On the trade-off between communication and control cost in event-triggered dead-beat control

Burak Demirel, Vijay Gupta, Daniel E. Quevedo and Mikael Johansson

Abstract—We consider a stochastic system where the communication between the controller and the actuator is triggered by a threshold-based rule. The communication is performed across an unreliable link that stochastically erases transmitted packets. To decrease the communication burden, and as a partial protection against dropped packets, the controller sends a sequence of control commands to the actuator in each packet. These commands are stored in a buffer and applied sequentially until the next control packet arrives. In this context, we study dead-beat control laws and compute the expected linear-quadratic loss of the closed-loop system for any given event-threshold. Furthermore, we provide analytical expressions that quantify the trade-off between the communication cost and the control performance of event-triggered control systems. Numerical examples demonstrate the effectiveness of the proposed technique.

Index terms— Event-triggering algorithms; Linear systems; Communication networks; Packet losses; Networked control systems

I. INTRODUCTION

Event-triggered implementations have emerged as an attractive alternative approach to the traditional periodic implementations since they can decrease the communication load in a networked control system compared to periodic ones while still guaranteeing closed-loop stability and performance; see e.g., the tutorial paper [1] and the references therein. However, quantifying the expected transmission rate of such implementations for a given closed-loop performance is challenging. A notable work in quantifying the relation between control performance and communication rate is that of Åström and Bernhardsson [2] who focused on the threshold-based event-triggered implementation of an impulse control of a single integrator system under Wiener process disturbances. They established that the event-based implementation gives a better performance than the traditional periodic implementation in terms of the state variance. Similarly, Henningsson et al. [3] proposed an event-triggered control scheme and compared its achievable performance with that of periodic control. Rabi [4] designed the jointly optimal event-triggering mechanism and the control law to minimize the average energy of the state signal. Likewise, Imer and Başar [5] formulated and solved an optimal control problem with limited control actions. However, the authors of [2]–[5] only considered scalar systems, which allows significant simplifications in the analysis. More recently, Meng and Chen [6] extended the work of [2] to a class of second-order stochastic systems, and showed that, for the same average transmission rate, the event-based impulse control outperforms the periodic one. Nevertheless, as stressed by the authors of [6], extending their analysis techniques to more general system dynamics remains an open and challenging problem.

B. Demirel was with the ACCESS Linnaeus Center, School of Electrical Engineering, KTH Royal Institute of Technology, Osquidas väg 10, SE 10044 Stockholm, Sweden. He is now with the Faculty of Electrical Engineering and Information Technology, University of Paderborn, Warburger Str. 100, 33098 Paderborn, Germany (e-mail: burak.demirel@upb.de).

D. E. Quevedo is with the Faculty of Electrical Engineering and Information Technology, University of Paderborn, Warburger Str. 100, 33098 Paderborn, Germany (e-mail: burak.demirel@upb.de, dqquevedo@ieee.org).

V. Gupta is with the Department of Electrical Engineering, University of Notre Dame, IN 46556 USA (e-mail: vgupta@nd.edu).

M. Johansson is with the ACCESS Linnaeus Center, School of Electrical Engineering, KTH Royal Institute of Technology, Osquidas väg 10, SE 10044 Stockholm, Sweden (e-mail: mikaelj@ee.kth.se).

The analysis of event-triggered control loops becomes even more involved when the network is unreliable. Blind and Allgöwer [7] extended the work of [2] to the case where transmissions from the sensor to the controller take place over an unreliable link. They derived analytical expressions for the control cost and the expected inter-event times for different packet loss rates. Rabi and Johansson [8] designed the optimal impulse control and the level triggering mechanism under packet losses with multiple loops sharing a common network. From a different perspective, Quevedo et al. [9] analyzed the stability of an event-triggered implementation of a controller in the presence of packet losses and limited processing resources. However, in all these works, packet losses only appear between the sensor and the controller, while the controller-actuator communication is reliable.

When the communication between the controller and the actuator is unreliable, sending a control sequence, which contains not only the control input to be used for the current time stamp, but also a few predicted future control actions, may provide robustness against networked imperfections, such as delays [10] and losses [11]–[13]. The key feature of packetized control scheme is that the controller generates a sequence of predicted control commands instead of a single control command, and transmits the entire sequence to the actuator. In case there is a packet drop from the controller to the actuator, the actuator can use previously received control actions that were designed to be applied at the current instant.

In this note, we consider a linear stochastic system where the communication between the controller and the actuator is dictated by a threshold-based event-triggering algorithm. The communication is across an erasure channel that stochastically erases transmitted data at any time step. As a control strategy, we apply a blend of packetized control (a well-known technique for dealing with unreliable communication channels) and cheap control (often used to minimize the state variance). Using a Markov renewal process-based framework, we are able to establish analytical expressions for the expected communication rate and control performance as measured by a linear-quadratic cost.

Notations: We write \( \mathbb{N} \) for the positive integers, \( \mathbb{N}_0 \) for \( \mathbb{N} \cup \{0\} \), and \( \mathbb{R} \) for the real numbers. Let \( \mathbb{R}^n \) be the set of real vectors of dimension \( n \). Vectors are written in bold lower case letters (e.g., \( \mathbf{u} \) and \( \mathbf{v} \)) and matrices in capital letters (e.g., \( A \) and \( B \)). If \( \mathbf{u} \) and \( \mathbf{v} \) are two vectors in \( \mathbb{R}^n \), the notation \( \mathbf{u} \leq \mathbf{v} \) corresponds to component-wise inequality. The set of all real symmetric positive semi-definite matrices of dimension \( n \) is denoted by \( \mathbb{S}_n^{\geq 0} \). We let \( \mathbf{0}_n \) be the \( n \)-dimensional column vectors of all zeros, \( \mathbf{1}_n \) be the vectors of all ones. For any given \( \mathbf{x} \in \mathbb{R}^n \), the \( \ell_\infty \)-norm is defined by \( \| \mathbf{x} \|_\infty = \max_{1 \leq i \leq n} |x_i| \).

