Sufficient Conditions for Stabilizability over Gaussian Relay and Cascade Channels

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Abstract—We present sufficient conditions for stabilizability of an unstable linear time invariant scalar system across an additive white Gaussian noise channel, with a relay assisting the observer. We consider two topologies: (i) a Gaussian relay channel, and (ii) a cascade of two Gaussian point to point channels. We assume that the control action is available at the relay, and that there are average power constraints on both the transmitter and the relay nodes. We propose coding schemes based on distributed stochastic approximation algorithms, and present sufficient conditions for the stabilizability of the plant through such schemes. The analysis suggests it may be useful to provide a relay node, even if the total transmission power remains the same.

I. INTRODUCTION

Networked control systems are now an active area of research (e.g., [1], [4] and the references therein). The performance of such systems is adversely affected by the detrimental effects such as random delays, data loss, data corruption, and so on introduced by the underlying communication network. The presence of various communication channel models in the control loop has been considered, including channels that introduce data loss (e.g., [6]), delay (e.g., [9]), digital noiseless channels (e.g., [11]), and additive white Gaussian noise channels (e.g., [3]).

In this paper, we are interested in stabilizability of a scalar unstable linear time invariant (LTI) discrete time system across an additive white Gaussian channel, when a relay node is present. Stability conditions for such systems when a relay is not present are available (e.g., [3], [10], [13]). Interesting parallels of the problem with schemes achieving the capacity of a Gaussian channel with feedback through the Schalkwijk-Kailath (SK) scheme [14], [15] are known [5]. Presence of relays can increase the capacity of a Gaussian channel, even if the total transmission power remains the same (e.g., [7]). Thus, it may be intuitively expected that the stabilizability region (in terms of the SNR required) in the presence of relays for AWGN channels will also be expanded. However, analytical results and coding schemes for this setup are not available. The problem is not trivial since, e.g., that there does not seem to be a distributed counterpart of the SK scheme for relay channels, that achieves a doubly exponential error decay required for mean squared stabilizability of an unstable plant [12].

We present sufficient conditions for stabilizing a scalar discrete-time LTI plant in the mean square sense by a remotely placed controller over a Additive White Gaussian (AWGN) channel in the presence of a relay node. We consider two arrangements: (i) when the controller has a direct path from the sensor (the relay channel), and (ii) when the controller does not have a direct path from the sensor (the cascade channel). We assume that the relay is full-duplex, i.e., it can send and receive at the same time. The coding schemes that we utilize are based on a distributed version of the Robbins-Munro stochastic approximation algorithm [2], [8]. Interestingly, our analysis suggests that the stability region may be increased by using a relay node even if the total transmission power remains the same.

II. PROBLEM SETUP

Consider an open loop unstable scalar linear time invariant process evolving as

\[ S(k + 1) = a S(k) + U(k), \]  

where \( S(k) \in \mathbb{R} \) is the state and \( U(k) \in \mathbb{R} \) is the control input. We assume that the initial condition \( S(0) \) is a random variable with a finite variance \( \sigma_0^2 \) and an arbitrary probability distribution. For ease of exposition, and without loss of generality, we assume that the state of the process is observed by the sensor. The state value \( S(k) \) is transmitted across a communication channel to the controller, that calculates the control input \( U(k) \) and applies it to the process in (1). The communication channel from the sensor to the controller is modeled as an additive white Gaussian noise channel with a relay node, while the communication from the controller to the process is assumed perfect. We consider two configurations of the relay channel:

1) In the setup depicted in Figure 1, a noisy version of the transmission from the sensor is received at both the relay and the controller. We refer to this setup as the Gaussian relay channel.

2) In the setup depicted in Figure 2, there is no direct path from the sensor to the controller. We refer to this setup as the cascade channel.

Note that in either of the setups, we assume that the relay node has access to the controller output via a perfect feedback path.

