Improving Control Performance across AWGN Channels using a Relay Node

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Consider an unstable linear time invariant system in which the sensor transmits information to a controller across an additive white Gaussian noise channel. The designer can optionally utilize a relay node to assist the controller; however the total transmission power consumed by the sensor and the relay node is constant. We consider two topologies: (i) a Gaussian relay channel, and (ii) a cascade of two Gaussian point to point channels. We propose coding schemes and present sufficient conditions for the stabilizability of the plant through such schemes. The analysis suggests that it is useful to utilize a relay node, even if the total transmission power remains the same.

Keywords: Networked control systems, additive white Gaussian noise channel, relay and cascade configurations

1. Introduction

Networked control systems are now an active area of research (see, e.g., (Antsaklis and Baillieuel 2004) and the references therein). The performance of such systems is adversely affected by the detrimental effects introduced by the underlying communication network, such as random delays, data loss, data corruption, and so on. The presence of various communication channel models in the control loop has been considered, including channels that introduce data loss (e.g., (Gupta et al 2009)), delay (e.g., (Luck and Ray 1990)), digital noiseless channels (e.g., (Nair et al 2007)), and additive white Gaussian noise channels (e.g., (Braslavsky et al 2007)). The works (Sahai and Mitter 2006; Martins and Dahleh 2008) have also considered stabilizability conditions and performance limitations induced by arbitrary communication channels.

In this short note, we are interested in an unstable linear time invariant (LTI) discrete time system being stabilized across an additive white Gaussian channel, when a relay node is present. If the relay is not present, the problem is quite well studied. Stabilizability conditions in terms of the signal to noise ratio (SNR) of the channel and the unstable eigenvalues of the system have been derived (e.g., (Tatikonda and Mitter 2004; Braslavsky et al 2007; Martins and Dahleh 2008)). Interesting parallels of the problem with the Schalkwijk-Kailath (SK) scheme that achieves capacity for a Gaussian channel with feedback are known (Elia 2004). A recent work (Zaidi et al 2010) has also addressed the problem of stabilizing two scalar linear time invariant systems over noisy multiple-access and broadcast communication channels.

It is known that relays can increase the capacity of a Gaussian channel, even if the total transmission power (consumed together by the encoder and the relay) remains the same (e.g., (Kumar et al 2009)). However, mean square stabilizability of an unstable process across a communication channel requires that the error in the estimate of the state value at the controller decays doubly exponentially (Sahai and Mitter 2006). Thus, it is unclear if relays can be used...
to expand the stabilizability region. Although coding schemes based on the SK scheme exist for the point to point Gaussian channel, there does not seem to be a distributed counterpart of the SK scheme for relay channels.

In this paper, we develop sufficient conditions for stabilizing a discrete-time LTI plant in the mean squared sense by a remotely placed controller over an Additive White Gaussian (AWGN) relay channel. We consider and propose coding schemes for two scenarios: (i) one in which the controller has a direct path from the sensor (the relay channel), and (ii) another in which the controller does not have a direct path from the sensor (the cascade channel). The coding schemes that we utilize are based on a distributed version of the Robbins-Munro stochastic approximation algorithm (Kushner and Yin 2003; Borkar 2008). A similar scheme was presented in (Kumar et al 2009) for the relay channel in the context of communication across a relay channel with feedback. However, the scheme in this paper is different since we do not consider a feedback path from the relay node to the sensor. We analyze the stability region of the closed loop system with the proposed scheme under a total average transmission power constraint for the sensor and the relay node. Interestingly, our analysis suggests that the stabilizability region can be increased by using a relay node even if the total transmission power remains the same.

The paper is organized as follows. The problem is setup formally in Section 2. The encoder and decoder design at the sensor, relay node and controller are presented and analyzed in Section 3. The stabilizability conditions are numerically illustrated in Section 4.

Notation We denote the set of real numbers by $\mathbb{R}$ and that of positive integers by $\mathbb{Z}_+$. Denote the $i$-th basis vector by $e_i$. Thus, $e_i \in \mathbb{R}^m$ has all elements 0, except for the $i$-th one which is unity. $I$ denotes an identity matrix of suitable dimensions. By $\log(x)$ we mean logarithm to the base 2. Denote the $m$ eigenvalues of matrix $M$ by $\lambda_j(M)$, $j = 1, \cdots, m$.