For a square matrix \( A \), \( \text{Tr}(A) \) denotes its trace, \( |A| \) its determinant and \( \lambda_{\text{max}}(A) \) its maximum eigenvalue in terms of magnitude. Let \( X = \mathbf{y} \mathbf{y}^\top (A, Q) \) denote the positive semi-definite solution of the discrete Lyapunov matrix equation: \( AXA^\top - X + Q = 0 \) for any given \( Q \in \mathbb{S}_n^{\geq 0} \) and \( A \in \mathbb{R}^{n \times n} \) with \( \lambda_{\text{max}}(A) < 1 \). We use the symbol \( \mathbb{P}(x_i \in \mathcal{A}) \) to denote the indicator function of the set \( \mathcal{A} \). We employ \( \mathbf{x}_{i,k} \) to denote a shorthand notation for \( \{x_{1,1}, \cdots, x_{1,n}\} \). The probability of \( \Omega \) and the conditional probability of \( \Omega \) given \( \Gamma \) are denoted by \( \mathbf{P}(\Omega) \) and \( \mathbf{P}(\Omega | \Gamma) \), respectively. When \( \chi \) is a stochastic variable, \( \mathbb{E}[\chi] \) stands for the expectation of \( \chi \) and \( \text{Cov}[\chi] \) stands for.
the covariance of \( \chi \). An \( n \)-dimensional vector of real-valued random variables \( \mathbf{x} = [x_1 \cdots x_n]^T \) follows a multivariate normal distribution with mean \( \mu \in \mathbb{R}^n \) and covariance matrix \( \Sigma \in \mathbb{S}_{n \times n} \), denoted by \( \mathcal{N}(\mu, \Sigma) \), if its probability density function is given by

\[
f(x; \mu, \Sigma) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)}.
\]

The cumulative distribution function \( F(e; \mu, \Sigma) \) is defined as

\[
F(e; \mu, \Sigma) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \int_{ \mathbb{R}^n } e^{-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)} dx.
\]

Suppose one or more variates of multivariate normal random variable \( x \) are subject to one-sided or two-sided truncation, i.e., \( e^- \leq x \leq e^+ \). Then, \( x \) has a truncated normal distribution and its probability density function is given by

\[
f(x; \mu, \Sigma, e^-, e^+) = \frac{e^{-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)}}{e^{-\frac{1}{2}(e^- - \mu)^T \Sigma^{-1} (e^+ - \mu)}}.
\]

The R package \texttt{mtmvnorm} [14] provides several efficient methods to work with truncated random variables.

II. PROBLEM FORMULATION

This section summarizes the control architecture of our event-triggered control scheme and introduces the assumptions under which we will develop the performance analysis.

Control architecture: We consider the feedback control loop shown in Fig. 1. A physical plant \( \mathcal{G} \), whose dynamics can be represented by a linear stochastic system, is being controlled. A sensor \( \mathcal{S} \) takes periodic samples of the plant state \( x_k \) and transmits these to the controller node. The controller \( \mathcal{C} \) is event-triggered and computes new actuation commands only at times when the event-triggering condition is met. The communication between the controller \( \mathcal{C} \) and the actuator \( \mathcal{A} \) is lossy, and control packets are dropped at any time step independently of each other, with probability \( p_t \in (0, 1) \). As partial protection against these losses, the controller sends a sequence of predicted commands in each packet. The predicted commands are placed in a buffer at the actuator. In the absence of new control packets, the actuator reads the predicted control command for the current time from the buffer and applies this input to the plant. In this context, we are interested in deriving analytical performance guarantees, both in terms of control performance and the number of transmission attempts on the communication link between the controller and the actuator.

Process model: The dynamics of the plant \( \mathcal{G} \) can be described by the stochastic discrete-time linear system:

\[
x_{k+1} = Ax_k + Bu_k + w_k,
\]

where \( x_k \in \mathbb{R}^n \) is the state vector, \( u_k \in \mathbb{R}^m \) is the control signal, and \( w_k \in \mathbb{R}^n \) is a discrete-time zero-mean Gaussian white noise with covariance \( \Sigma_w \in \mathbb{S}_{n \times n} \), i.e., \( w_k \sim \mathcal{N}(0, \Sigma_w) \). The initial state \( x_0 \) is modeled as a random variable having a normal distribution with zero mean and covariance \( \Sigma_{x_0} \in \mathbb{S}_{n \times n} \), i.e., \( x_0 \sim \mathcal{N}(0, \Sigma_{x_0}) \). The process noise \( \{w_k\}_{k \in \mathbb{N}_0} \) is independent of the initial condition \( x_0 \).

The system matrix \( A \in \mathbb{R}^{n \times n} \) and the input matrix \( B \in \mathbb{R}^{n \times m} \) are constant, and \( B \) is assumed to be of full column-rank. Furthermore, the system (4) with \( w_k = 0 \), for all \( k \in \mathbb{N}_0 \) is assumed to be completely \( \nu \)-step controllable for some \( \nu \leq n \). In other words, for every \( x_k \in \mathbb{R}^n \), there exists a control sequence \( u_k = \{u_k, u_{k+1}, \ldots, u_{k+\nu-1}\} \) that transfers the state from \( x_k \) to the origin in \( \nu \) time steps. When \( A \) is non-singular (which it will be if it is obtained by sampling a delay-free continuous-time linear system), the system (4) is completely \( \nu \)-step controllable if and only if

\[
\text{rank}(A^1B, A^2B, \ldots, A^\nu B) = n.
\]

Controller design and performance criterion: We quantify the closed-loop performance using the quadratic cost

\[
J_{\infty} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \langle x_k (Q_k x_k + \rho u_k) \rangle_Q u_k,
\]

for given symmetric positive semidefinite matrix \( Q_k \) and symmetric positive definite matrix \( Q_u \). We are particularly interested in the case when \( \rho \to 0 \), sometimes called the cheap control scenario; see, e.g., [16]. It is well known that the optimal controller for the cheap control scenario is a dead-beat controller which ensures that, in the absence of process noise, the state converges to zero in a finite number of steps [15]. Our analysis technique considers a standard linear dead-beat controller

\[
u_k = K x_k,
\]

and a cost of the form (5). When the system is completely \( \nu \)-step controllable, one can always find such a controller \( K \) that drives the system state to zero in \( \nu \) steps (see e.g., [15]).