We adopt the following notation:

- The sensor, relay, and the controller are denoted by the nodes 1, 2, and 3 respectively.
- The transmission from node \( i \) across a channel is denoted by \( X_i \).
- The quantity received by node \( i \) from a channel is denoted by \( Y_i \).
- The noise over the sensor-relay channel (i.e., noise at node 2) is denoted by \( Z_2 \).
Throughout the paper, we assume the following:

- The noise corrupting the reception at the controller (i.e. noise at node 3) is denoted by $Z_3$.
- Every communication link is a zero-mean Additive White Gaussian Noise (AWGN) channel. Moreover, the noises on the various links are mutually independent and white.
- The signal from node $i$ to node $j$ is scaled by an attenuation factor $g_{i,j}$ that depends on the distance $d_{i,j}$ between nodes $i$ and $j$ as
  
  $$
  g_{i,j} = \begin{cases} 
  bd_i^{-\eta/2}, & i \neq j, \\
  0, & i = j, 
  \end{cases}
  $$

  where $\eta$ is the path loss exponent (typically $2 \leq \eta \leq 4$ for wireless communication), and $b$ is a constant. Every node knows the distance of the other two nodes from itself and hence knows the attenuation factors.

Thus, the outputs at the two channels can be written in a compact form as

$$
\begin{bmatrix}
Y_2(k) \\
Y_3(k)
\end{bmatrix} =
\begin{bmatrix}
g_{1,2} & 0 \\
g_{1,3} & g_{2,3}
\end{bmatrix}
\begin{bmatrix}
X_1(k) \\
X_2(k)
\end{bmatrix} +
\begin{bmatrix}
Z_2(k) \\
Z_3(k)
\end{bmatrix}.
\tag{2}
$$

We impose three constraints on the encoder and controller design:

- **Constraint $C_1$:** The control action must satisfy a controller cost constraint, $\sum_{k=0}^{\infty} \mathbb{E}[U(k)^2] < \infty$.
- **Constraint $C_2$:** There is an average power constraint imposed on the signals transmitted by the sensor and the relay. Thus, the encoding schemes must be such that the transmitted signals satisfy
  
  $$
  \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[X_1^2(k)] \leq P_1,
  $$

  $$
  \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[X_2^2(k)] \leq P_2.
  $$

- **Constraint $C_3$:** The encoders are assumed to be causal, but otherwise unconstrained in terms of computation and memory. The information structure at the encoders is as follows. If $h_1$ and $h_2$ be the encoding policies at the sensor and relay respectively, then

  $X_1(k) = h_1(S(0), \ldots, S(k), U(0), \ldots, U(k-1)),$

  $X_2(k) = h_2(Y_2(0), \ldots, Y_2(k), U(0), \ldots, U(k-1)).$

The problem that we are interested in this paper is to design the maps $h_1$ and $h_2$, and the controller $U(k)$ so that the process (1) is mean square stabilized, while satisfying the constraints $C_1$, $C_2$, and $C_3$. Recall that a system is said to be mean square stabilized if and only if irrespective of the initial state $S(0)$, the following conditions are satisfied:

$$
\mathbb{E}[S(k)] = 0,
\lim_{k \to \infty} \mathbb{E}[S^2(k)] = 0.
\tag{3}
$$

In Section III, we will present a recursive coding scheme that achieves mean square stability of the process. The coding scheme is based on distributed stochastic approximation. Distributed stochastic approximation [2], [8] is a distributed method used to calculate the zeros of a vector function $f(T) : \mathbb{R}^m \to \mathbb{R}^m$ from noisy observations of the function. Denote the $i$-th component of the function by $f_i$ and the current estimate of the zeros to be $\hat{T}(k)$. The method works as follows. Noisy observations of the form $f_i(\hat{T}(k)) = f_i(\hat{T}(k)) + N_i(k)$ are obtained and used to update each component of the estimate $\hat{T}(k)$ as

$$
\hat{T}_i(k) = \hat{T}_i(k - 1) - \beta_i(k)f_i(\hat{T}(k)),
\tag{4}
$$

where $\hat{T}_i(k)$ is the $i$-th component of $\hat{T}(k)$, $\beta_i(k)$ is a step size that needs to satisfy some technical constraints [2], and the initial estimate $\hat{T}_i(0)$ can be assumed to be any arbitrary finite value.
III. MAIN RESULTS