2. Problem Setup

Consider Figs. 1 and 2 with the following assumptions:

**Process:** Consider an open loop unstable linear time invariant process evolving as

$$S(k + 1) = AS(k) + BU(k),$$  \hspace{1cm} (1)

where $S(k) \in \mathbb{R}^m$ is the state and $U(k) \in \mathbb{R}$ is the control value. The initial condition $S(0)$ is a random variable with an arbitrary probability distribution and a finite covariance that we assume for simplicity to be $\sigma^2_S I$, where $\sigma^2_S > 0$. Without loss of generality, we assume that the matrix $A$ is in the Jordan normal form and can be expressed as

$$A = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix},$$

where $A_s \in \mathbb{R}^{(m-n) \times (m-n)}$ and $A_u^{-1} \in \mathbb{R}^{n \times n}$ are Schur stable. Note that $0 \leq n \leq m$, and we assume that an empty $A_s$ (resp. $A_u$) corresponds to $n = m$ (resp. $n = 0$). For simplicity, we assume that the process state is observed\(^1\), that the pair $(A, B)$ is controllable, and all the eigenvalues $\lambda_j(A)$ of $A$ are such that $\log(|\lambda_j(A)|)$ is rational.

**Communication channel:** We model a wireless link as an additive white Gaussian noise channel:

$$Y(k) = gX(k) + Z(k),$$  \hspace{1cm} (2)

\(^1\)All the results can be extended to the case when the state is only partially observed at the expense of more notation; see Remark 1.
where \( X(k) \in \mathbb{R} \) and \( Y(k) \in \mathbb{R} \) are the input and the output of the channel at time \( k \) respectively, and \( Z(k) \) is white Gaussian noise with mean 0 and covariance \( \Sigma_Z > 0 \). The factor \( g \) is an attenuation factor that depends on the distance \( d \) between the transmitter and receiver. A typical model is \( g = bd^{-\eta/2} \), where \( \eta \) is the path loss exponent (typically \( 2 \leq \eta \leq 4 \) for wireless communication), and \( b \) is a constant. This model usually holds only for distances \( d \), for which \( bd^{-\eta/2} \ll 1 \). However, since the constant terms \( b \) and \( \eta \) do not affect the following analysis in terms of relative performance, we have assumed the same path loss exponent model.

**Sensor to controller communication:** In the baseline case in which the relay node is not utilized, the sensor transmits to the controller (that is located a distance \( d \) away) across a channel of the form (2). Thus, at every time \( k \), it transmits a scalar \( X_1(k) = f(S(0), S(1), \ldots, S(k-1)) \), with a suitably designed function \( f : \mathbb{R}^k \to \mathbb{R} \). The transmitted variable must satisfy a time and ensemble average power constraint of the form

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[|X_1^2(k)|] \leq P. \tag{3}
\]

If the relay is utilized, it is considered to be placed at a distance \( d_1 \) from the sensor and \( d_2 \) from the controller. The sensor transmits a quantity \( X_1(k) \in \mathbb{R} \) which is once again a causal function \( X_1(k) = \bar{f}(S(0), \ldots, S(k-1)) \) of the information to which it has access, while satisfying a power constraint

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[|X_1^2(k)|] \leq P_1, \tag{4}
\]

for a value \( P_1 \leq P \) that is under the control of the designer. We consider two cases. In the first case (the relay channel) shown in Figure 1, a noisy version of \( X_1(k) \) is received at both the relay and the controller. In the second case (the cascade channel) shown in Figure 2, \( X_1(k) \) is not received at the controller. We denote the quantity received at time \( k \) at the relay from the sensor by

![Figure 1](image1.png)  
Figure 1. Block diagram for an unstable plant being controlled by a controller across an AWGN relay channel  

![Figure 2](image2.png)  
Figure 2. Block diagram for an unstable plant being controlled by a controller across an AWGN cascade channel
for a power $P_2 \leq P$ as specified by the designer. In the relay channel case, this transmission interferes with transmission from the sensor. Thus, the controller receives $Y_3(k) = g_{13}X_1(k) + g_{23}X_2(k) + Z_3(k)$, where $g_{13} = bd^{-n/2}$ and $g_{23} = bd_2^{-n/2}$. In the cascade channel case, $Y_3(k) = g_{23}X_2(k) + Z_3(k)$. We assume that the noises $Z_2(\cdot)$ and $Z_3(\cdot)$ are mutually independent and white with mean 0 and variances $\sigma_2^2$ and $\sigma_3^2$ respectively. If $P_1 + P_2 = P$, the relay and the sensor nodes share the power used in the baseline case. It is unclear if a relay node can expand the stabilizability region with this constraint. Resolving this question is a major motivation of this work.