We use a packetized dead-beat controller to reduce transmissions over the communication channel and to guard against losses. If the event-triggering rule leads to the controller executing the control algorithm at time \( k \in \mathbb{N}_0 \), it computes and transmits a sequence of control commands

\[
U_k = \{K x_k, K(A + BK)x_k, \ldots, K(A + BK)^{\nu-1} x_k\}
\]

which would transfer the process state of (4) from \( x_k \) to the origin in at most \( \nu \leq n \) time steps in the absence of process noise. We assume the presence of a buffer of length \( \nu - 1 \) at the actuator. When a new set of control actions arrives from the controller, the actuator immediately applies the first control action in the set and stores the rest of the control actions in a buffer; see Fig. 1. In the next \( \nu - 1 \) time steps, the controller issues no transmissions even if the event-triggered condition is met. Rather, the actuator applies the control commands sequentially from the buffer. If the buffer is empty, the actuator applies zero input (cf. [17]). Note that this is consistent with the dead-beat assumption: in the absence of noise, the state would be at the origin after \( \nu \) steps and (6) would evaluate to zero.

After each successful packet transmission, the controller is switched off for \( \nu - 1 \) time steps, and then it is switched on again. The controller uses a simple threshold-based rule

\[
\|x_k\|_2 > \varepsilon
\]

to determine if a new control sequence should be computed, and a transmission should be attempted between the controller and

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{Block diagram of the event-triggered control system with a (linear) plant \( \mathcal{G} \), a controller \( \mathcal{C} \), a sensor \( \mathcal{S} \), an actuator \( \mathcal{A} \), a buffer with queue size \( \nu - 1 \), a comparator, and an unreliable communication link.}
\end{figure}
In the latter case, it calculates and sends a sequence of control signals, i.e., $U_k$. If this data packet is successfully received and acknowledged, then the controller moves to Sleep mode. Otherwise, the controller moves into RTx mode. In this mode of operation, even if the triggering condition is not satisfied, the controller computes and transmits a new control sequence, i.e., $U_{k+1}$. However, this must be done to the actuator by using the latest state information. Whenever a successful transmission occurs, the controller goes into Sleep mode. A key feature of the Sleep mode is that the actuator continues applying control commands stored in the buffer. The controller stays in this mode of operation until the buffer runs out of data. After that, the controller switches back to StandBy mode.

To elucidate the situation, it is convenient to introduce the variables $r_k, \gamma_k$, and $\eta_k$. The integer variable $r_k$ describes how many time steps the controller stays in StandBy mode where $x_{k+1} = Ax_k + w_k$. Thereby, the value of $r_k$ influences the probability of moving out of StandBy mode. The integer variable $\gamma_k$ denotes how long the controller remains in RTx mode; this variable also represents the number of consecutive transmission failures since leaving StandBy mode. The probability of staying in RTx mode is $P(\gamma_k = n) = p_k^n, n \geq 1$. The integer variable $\eta_k$ is a countdown timer, which denotes the amount of calculated plant input values stored in the buffer. Since each control packet transmitted from the controller to the actuator comprises a fixed number of plant input values whenever the controller moves to Sleep mode, the countdown timer evolves deterministically.

Under the assumption of the use of packetized dead-beat control actions, defined in (7), the evolution of the process $\{x_k\}_{k \in \mathbb{N}_0}$ can be rewritten as:

$$x_{k+1} = (A + BK)^t \left[ A^k + \gamma_k A^m w_{k-r_k} - t_{k-m} + \sum_{m=0}^{\gamma_k} A^m w_{k-m} - \sum_{m=0}^{\gamma_k} A^m w_{k-m} \right],$$

where

$$t_k = \begin{cases} 0 & \text{if } r_k < 0 \land \gamma_k \neq 0 \text{ and } \eta_k = 0, \\ \nu & \text{if } r_k = 0 \land \gamma_k = 0 \land \eta_k = 0. \end{cases}$$

It is worth noting that $x_{k+1}$ depends stochastically on $r_k, \gamma_k$, and $\eta_k$. These processes evolve as (see Fig. 2):

$$\begin{bmatrix} r_{k+1} \\ 0 \end{bmatrix} = \begin{bmatrix} \min(r_k, T_k) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $r_k$ is the number of consecutive transmission failures since leaving StandBy mode.

A key feature of our method is that, despite $r_k$ depending on $x_k$, the behavior of the event-triggered unreliable communication can be described by the Markov chain $\{\Omega_k\}_{k \in \mathbb{N}_0} \triangleq \{(r_k, \gamma_k, \eta_k)\}_{k \in \mathbb{N}_0}$. 

III. MAIN RESULTS

The proposed controller adapts its mode of operation to the system state and the communication outcomes. Thereby, it achieves an improved trade-off between the control performance and the communication (or energy) expenditure. As can be seen in Fig. 2, the controller operates in three distinct modes: StandBy, RTx, and Sleep. In StandBy mode, the controller listens to the sensor but does not transmit any control messages unless the triggering condition is met.
Lemma 1 The process \( \{\Omega_k\}_{k \in \mathbb{N}_0} \) is an ergodic, time-homogeneous Markov chain in a countably infinite state-space \( B = \{0, 1, \ldots, T\} \times \mathbb{N}_0 \times \{0, 1, \ldots, \nu - 1\} \) and has a unique invariant distribution \( \pi \).