A. Preliminary Results

As stated earlier, the coding scheme we present is based on distributed stochastic approximation. In our design, every node $i$ updates and stores an estimate $\hat{S}_i(k)$ of the initial state $S(0)$. Since node 1 has perfect knowledge of the state value $S(0)$, the estimate $\hat{S}_1(k) = S(0)$ identically. We define the following estimation errors:

\[
\epsilon_2(k) := \hat{S}_2(k) - S(0), \\
\epsilon_3(k) := \hat{S}_3(k) - S(0), \\
\epsilon_{3,2}(k) := \hat{S}_3(k) - \hat{S}_2(k).
\]

The basic idea of the coding scheme is as follows. The sensor, relay, and the controller implement a distributed stochastic approximation scheme to calculate the zeros of the function $f : \mathbb{R}^3 \to \mathbb{R}^3$, the $i$-th component of which is given by

\[
f_i(T_1, T_2, T_3) = \begin{cases} 0 & \text{for } i = 1, \\
\sum_{j \neq i} \mu_{ij} f_{ij}(T_i, T_j) & \text{for } i = 2, 3,
\end{cases}
\]

where $f_{ij}(T_i, T_j) := (T_i - T_j)$, the variable $T_i$ is associated with the $i$-th node, $T_1 = S(0)$, and $\mu_i$ is a term suitably designed to satisfy the transmission power constraint $C_2$. The estimates $\hat{S}_2(k)$ and $\hat{S}_3(k)$ of the variables $T_2$ and $T_3$ are updated as in (4), while the estimate $\hat{S}_1(k)$ of the variable $T_1$ is fixed at the message $S(0)$, i.e., $\hat{S}_1(k) = S(0)$ identically. Thus, the zero of the function $f$ is attained by all estimates $\hat{S}_i$’s converging to $S(0)$. In other words, the distributed stochastic approximation algorithm would eventually yield the estimate at the controller being equal to the initial state $S(0)$.

However, the crucial additional property that needs to be satisfied for mean square stability is that the estimate $\hat{S}_3(k)$ converges to $S(0)$ at a rapid enough rate as shown by the following result.

**Lemma 1:** The LTI system in (1) can be mean square stabilized over a communication channel if the following conditions are satisfied:

\[
\mathbb{E}[\epsilon_3(k)] = 0, \\
\lim_{k \to \infty} a^{2k} \mathbb{E}[\epsilon_3^2(k)] = 0.
\]

**Proof:** Since the controller does not the know the state value $S(0)$ exactly, the controller takes actions using the estimates $\hat{S}_3(n)$. The controller actions are defined as

\[
U(k) = \begin{cases} -a\hat{S}_3(0), & k = 0, \\
a^{k+1}(\hat{S}_3(k) - \hat{S}_3(k-1)), & k \geq 1.
\end{cases}
\]

Thus, the state $S(k)$ evolves as

\[
S(k+1) = a^{k+1} \left( \hat{S}_3(k) - S(0) \right) = a^{k+1} \epsilon_3(k).
\]

The mean value and mean squared value of $S(k+1)$ is given by

\[
\mathbb{E}[S(k+1)] = -a^{k+1} \mathbb{E}[\epsilon_3(k)], \\
\mathbb{E}[S^2(k+1)] = a^{2(k+1)} \mathbb{E}[\epsilon_3^2(k)].
\]

Thus, if the conditions in (6) are satisfied, the process is mean square stabilized.  

The above result also presents the controller design. We will now present our coding scheme and show that it does satisfy the constraints in (6).