**Controller to actuator communication:** The controller calculates a control input according to a causal function $U(k) = f_2(U(0), \cdots, U(k-1), Y_3(0), \cdots, Y_3(k))$ and transmits it to the actuator. The control input must satisfy the following constraint.

**Constraint $C_1$:** The control inputs must satisfy the cost constraint, $\sum_{k=0}^{\infty} \mathbb{E}[U^2(k)] < \infty$.

We assume that the controller is not power limited; thus, the actuator receives $U(k)$ without corruption. Due to the broadcast nature of the communication, the sensor and the relay also receive the control $U(k)$.

**Problem statement:** We are interested in designing the quantities $X_1(\cdot)$ and $X_2(\cdot)$ (that we refer to as the design of the coding scheme), and $U(\cdot)$ (that we refer to as the design of the controller), such that the process (1) is mean squared stabilized while the control cost constraint $C_1$, and the transmission power constraints (4) and (5) are satisfied. In particular, we are interested in the question whether the stabilizability region of the process can be expanded using a relay node even if the total transmission power used by the sensor and the relay node stays constant?

### 3. Main Results

**Baseline Case:** The stabilizability condition when the sensor and controller communicate across a point-to-point AWGN channel is as follows (Tatikonda and Mitter 2004; Braslavsky et al 2007; Martins and Dahleh 2008).

**Theorem 3.1.** Consider the problem formulation presented in Section 2 with the process given by (1) without a relay node. The process (1) can be mean square stabilized if

$$
\sum_{i=1}^{m} \max \{0, \log(\lambda_i(A))\} < \frac{1}{2} \log \left(1 + \frac{g_{13}^2 \sigma_2^2 P}{\sigma_3^2} \right). 
$$

**Case with Relay:** The following is the main result for the case of a relay channel.

**Theorem 3.2.** Consider the problem formulation presented in Section 2. There exists a coding scheme and controller design such that the process (1) is mean square stabilized over the AWGN relay channel if the following condition is satisfied:

$$
\sum_{i=1}^{m} \max \{0, \log |\lambda_i(A)|\} < \liminf_{k \to \infty} \frac{1}{2} \log \frac{1}{q(k)}.
$$
where

\[
q(k) = \frac{g_{2,3}^2 P_2 (1 - \rho^2(k)) + \sigma_3^2}{g_{1,3}^2 P_1 + g_{2,3}^2 P_2 + 2g_{1,3}g_{2,3}\sqrt{P_1 P_2} \rho(k) + \sigma_3^2} \tag{8}
\]

\[
\rho(k) = \frac{\alpha_3(k) - \gamma(k)}{\sqrt{\alpha_3(k)(\alpha_3(k) + \alpha_2(k) - 2\gamma(k))}} \tag{9}
\]

\[
\gamma(k) = r(k-1) (\gamma(k-1) - \beta_3(k)) \left[ g_{1,3} \sqrt{\frac{P_1}{\alpha_3(k-1)}} \gamma(k-1) \right. \\
+ g_{2,3} \sqrt{\frac{P_2}{(\alpha_3(k-1) + \alpha_2(k-1) - 2\gamma(k-1))}} (\gamma(k-1) - \alpha_2(k-1)) \left] \right) \tag{10}
\]

\[
r(k) = \left( \frac{\sigma_1^2}{g_{1,2}^2 P_1 (P_1^{1/2}) \alpha_2(k) + \sigma_2^2} \right) \tag{11}
\]

\[
\beta_3(k) = \frac{\sqrt{\alpha_3(k-1)}}{g_{1,3}^2 P_1 + g_{2,3}^2 P_2 + 2g_{1,3}g_{2,3}\sqrt{P_1 P_2} \rho(k-1) + \sigma_3^2}, \tag{12}
\]

with the recursions \( \alpha_3(k) = \alpha_3(k-1)q(k-1) \), \( \alpha_2(k) = \alpha_2(k-1)r(k-1) \), and the initial conditions \( \alpha_3(0) = \frac{\sigma_{2,3}^2}{g_{1,3}^2 P_1} \), \( \alpha_2(0) = \frac{\sigma_{2,1,2}^2}{g_{1,3}^2 P_1} \) and \( \gamma(0) = 0 \).