Bear in mind that at the time instant when the controller enters StandBy mode (i.e., \( r_k = \gamma_k = \eta_k = 0 \)), \( t_k \) becomes \( \nu \) and, in turn, the state \( x_{k+1} \) only relies on the noise sequence \( \{w_{k-v+1}, \ldots, w_k\} \) as described in (9). Similarly, during the time when the controller is in StandBy mode, the process runs without feedback. As a result, the state \( x_{k+1} \) also only depends on the noise trajectory \( \{w_{k-v+1}, \ldots, w_{k+1}\} \).

To characterize the transition probabilities of the Markov chain \( \{\Omega_k\}_{k \in \mathbb{N}_0} \) in StandBy mode, we introduce the random processes:

\[
\xi(k) = \begin{cases} \sum_{i=0}^{v-1} A^i w_{k-i} & \text{if } i = 0, \\ \sum_{i=0}^{v-1} A^i w_{k-i} + \sum_{j=0}^{v-1} A^j w_{k-i-j} & \text{if } i \neq 0, 
\end{cases}
\]

(11)

for all \( i \in \{0, 1, \ldots, T - 1\} \). The probability density functions of \( \xi(k) \) are time-invariant since the noise sequence \( \{w_{k}\}_{k \in \mathbb{N}_0} \) is white and stationary. Hence, we can drop the time index \( k \) to simplify notation. The vector-valued random variable \( \Delta_{\text{in}} \triangleq [\xi(0) \xi(1) \cdots \xi(\nu-1)]^\top \) has a multi-variate normal distribution with mean \( \Delta_{\text{in}} = 0_{\nu \times 1} \) and covariance matrix \( \Sigma_{\text{in}} = \Sigma : \begin{bmatrix} \Sigma_1 & \Sigma_2 & \cdots & \Sigma_{\nu-1} \\ \Sigma_2 & \Sigma_1 & \cdots & \Sigma_{\nu-2} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{\nu-1} & \Sigma_{\nu-2} & \cdots & \Sigma_0 \end{bmatrix} \),

where \( \Sigma \triangleq \sum_{i=0}^{v-1} A^i \Sigma_{w} A^\top i \). For each \( i \in \{1, \ldots, T\} \), we define the events:

\[
\mathcal{F}_i = \bigcap_{j=0}^{i-1} \{ \| \xi(j) \|_\infty \leq \varepsilon \},
\]

(12)

with the convention \( \mathcal{F}_0 \) is a sure event. Thus, we have:

\[
P(\mathcal{F}_i) = F(\varepsilon 1_{\nu \times 1}; 0_{\nu \times 1}, \Sigma_{\text{in}})
\]

(13)

with \( P(\mathcal{F}_0) = 1 \). It is worth noting that transition probabilities in RTx mode are \( p_{i, j} \) for \( i \neq j \) since the packet loss process is i.i.d., and transition probabilities in Sleep mode are equal to one because, in this mode, the controller acts deterministically.

For the transition probabilities, we use the shorthand notation:

\[
P\left( (i_1, j_1, l_1) \mid (i_0, j_0, l_0) \right) \triangleq P\left( (r_{k+1}, \gamma_k+1, \eta_{k+1}) \mid (i_0, j_0, l_0) \right)
\]

\[
= (i_1, j_1, l_1) \mid (r_k, \gamma_k, \eta_k) = (i_0, j_0, l_0).
\]

The transition probabilities can be computed by using the following lemma:

Lemma 2 The non-null transition probabilities of the Markov chain \( \{\Omega_k\}_{k \in \mathbb{N}_0} \) are given by

\[
P\left( (0, 0, 0) \mid (i, j, l) \right) = 1
\]

\[
P\left( (i, l, l+1) \mid (i, j, l) \right) = 1
\]

\[
P\left( (0, \nu - 1) \mid (i, 0, 0) \right) = (1 - p_{l}) p_{i}
\]

\[
P\left( (i + 1, 0, 0) \mid (i, 0, 0) \right) = 1 - p_{l}
\]

\[
P\left( (i, 1, 0) \mid (i, 0, 0) \right) = p_{l} p_{i}
\]

\[
P\left( (i, 0, 1) \mid (i, j, 0) \right) = p_{l}
\]

\[
P\left( (i, j, 1) \mid (i, j, 0) \right) = 1 - p_{l}
\]

for all \( i \in \{0, \ldots, T\}, j \in \mathbb{N}_0 \) and \( l \in \{2, \ldots, \nu - 1\} \).

A visit of the Markov chain \( \{\Omega_k\}_{k \in \mathbb{N}_0} \) to any state \((i, j, \nu - 1)\) with \( i \in \{0, \ldots, T\} \) and \( j \in \mathbb{N}_0 \) corresponds to a successful transmission of the control signal from the controller to the actuator, and therefore, in view of the ergodic theorem,

\[
\pi_{ST} = \lim_{n \to \infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{i=0}^{T-1} \sum_{j=0}^{\nu-1} \mathbb{I}(r_k = i, \gamma_k = j, \eta_k = \nu - 1),
\]

(14)

is the empirical frequency of successful reception. With the transition probabilities of the Markov chain at hand, we can give an explicit characterization of the expected (successful and attempted) communication rates of the event triggered control system:

Theorem 1 (Communication rate) Under the event-triggered rule (defined in § II), the expected rate of successful reception of control packets at the actuator is

\[
\pi_{ST} = \frac{1}{1 - p_{l}}
\]

and the expected rate of attempt transmissions between the controller and the actuator is

\[
\pi_{AT} = \frac{\pi_{ST}}{1 - p_{l}}.
\]

Remark 1 Using the MATLAB command mwcndf or toolboxes related to the double truncated multivariate normal distribution (e.g., [14]), it is possible to compute the transition probabilities in Lemma 2 with ease. These can, then, be used in (15) to calculate the communication rate.