B. Coding Scheme

We now proceed to explain the coding scheme used at the sensor and the relay. We concentrate on the relay channel while noting that the coding scheme and its analysis carry over to the cascade channel if we set the channel gain $g_{1,3} = 0$. The code works as follows.

- **Initialization:** At time step $k = 0$,
  - The observer node 1 observes $S(0)$ and sends the input $X_1(0)$ given by
    \[
    X_1(0) = \sqrt{\frac{P_1}{\sigma_{S(0)}^2}} S(0).
    \]
  - The relay node 2 sends nothing, while it receives
    \[
    Y_2(0) = g_{1,2} X_1(0) + Z_2(0).
    \]

It then forms an estimate of $S(0)$ by scaling as follows:

\[
\hat{S}_2(0) = \frac{1}{g_{1,2}} \sqrt{\frac{\sigma_{S(0)}^2}{P_1}} Y_2(0).
\]

The estimation error $\epsilon_2(0)$ is given by

\[
\epsilon_2(0) = \frac{1}{g_{1,2}} \sqrt{\frac{\sigma_{S(0)}^2}{P_1}} Z_2(0).
\]

Clearly, $\epsilon_2(0)$ is zero-mean Gaussian with variance $\alpha_2(0)$, given by

\[
\alpha_2(0) = \frac{\sigma_{S(0)}^2}{g_{1,2}^2 P_1}.
\]

- The controller node 3 receives
  \[
  Y_3(0) = g_{1,3} X_1(0) + Z_3(0)
  \]
  and forms an estimate of $S(0)$ given by
  \[
  \hat{S}_3(0) = \frac{1}{g_{1,3}} \sqrt{\frac{\sigma_{S(0)}^2}{P_1}} Y_3(0).
  \]
  The estimation error $\epsilon_3(0)$ is again zero-mean Gaussian with variance $\alpha_3(0)$, given by
  \[
  \alpha_3(0) = \frac{\sigma_{S(0)}^2}{g_{1,3}^2 P_1}.
  \]

The controller calculates the control $U(0)$ according to Equation (7) and transmits it to the process.

- **Update:** At every time step $k \geq 1$,
  - Using the message transmitted by the controller at time $k-1$, the sensor and the relay can both calculate $\hat{S}_3(k-1)$. Moreover, the sensor can calculate $\epsilon_3(k)$, while the relay can calculate $\epsilon_{3,2}(k)$. Note that by
the sensor transmits minimum mean squared error estimate of $\epsilon_3(k)$ and $\epsilon_{3,2}(k)$ are jointly Gaussian with zero means, respective variances $\alpha_3(k)$ and $\alpha_{3,2}(k)$ and correlation coefficient $\rho(k)$ given by

$$\rho(k) = \frac{E[\epsilon_3(k)\epsilon_{3,2}(k)]}{\sqrt{\alpha_3(k)\alpha_{3,2}(k)}}.$$  \hspace{1cm} (11)

The quantities $\alpha_2(k)$, $\alpha_3(k)$, $\alpha_{3,2}(k)$ and $\rho(k)$ are deterministic and can be calculated prior to transmission by all the nodes. In particular,

$$\alpha_{3,2}(0) = \alpha_3(0) + \alpha_2(0)$$
$$\rho(0) = \frac{\alpha_3(0)}{\sqrt{(\alpha_3(0) + \alpha_2(0))\alpha_3(0)}}.$$  

We present recursive expressions for these quantities in Section III-C, and assume for the rest of the presentation that these quantities can be calculated by all the nodes.

