogenously removed, nor the likelihood of a partner buyer being removed. Since the buyer strategies funnel them towards the most attractive seller when they are selected by nature to act, only the most attractive seller will be approached by buyers. Since buyers accept a proposal whenever the seller is strictly more profitable, there is no loss to randomly selecting from any of the buyers who would accept. In particular, targeting buyers who are unmatched or matched to the worst seller, or the next worst seller, provides no benefit because the set of buyers who will accept a proposal depends only on whether the seller is more attractive than their current partner, and the size of this pool at the end of the next period will be the same regardless of where the buyer is poached from in the current period.

Therefore, the proposed strategies are a Markov perfect equilibrium of the game. □