The next theorem characterizes the expected linear-quadratic cost.

Theorem 2 (Control performance) Consider the problem formulation in § II with the event-triggering algorithm and the dead-beat controller described in § II. Suppose that

\[
p_{l} \lambda_{\text{max}}(A)^2 < 1.
\]

Then, for a given event-threshold \( \varepsilon > 0 \), the empirical average of the control loss function is

\[
J_{\infty} = \left[ p_{l} \sum_{i=0}^{\nu-1} X_{0i} + p_{l} Y_{l} \right] + \sum_{i=0}^{T-1} \mathbb{E}(Q \Sigma_{i-1})
\]

\[
+ p_{l} \left( \sum_{j=0}^{\nu-1} X_{ij} + p_{l} Y_{l} \right) \prod_{i=0}^{l-1} (1 - p_{l}) \pi_{ST},
\]

(17)

where

\[
X_{ij} = \mathbb{E}(\Theta_{ij}((1 - p_{l}) \Sigma_{i} + p_{l} \Sigma_{w})),
\]

\[
Y_{l} = \mathbb{E}(\Psi((1 - p_{l}) \Sigma_{l} + p_{l} \Sigma_{w})),
\]

\[
Z = \sum_{j=0}^{\nu-2} \sum_{i=0}^{T-1} \mathbb{E}(A^\top Q_{i} A^\top \Sigma_{w}),
\]

with, for all \( j \in \{0, \ldots, \nu - 1\} \),

\[
\Psi = \lambda_{\text{yapk}}(\sqrt{p_{l}} A^\top, Q_{a}),
\]

\[
\Theta_{ij} = \lambda_{\text{yapk}}(\sqrt{p_{l}} A^\top, (A + BK)Q_{i}^\top (A + BK)^\top),
\]

where \( Q_{a} = p_{l} K^\top Q_{a} K \) and the truncated variances:

\[
\Sigma_{i} = \text{Cov}(\xi(i) \mid \mathcal{F}_{i+1}),
\]

\[
\Sigma_{i}^\infty = \mathbb{E}(\xi(i) \mid \|\xi(i)\|_\infty > \varepsilon, \mathcal{F}_{i}),
\]

\[
\Sigma_{\mathcal{F}_{i}} = A \text{Cov}(\xi(T - 1) \mid \mathcal{F}_{i}) A^\top + \Sigma_{w},
\]

and \( \mathcal{F}_{i} \) defined as in (12).
As shown in Fig. 3, the analytical results obtained in Theorem 1 and 2 match Monte Carlo simulations perfectly. Note that the successful communication rate is limited to 0.5 since $\nu = 2$. Therefore, the event-triggered control sends at most one packet every two sampling instances. Extensive numerical simulations confirm that the effect of time-out on performance diminishes with increased value of time-out. While the event-threshold $\varepsilon$ is increasing, the change in the communication rate and control performance becomes more and more apparent.

Fig. 4(a) highlights the differences between our packetized event-based control algorithm and the baseline threshold-based event-triggered control algorithm, formed as

$$u_k = \begin{cases} K x_k & \text{if } ||x_k||_\infty > \varepsilon \text{ or } r_k > T, \\ 0 & \text{otherwise} . \end{cases} \quad (18)$$

For the lossless case ($p_\ell = 0$), Fig. 4(a) shows that the packetized event-triggered control outperforms the baseline threshold-based implementation (18) as long as the average attempted transmission rate is smaller than 0.5. If we allow for higher communication rates, then the baseline implementation performs better on average.

The differences between the two variations are more striking in the case of packet losses. Fig. 4(b) compares communication vs. control trade-off for the packetized and the baseline implementation of the threshold-based event-triggered control when the loss rate is 20%. In this case, the packetized implementation strictly dominates the baseline implementation. When a comparable performance is searched for, this can be done at a dramatic decrease in communication cost. This performance improvement can be understood by observing that in order to reset the state $x_k$ in absence of the process noise, it is necessary to apply two consecutive control commands computed by the dead-beat controller. Whenever the packetized controller succeeds in transmitting a packet, this sequence of control commands will be available to the actuator and can be applied without interruption. In the baseline implementation (18), on the other hand, the likelihood that consecutive packet transmissions will be successful is only $(1 - p_\ell)^2$. Therefore, the state will often not be brought back close to the origin after an event triggering.

To conform with the guidelines of reproducible research, the R/\texttt{Matlab} file to generate the results presented above is publically available at https://people.kth.se/~demirel/papers_html/TAC15.html.

V. Conclusions

In this paper, we developed a theoretical framework to analyze the trade-off between the communication cost and the control performance of an event-triggering algorithm for control over an unreliable network. We assumed that a threshold-based event-triggering algorithm governs the decision of when to transmit the information from the controller to the actuator. Additionally, we assumed the presence of a buffer at the actuator to store the control command sequence received from the controller to mitigate the detrimental effect of packet loss. We developed a multi-dimensional Markov chain model which characterizes the attempted and successful transmissions of control signals over an unreliable communication link. By combining this communication model with an analytical model of closed-loop performance we provided a systematic way to analyze the trade-off between the communication cost and control performance by appropriately selecting an event threshold.