- The sensor transmits

$$X_1(k) = \sqrt{\frac{P_1}{\alpha_3(k-1)}} \left( \hat{S}_3(k-1) - S(0) \right)$$
$$= \sqrt{\frac{P_1}{\alpha_3(k-1)}} \epsilon_3(k-1).$$  \hspace{1cm} (12)

- The relay transmits

$$X_2(k) = \sqrt{\frac{P_2}{\alpha_{3,2}(k-1)}} \left( \hat{S}_3(k-1) - \hat{S}_2(k-1) \right)$$
$$= \sqrt{\frac{P_2}{\alpha_{3,2}(k-1)}} \epsilon_{3,2}(k-1).$$  \hspace{1cm} (13)

Moreover, it updates its estimate as follows. Using the feedback from the controller and the received signal from the sensor, it first calculates

$$Y_2'(k)$$
$$= Y_2(k) - g_{1,2} \sqrt{\frac{P_1}{\alpha_3(k-1)}} \left( \hat{S}_3(k-1) - \hat{S}_2(k-1) \right)$$
$$= g_{1,2} \sqrt{\frac{P_1}{\alpha_3(k-1)}} \left( \hat{S}_2(k-1) - S(0) \right) + Z_2(k).$$

Using this quantity, the relay calculates the linear minimum mean squared error estimate of $S(0)$ given $Y_2'(k)$ and $\hat{S}_2(k-1)$ as

$$\hat{S}_2(k) = \hat{S}_2(k-1) - \frac{E[Y_2'(k)\epsilon_2(k-1)]}{E[Y_2'^2(k)]} Y_2'(k).$$ \hspace{1cm} (14)

Notice that the step size $\beta_2(k)$ used by the relay in the distributed stochastic approximation iteration (4) is given by $\beta_2(k) = \frac{E[Y_2'(k)\epsilon_2(k-1)]}{E[Y_2'^2(k)]}$.

- At time $k \geq 1$, the controller calculates the linear minimum mean squared error estimate of $S(0)$ given $Y_2'(k)$ and $\hat{S}_2(k-1)$ as

$$\hat{S}_3(k) = \hat{S}_3(k-1) - \frac{E[Y_3'(k)\epsilon_3(k-1)]}{E[Y_3'^2(k)]} Y_3(k).$$ \hspace{1cm} (15)

The step size $\beta_3(k)$ used by the controller is thus

$$\beta_3(k) = \frac{E[Y_3'(k)\epsilon_3(k-1)]}{E[Y_3'^2(k)]}.\hspace{1cm} (16)$$

Moreover, the controller then calculates the control input $U(k)$ using the equation (7) and transmits it.

Note that the inputs $X_1(k)$ to the various channels satisfy their respective power constraints.

**C. Calculation of the Variances**

We now proceed to evaluate the recursive expressions of $\alpha_2(k)$, $\alpha_3(k)$, $\alpha_{3,2}(k)$, and $\rho(k)$ as used in the coding scheme presented above. The following recursions do not depend on the data, and can be executed by any node.

- **Variance $\alpha_2(k)$ of the error at the relay node:** Since $\epsilon_2(k)$ is defined as $\hat{S}_2(k) - S(0)$, from (14) we obtain

$$\epsilon_2(k) = \epsilon_2(k-1) - \frac{E[Y_2'(k)\epsilon_2(k-1)]}{E[Y_2'^2(k)]} Y_2'(k).$$  \hspace{1cm} (17)

The variance of $\epsilon_2(k)$ can be obtained as

$$\alpha_2(k) = \alpha_2(k-1) - \frac{E[2Y_2'(k)|\epsilon_2(k-1)]}{E[Y_2'^2(k)]},$$ \hspace{1cm} (18)

with the initial condition in Equation (9). The terms in (17) can be further evaluated to be

$$E[Y_2'^2(k)] = g_{1,2}^2 \frac{P_1}{\alpha_3(k-1)} \alpha_2(k-1) + \sigma_2^2,$$

and

$$E[Y_2'(k)\epsilon_2(k-1)] = g_{1,2} \frac{P_1}{\alpha_3(k-1)} \alpha_2(k-1).$$  \hspace{1cm} (19)

Substituting (18) and (19) into (17), we get

$$\alpha_2(k) = \alpha_2(k-1) - r(k-1),$$ \hspace{1cm} (20)

where $r(k-1) = \left( \frac{g_{1,2}^2 \frac{P_1}{\alpha_3(k-1)} + \sigma_2^2}{g_{1,2} \frac{P_1}{\alpha_3(k-1)}} \right)$.

- **Variance $\alpha_3(k)$ of the error at the controller node:** Since $\epsilon_3(k)$ is defined as $\hat{S}_3(k) - S(0)$, from (15) we obtain

$$\epsilon_3(k) = \epsilon_3(k-1) - \frac{E[Y_3'(k)\epsilon_3(k-1)]}{E[Y_3'^2(k)]} Y_3(k).$$ \hspace{1cm} (21)

The variance of $\epsilon_3(k)$ can be obtained as

$$\alpha_3(k) = \alpha_3(k-1) - \frac{E[2Y_3'(k)|\epsilon_3(k-1)]}{E[Y_3'^2(k)]},$$ \hspace{1cm} (22)

with the initial condition in Equation (10). The terms in (22) can be simplified as

$$E[Y_3'^2(k)] = g_{1,3}^2 P_1 + g_{2,3}^2 P_2$$
$$+ 2g_{1,3}g_{2,3} \sqrt{P_1P_2} \rho(k-1) + \sigma_3^2.$$  \hspace{1cm} (23)
and
\[
E[Y_3(k)\epsilon_3(k-1)] = g_{1,3}\sqrt{P_1}\alpha_3(k-1)
\]
\[
+ g_{2,3}\sqrt{\frac{P_2}{\alpha_{3,2}(k-1)}}E[\epsilon_3(k-1)\epsilon_{3,2}(k-1)]
\]
\[
= \sqrt{\alpha_3(k-1)}\left(g_{1,3}\sqrt{P_1} + g_{2,3}\sqrt{P_2}\rho(k-1)\right). \quad (24)
\]
Substituting (23) and (24) into (22), we get
\[
\alpha_3(k) = \alpha_3(k-1)q(k-1), \quad (25)
\]
where
\[
q(k-1) = \frac{g_{2,3}^2P_2\left(1 - \rho^2(k-1)\right) + \sigma_3^2}{g_{1,3}^2P_1 + g_{2,3}^2P_2 + 2g_{1,3}g_{2,3}\sqrt{P_1P_2}\rho(k-1) + \sigma_3^2}. \quad (26)
\]