VI. Acknowledgment

The authors are grateful to Prof. Serdar Yüksel for many fruitful discussions. The authors would also like to thank the associate editor and anonymous reviewers for suggestions that helped to improve the quality of the manuscript.
This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2016.2606590, IEEE Transactions on Automatic Control

VII. APPENDIX

Proof of Lemma 1: We begin by proving that the process \( \{ \Omega_k \} \in \mathbb{N}_0 \) is a Markov chain. A key feature of this proof is that since we use predictive deadbeat control actions, provided in (7), for \( \nu \) consecutive time steps, at the instant when the controller enters the StandBy mode (i.e., \( r_k = \gamma_k = \eta_k = 0 \)), \( x_{k+1} \) only depends on the noise trajectory \( \{ w_{k-v+1}, \ldots, w_k \} \) as described in (9). In fact, we have: 
\[
\begin{align*}
\mathbb{P}(x_{k+1} = \gamma_{k+1}, \eta_{k+1}, r_{k+1} | r_{0:k}, \gamma_{0:k}, \eta_{0:k}) &= \int_{\mathbb{R}^n} \mathbb{P}(r_{k+1}, \gamma_{k+1}, \eta_{k+1}, x_{k+1} | r_{0:k}, \gamma_{0:k}, \eta_{0:k}) dx_{k+1} \\
&= \int_{\mathbb{R}^n} \mathbb{P}(r_{k+1}, \gamma_{k+1}, \eta_{k+1}, x_{k+1} | r_{0:k}, \gamma_{0:k}, \eta_{0:k}) dx_{k+1} \\
&= \int_{\mathbb{R}^n} \mathbb{P}(x_{k+1} = \gamma_{k+1}, \eta_{k+1}, r_{k+1} | r_{0:k}, \gamma_{0:k}, \eta_{0:k}) dx_{k+1} \\
&= \int_{\mathbb{R}^n} \mathbb{P}(r_{k+1}, \gamma_{k+1}, \eta_{k+1}, x_{k+1} | r_{k}, \gamma_{k}, \eta_{k}) dx_{k+1} \\
&= \mathbb{P}(r_{k+1}, \gamma_{k+1}, \eta_{k+1} | r_k, \gamma_k, \eta_k),
\end{align*}
\]
where (a) and (e) come from the definition of the conditional probability. The equality in (b) holds because \( x_{k+1} \) depends stochastically on \( r_k, \gamma_k \) and \( \eta_k \); \( r_{k+1}, \gamma_{k+1}, \) and \( \eta_{k+1} \) depend on \( x_{k+1}, r_k, \gamma_k \) and \( \eta_k \). Notice that knowing \( r_k = i, \gamma_k = j, \) and \( \eta_k = l \) implies knowing \( r_k = i, \ldots, r_{k-i} = 0, \gamma_k = j, \gamma_{k-i} = 0, \) and \( \eta_k = l, \ldots, \eta_{k-\nu-1} = 0 \). This concludes that the process \( \{ \Omega_k \} \in \mathbb{N}_0 \) is a Markov chain.

The proof is completed by showing the ergodicity of this Markov chain. This chain is clearly irreducible because it is possible to move from one state to another when \( \epsilon > 0 \) and \( 0 < p_{\ell} < 1 \). The chain is also aperiodic since it is irreducible, and the state \( (r_k, \gamma_k, \eta_k) = (0, 0, 0) \) has a non-zero returning loop for \( \epsilon > 0 \). The distribution of the return time to the state \((0, 0, 0)\) is, for \( n = 0 \) and \( m \geq 0 \),
\[
\mathbb{P}_0(T_0 = \nu + m) = (1 - p_{\ell})p_{\ell}^m p_0, 
\]
and, for \( n \geq 1 \) and \( m \geq 0 \),
\[
\mathbb{P}_0(T_0 = \nu + n + m) = (1 - p_{\ell})p_{\ell}^m p_0 \prod_{j=0}^{n-1}(1 - p_j).
\]
Using the distribution mentioned above, we write the expected return times as
\[
\mathbb{E}_0[T_0] = \sum_{n=0}^{\nu} \sum_{m=0}^{\nu} (\nu + n + m) \mathbb{P}_0(T_0 = \nu + n + m) \\
= \frac{pe_l}{1 - p_l} + \nu p_0 + \sum_{i=1}^{\nu} (\nu + i)p_i \prod_{j=0}^{i-1}(1 - p_j) \\
\leq \frac{pe_l}{1 - p_l} + \frac{T^2 + (1 + 2\nu)T + 2\nu}{2} < \infty;
\]
therefore, the chain is positive recurrent. The process \( \{ \Omega_k \} \in \mathbb{N}_0 \) is an ergodic Markov chain because it is irreducible, aperiodic, and positive recurrent. By [25, Thm 2.1], we conclude that this Markov chain has an invariant distribution. ■