- Variance \(\alpha_{3,2}(k)\) and the correlation coefficient \(\rho(k)\): To begin with, define
\[
\gamma(k) := E[\epsilon_3(k)\epsilon_2(k)],
\]
so that
\[
E[\epsilon_{3,2}(k)\epsilon_2(k)] = \gamma(k) - \alpha_2(k).
\]
We will now find a recursive relation for \(\gamma(k)\). From (16) and (21), we can write
\[
\gamma(k) = E\left[(\epsilon_2(k-1) - \beta_2(k)\epsilon_2(k))\right]
\]
\[
\left((\epsilon_3(k-1) - \beta_3(k)Y_3(k))\right)
\]
\[
= \gamma(k-1) - \beta_2(k)E[\epsilon_3(k-1)Y_2(k)]
\]
\[
- \beta_3(k)E[\epsilon_2(k-1)Y_3(k)]
\]
\[
+ \beta_3(k)\beta_2(k)E[\epsilon_2(k-1)\epsilon_3(k-1)Y_2(k)Y_3(k)]. \quad (27)
\]
We need to compute the three terms \(E[\epsilon_3(k-1)Y_2(k)]\), \(E[\epsilon_2(k-1)Y_3(k)]\), and \(E[\epsilon_2(k-1)\epsilon_3(k-1)Y_2(k)Y_3(k)]\) to evaluate the above expression. Using the expressions from the coding scheme, we obtain
\[
E[\epsilon_3(k-1)Y_2(k)] = g_{1,2}\sqrt{\frac{P_1}{\alpha_3(k-1)}}\gamma(k-1)
\]
\[
E[\epsilon_2(k-1)Y_3(k)] = g_{1,3}\sqrt{\frac{P_1}{\alpha_3(k-1)}}\gamma(k-1)
\]
\[
+ g_{2,3}\sqrt{\frac{P_2}{\alpha_{3,2}(k-1)}}(\gamma(k-1) - \alpha_2(k-1)),
\]
\[
E[Y_2(k)Y_3(k)] = g_{1,3}g_{1,2}\frac{P_1}{\alpha_3(k-1)}\gamma(k-1)
\]
\[
+ g_{1,2}g_{2,3}\sqrt{\frac{P_1P_2}{\alpha_3(k-1)\alpha_{3,2}(k-1)}}(\gamma(k-1) - \alpha_2(k-1)).
\]
Using the expectations above and substituting the value of \(\beta_3(k)\) from (14), we can simplify the sum of first and second terms of (27).
\[
\gamma(k-1) - \beta_2(k)E[\epsilon_3(k-1)Y_2(k)] = r(k-1)\gamma(k-1). \quad (28)
\]
Similarly, the third and fourth terms of (27) can be simplified to obtain
\[
- \beta_3(k)E[\epsilon_2(k-1)Y_3(k)] + \beta_2(k)\beta_3(k)E[Y_2'(k)Y_3(k)]
\]
\[
= -\beta_3(k)r(k-1)\left(g_{1,3}\sqrt{\frac{P_1}{\alpha_3(k-1)}}\gamma(k-1)
\]
\[
+ g_{2,3}\sqrt{\frac{P_2}{\alpha_{3,2}(k-1)}}(\gamma(k-1) - \alpha_2(k-1))\right). \quad (29)
\]
Substituting (28) and (29) in (27), we get a recursive relation for \(\gamma(k)\), with the initial value \(\gamma(0) = 0\). Given \(\alpha_2(k), \alpha_3(k)\), and \(\gamma(k)\), we can calculate \(\alpha_{3,2}(k)\) and \(\rho(k)\) using
\[
\alpha_{3,2}(k) = \alpha_3(k) + \alpha_2(k) - 2\gamma(k),
\]
\[
\rho(k) = \frac{\alpha_3(k) - \gamma(k)}{\sqrt{\alpha_3(k)\alpha_{3,2}(k)}}.
\]

D. Stability Analysis

We now present the stability conditions when the coding scheme described above is used to stabilize the process (1).

Theorem 2: Consider the problem formulation presented in Section II with the coding scheme presented in Section III-B. The process (1) is mean square stabilized over the AWGN relay channel if
\[
\log(a) < \lim_{n \to \infty} \frac{1}{2n} \sum_{k=1}^{n} \log \frac{1}{q(k-1)},
\]
where
\[
q(k-1) = \frac{\alpha_3(k)}{\alpha_3(k-1)}
\]
is calculated as in Equation (26).

Proof: It is easy to see that \(E[\epsilon_2(0)] = E[\epsilon_3(0)]\). It is known that the linear minimum mean squared error is an unbiased estimator. Thus, \(E[\epsilon_3(k)] = 0\) for all \(k \geq 0\), which is the first condition in (6). For the second condition, from (25) we can see that
\[
a^{2n}\alpha_3(n) = a^{2n}\alpha_3(0) \prod_{k=1}^{n} q(k-1).
\]
If the condition in (30) is satisfied, then \(a^{2n}\alpha_3(n) \to 0\) and mean square stability is obtained.

It can be easily shown that the bound presented in Equation (30) is non-trivial.

Proposition 3: The right hand side of the equation (30) is finite.

Proof: First note that the quantity \(1/q(k-1)\) from (26) is an increasing function of \(\rho(k-1)\) and hence is maximized by choosing \(\rho(k-1) = 1\). Thus,
\[
\frac{1}{q(k-1)} \leq \frac{g_{1,3}^2P_1 + g_{2,3}^2P_2 + 2g_{1,3}g_{2,3}\sqrt{P_1P_2} + \sigma_3^2}{\sigma_3^2}.
\]
This implies that
\[
\lim_{n \to \infty} \frac{1}{2n} \sum_{k=1}^{n} \log \frac{1}{q(k-1)} \leq \frac{1}{2} \log \left( 1 + \frac{g_1 \rho_1 + g_2 \rho_2 + 2g_1 g_2 \sqrt{\rho_1 \rho_2}}{\sigma_3^2} \right).
\]

The constraints \( C_2 \) and \( C_3 \) are satisfied by construction of the coding scheme. We can also show that the constraint \( C_1 \) is satisfied by the proposed design.

**Proposition 4:** The controller satisfies the cost constraint \( \sum_{k=0}^{\infty} \mathbb{E}[U(k)^2] < \infty \).

**Proof:** From (7) and (15), we can write
\[
U(k) = -a^{k+1}(S_1(k) - S_2(k - 1)) = a^{k+1} \mathbb{E}[Y_3(k)c_3(k - 1)] Y_3(k) \Rightarrow \mathbb{E}[U^2(k)] = a^{2(k+1)} \mathbb{E}[Y_3(k)c_3(k - 1)] \mathbb{E}[Y_3^2(k)] = a^{2(k+1)} c_3(k - 1) c = a^{2} c_3(0) \prod_{j=0}^{k-1} [a^2 q(j)],
\]
where
\[
c = \frac{g_1 \rho_1 + g_2 \rho_2 + 2g_1 g_2 \sqrt{\rho_1 \rho_2}}{\sigma_3^2}. \]

Using (30), the result follows.

The special case of the cascade channel can be derived from the above results that have been presented for the relay channel.

**Theorem 5:** The LTI system in (1) can be mean square stabilized over the cascade channel if (30) is satisfied with \( g_{1,3} = 0 \).

**Proof:** The cascade channel is the same as the relay channel, except that there is no path from the sensor to the controller. The coding scheme and the analysis in Theorem 2 thus hold for the cascade channel with \( g_{1,3} = 0 \).

**IV. Numerical Example**

Consider the process (1) where the controller is placed at a (spatial) distance of 2 units from the sensor. The relay is placed at the midpoint of the line joining the sensor and controller. The system parameters are considered to be \( P_1 = P_2 = 0.5, \sigma_3^2 = 0.5, \sigma_2^2 = 1, \text{ and } \eta = 2 \). All the logarithms are taken to the base 2. Under these constraints, the limit in (30) can be evaluated to be 0.2410 for the relay channel. Under the same constraints, the limit for the cascade channel is 0.1995. If we consider that there is no relay node and use the power \( P_1 + P_2 = 1 \) at the sensor node, the sufficient condition for mean square stability (e.g., [3]) is \( \log(a) < 0.1610 \). This shows that using a relay node may be beneficial even if the total power constraint on transmitting nodes remains the same, i.e., even if the power used by the relay node is at the expense of the power used by the sensor node.

**V. Conclusions and Future Work**

In this paper, we derived sufficient conditions for mean square stabilizability of a scalar linear time invariant open loop unstable plant over a relay AWGN channel and a cascade of two point-to-point AWGN channels. We proposed a coding scheme, which makes use of distributed stochastic approximation algorithms to ensure appropriately fast rate of convergence of the estimate of the initial state at the controller to the correct value. The scheme can also be interpreted as a distributed version of the Schalkwijk-Kailath coding scheme for point-to-point channels with both the sensor and the relay transmitting innovations with respect to the estimate at the controller at every step. We analyzed the stability region of the closed loop system with the proposed scheme under average transmission power constraints for both the sensor and the relay node. Interestingly, the stability region may be increased by using a relay node even if the total transmission power remains the same.

An immediate extension of this work would be to consider a half-duplex rather than a full-duplex relay channel. Similarly, the effect of process noise in (1) can be considered. Another direction of work can be to utilize the distributed approach used in this paper to consider the problem of stabilizing a plant over more general networks.

**References**