Proof of Lemma 2: The only interesting case is when the controller moves from StandBy mode to either RTx or Sleep mode because remaining in RTx mode and moving from RTx to Sleep mode have a known geometric distribution, and staying in Sleep mode is a deterministic event. Therefore, we first focus on the case when the controller moves from StandBy mode to Sleep mode. Since \( r_k = \gamma_k = \eta_k = 0 \) is equivalent to \( t_k = \nu \) (see (10)), we have:
\[
\mathbb{P}(r_{k+1} = 0, \gamma_{k+1} = 0, \eta_{k+1} = 0 | r_k, \gamma_k, \eta_k = 0) = \mathbb{P}(\| \xi_k(0) \|_\infty > \epsilon, \delta_{k+1} = 1 | t_k = \nu) \\
\overset{(a)}{=} \mathbb{P}(\| \xi_k(0) \|_\infty > \epsilon, \delta = 1) \\
\overset{(b)}{=} \mathbb{P}(\delta = 1) \mathbb{P}(\| \xi_k(0) \|_\infty > \epsilon) = (1 - p_0) p_0,
\]
where (a) holds because \( t_k \) is independent of the noise trajectory \( \{ w_{\nu-v+1}, \ldots, w_k \} \) and the binary variable \( \delta_{k+1} \), and (b) is true since \( \delta_{k+1} \) is independent of the process noise. Similarly, for any \( i \in \{1, \ldots, T - 1\} \), we have:
\[
\mathbb{P}(r_{k+1} = i, \gamma_{k+1} = 0, \eta_{k+1} = 0 | r_k = i, \gamma_k = 0, \eta_k = 0) \\
\overset{(a)}{=} \mathbb{P}(r_{k+1} = i, \gamma_{k+1} = 0, \eta_{k+1} = 0 | r_k = i, \gamma_k = 0, \eta_k = 0) \\
\overset{(d)}{=} \mathbb{P}(\| \xi_k(i) \|_\infty > \epsilon, \delta_{k+1} = 1 | \| \xi_{k-i}(-j) \|_\infty \leq \epsilon, 0 \leq j < i, t_{k+1} = \nu) \\
\overset{(e)}{=} \mathbb{P}(\| \xi_k(i) \|_\infty > \epsilon, \delta = 1 | \| \xi_k(i) \|_\infty > \epsilon) \\
\overset{(f)}{=} \mathbb{P}(\| \xi_k(i) \|_\infty > \epsilon | \| \xi_k(i) \|_\infty \leq \epsilon, 0 \leq j < i) \mathbb{P}(\delta = 1) \\
\overset{(g)}{=} (1 - p_0) \mathbb{P}(\| \xi_k(i) \|_\infty > \epsilon, F_i) = (1 - p_0) \left( 1 - \frac{\mathbb{P}(F_{k+1})}{\mathbb{P}(F_i)} \right),
\]
where (c) comes from the Markov property, (d) follows from the definitions in (11), (e) holds because \( t_{k+1} \) is independent of the process noise after the time step \( k - \nu - i + 1 \) and the binary variable
δ_{k+1}, and (f) is true since δ_{k+1} is also independent of the process noise. Lastly, the combination of these expressions and (13) yields the desired result, expected by Gaussian integrals (2).

In a similar fashion, we can derive the transition probabilities from StandBy mode to RTx mode by replacing \( P(\delta = 1) \) with \( P(\delta = 0) \) in the aforementioned equations.

**Proof of Theorem 1:** The Markov chain \( \{\Omega_k\}_{k \in \mathbb{N}_0} \) is ergodic; therefore, it has a stationary distribution. Thus, using the ergodicity property, we write:

\[
\pi_{00} = (1 - p_1)p_0\pi_{00}, \quad \pi_{ij} = (1 - p_1)p_{ij}0, \quad \pi_{i0} = p_1p_{00}, \quad \pi_{i1} = p_1p_{i0}, \quad \pi_{i(j+1)} = p_1\pi_{ij0}, \quad \pi_{i(j+1)0} = (1 - p_1)\pi_{i00}, \quad \pi_{ij1}, \quad \pi_{01}, \quad \pi_{0(i+1)}.
\]

Since

\[
p_0 + \sum_{i=1}^{T} p_i \prod_{j=0}^{i-1} (1 - p_j) = 1,
\]

when \( p_T = 1 \), using equations derived above, we obtain:

\[
\sum_{i=0}^{T} \sum_{j=0}^{\nu-1} \pi_{ij0} = \frac{p_1}{1 - p_1} \pi_{000}, \quad \sum_{i=0}^{T} \sum_{j=0}^{\nu-1} \pi_{ij1} = (\nu - 1)\pi_{000}, \quad \sum_{i=0}^{T} \sum_{j=0}^{\nu-1} \pi_{ij0} = \sum_{i=0}^{T} \sum_{j=0}^{\nu-1} (1 - p_j) \pi_{000}.
\]

Using the balance equation, i.e., \( \sum_{i=0}^{T} \sum_{j=0}^{\nu-1} \pi_{ij0} = 1 \), and equations (20), (21), and (22), we compute:

\[
\pi_{000} = \frac{1}{1 - p_1} + \sum_{i=0}^{T} \sum_{j=0}^{\nu-1} (1 - p_j).
\]

The expected rate of successful transmissions is calculated as

\[
\pi_{ST} = \sum_{i=0}^{T} \sum_{j=0}^{\nu-1} \pi_{ij0} = \pi_{000}.
\]

The expected rate of attempted transmissions can be computed as 
\( \pi_{AT} = \pi_{ST}^{\nu}/(1 - p_1) \) by combining (20) and (23).

**Lemma 3 (Lyapunov equation [26]):** The discrete-time algebraic Lyapunov equation given by

\[
X = p_1A^TXA = Q,
\]

where \( A \in \mathbb{R}^{n \times n} \) and \( X, Q \in \mathbb{S}^{n}_{\geq 0} \), has a unique solution if and only if \( p_1n(A^T)^2 < 1 \) with \( 0 < p_1 < 1 \). The unique solution of (24) can be expressed as an infinite series:

\[
X = \sum_{k=0}^{\infty} p_1^k (A^T)^k QA^k.
\]

**Proof of Theorem 2:** The Markov chain \( \{\Omega_k\}_{k \in \mathbb{N}_0} \) is aperiodic and positive recurrent on the countable state space \( \mathcal{B} = \{0, \cdots, T\} \times \mathbb{N}_0 \times \{0, \cdots, \nu - 1\} \). Therefore, the process \( \{\Omega_k\}_{k \in \mathbb{N}_0} \) is an ergodic Markov chain with stationary distribution \( \pi_{ijl} \triangleq \lim_{k \to \infty} \mathbb{P}(r_k = i, \gamma_k = j, \eta_k = l) \) for all \( i, j, l \in \mathcal{B} \). By the ergodic theorem for Markov chains [25, Thm. 4.1], the linear-quadratic loss (5) converges to

\[
J_\infty = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} J_k = \lim_{k \to \infty} \mathbb{E}_{\pi}[J_k],
\]

where

\[
J_k \triangleq x_k^TQ_x x_k + \rho u_k^TQ_u u_k.
\]

Note that (26) holds since \( x_{k+1} \) is a function of \( r_k, \gamma_k, \) and \( \eta_k \) as described in (9). By the law of total expectation, we have:

\[
\mathbb{E}_{\pi}[J_k] = \sum_{i,j,l \in \mathcal{B}} \mathbb{E}[J_k | r_k = i, \gamma_k = j, \eta_k = l] \pi_{ijl},
\]

where \( \pi_{ijl} \triangleq \mathbb{P}(r_k = i, \gamma_k = j, \eta_k = l) \). Using the law of total expectations and Baye’s rule, for \( i = l = 0 \) and \( j \in \mathcal{N} \), we have:

\[
H_{0j}^k = p_0p_1^k \mathbb{E}[J_k || x_{k+1-j} || \infty > \varepsilon, \eta_{k+1-j} = 0]_{\pi_{000}}^{\nu - 1-j} = \left[ \text{Tr}\left((A^{\nu-1})^T Q_x (A^{\nu-1})^T \mathbb{E}[\xi_{k+1-j} | 0] \mathbb{E}[\xi_{k+1-j} | 0] \right) \right] \}
\]

where \( \pi_{000} \triangleq \mathbb{P}(r_k = i, \gamma_k = j, \eta_k = l) \). Using the law of total expectations and Baye’s rule, for \( i = l = 0 \) and \( j \in \mathcal{N} \), we have:

\[
H_{0j}^k = p_0p_1^k \mathbb{E}[J_k || x_{k+1-j} || \infty > \varepsilon, || x_{k+1-j} || \infty \leq \varepsilon, 1 \leq n \leq i, \eta_{k+1-j} = 0]_{\pi_{000}}^{\nu - 1-j} = \left[ \text{Tr}\left((A^{\nu-1})^T Q_x (A^{\nu-1})^T \mathbb{E}[\xi_{k+1-j} | 0] \mathbb{E}[\xi_{k+1-j} | 0] \right) \right] \}
\]

Similarly, for \( i \in \{1, \cdots, T - 1\} \), \( j \in \mathcal{N} \) and \( l = 0 \), we have:

\[
H_{jl}^k = p_0p_1^k \mathbb{E}[J_k || x_{k+1-j} || \infty > \varepsilon, || x_{k+1-j} || \infty > \varepsilon, 1 \leq n \leq i, \eta_{k+1-j} = 0]_{\pi_{000}}^{\nu - 1-j} = \left[ \text{Tr}\left((A^{\nu-1})^T Q_x (A^{\nu-1})^T \mathbb{E}[\xi_{k+1-j} | 0] \mathbb{E}[\xi_{k+1-j} | 0] \right) \right] \}
\]

In a similar way, for \( i = T, j \in \mathcal{N} \) and \( l = 0 \), we have:

\[
H_{Tj}^k = p_0p_1^k \mathbb{E}[J_k || x_{k+1-j} || \infty \leq \varepsilon, 1 \leq n \leq T, \eta_{k+1-j} = 0]_{\pi_{000}}^{\nu - 1-j} = \left[ \text{Tr}\left((A^{\nu-1})^T Q_x (A^{\nu-1})^T \mathbb{E}[\xi_{k+1-j} | 0] \mathbb{E}[\xi_{k+1-j} | 0] \right) \right] \}
\]

Similarly, for \( i = 0, j \in \mathcal{N}_0 \) and \( l \in \{1, \cdots, \nu - 1\} \), we derive:

\[
H_{0ji} = \mathbb{E}[J_k || x_{k-\nu+1-j+l} || \infty > \varepsilon, \eta_{k-\nu+1+l} = 0]_{\pi_{000}}^{\nu - 2-i} = \left[ \text{Tr}\left((A^{\nu-1})^T Q^* (A^{\nu-1})^T \mathbb{E}[\xi_{k-\nu+1+l} | 0] \mathbb{E}[\xi_{k-\nu+1+l} | 0] \right) \right] \}
\]

where \( Q^* = Q_x + \rho K^T Q_u K \) and \( A_c = A + BK \).

Note that the equations for the remaining cases of \( H_{ij}^k \) can be derived by following the same procedure used earlier in this proof.
Let’s define $H_{ijl}^\infty \triangleq \lim_{k \to \infty} H_{ijl}^k$ for all $i, j, l \in B$. Letting $k \to \infty$ and using Lemma 3, we obtain:

$$H_{i,j,l}^\infty + \sum_{i=0}^{T} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} H_{ijl}^k = \sum_{i=0}^{T} \left\{ (1 - p_i) p_l \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} T \left( (A^n)^T Q(A^n) \Sigma_i \right)^2 \right\} + \sum_{i=0}^{T} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \left( (A^n)^T Q(A^n) \Sigma_w \right),$$

with $\Theta_j = 2\gamma_2 \sqrt{\nu} A^T \Sigma_i^T + \nu_1 (A^T \Sigma_w)$. Similarly, using Lemma 3, we get:

$$\sum_{i=0}^{T} \sum_{j=0}^{\infty} H_{ijl}^\infty = \sum_{i=0}^{T} \sum_{l=0}^{\infty} p_i p_l \left( (A^n)^T Q(A^n) \Sigma_i \right)^2 + \sum_{i=0}^{T} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \left( (A^n)^T Q(A^n) \Sigma_w \right),$$

where $\Psi = \left\{ \begin{array}{ll} 1 & \text{if } j = 0, \ldots, T - 1, \\
\nu_1 & \text{if } j = 0, \ldots, T - 1, \\
\nu_2 & \text{if } j = 0, \ldots, T - 1. \end{array} \right.$

We conclude the proof by replacing $\pi_{000}$ with $\pi_{ST}$ in $J_\infty = \sum_{i=0}^{T} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} H_{ijl}^k$.  

REFERENCES