Before we prove this result, we notice the following properties of the result. Firstly, since the cascade channel is the same as the relay channel, except that there is no path from the sensor to the controller, the stabilizability conditions for the cascade channel can be derived from Theorem 3.2 by substituting \( g_{1,3} = 0 \).

**Theorem 3.3.** The LTI system in (1) can be mean square stabilized over the cascade channel if (7) holds with \( g_{1,3} = 0 \).

Secondly, we note that the bound in equation (7) is non-trivial.

**Corollary 3.4.** The right hand side of (7) is finite.

**Proof.** From (8), we see that the choice \( \rho(k) = 1 \) simultaneously minimizes the numerator and maximizes the denominator\(^1\). Thus, \( 1/q(k) \) is maximized by choosing \( \rho(k) = 1 \), or in other words,

\[
\frac{1}{q(k)} \leq \frac{g_{1,3}^2 P_1 + g_{2,3}^2 P_2 + 2g_{1,3}g_{2,3}\sqrt{P_1 P_2} + \sigma_3^2}{\sigma_3^2},
\]

from which the result follows. \( \square \)

The proof of Theorem 3.2 has two components. We develop an encoding scheme that ensures that at every time \( k \), the controller can calculate an estimate \( \hat{S}_3(k) \) of the state value \( S(0) \) such that the error \( \epsilon_3(k) = \hat{S}_3(k) - S(0) \) has a covariance that decreases geometrically in \( k \). We then show that using such a \( \hat{S}_3(k) \), the controller can calculate a control input that stabilizes the process in the mean squared sense.

**Controller Design:** We begin by proving the second part.

\(^1\)Note that \( \rho(k) \) as discussed further when we describe the coding scheme is a cross-correlation coefficient and hence cannot be larger than one.
**Theorem 3.5. (Controller Design)** Consider the problem formulation stated in Section 2. Assume that the controller can calculate an estimate \( \hat{S}_3(k) \) of the state value \( S(0) \) such that the error \( \epsilon_3(k) = \hat{S}_3(k) - S(0) \) satisfies the following properties:

\[
\mathbb{E}[\epsilon_3(k)] = 0, \quad (13)
\]

\[
\lim_{k \to \infty} A^k \mathbb{E}[\epsilon_3(k)\epsilon_3^T(k)](A^T)^k = 0. \quad (14)
\]

Then, the process (1) can be mean squared stabilized by a suitable choice of the controller.

**Proof.** Let \( K \) be such that the closed loop matrix \( A + BK \) is Schur-stable. Such a \( K \) exists since the pair \((A, B)\) is controllable. Consider the controller \( U(k) = K\hat{S}(k) \), where

\[
\hat{S}(k) = A^k\hat{S}_3(k) + \sum_{j=1}^{k} A^{k-j}BU(j-1). \quad (15)
\]

With this controller, the process (1) evolves as

\[
S(k + 1) = (A + BK)S(k) + BK\delta(k), \quad (16)
\]

where \( \delta(k) = S(k) - \hat{S}(k) = -A^k\epsilon_3(k) \). Since \( S(0) \) is zero mean, (13) implies that \( \mathbb{E}[\delta(k)] = 0 \) and, in turn, \( \mathbb{E}[S(k)] = 0 \) at every \( k \). Moreover, if (14) is satisfied, \( \mathbb{E}[\delta(k)\delta^T(k)] \to 0 \) as \( k \to \infty \). Thus, since \( A + BK \) is stable, (16) yields that \( \lim_{k \to \infty} \mathbb{E}[S(k)S^T(k)] = 0 \). \( \square \)

This result also provides the controller design. We would like to remark that although a certainty equivalence controller is used in the above result, the controller does not generate its estimate using a Kalman filter.

Next, we provide a coding scheme that ensures that the relations (13) and (14) are satisfied.

**Coding Scheme:** We concentrate on the relay channel while noting that the coding scheme and its analysis carry over to the cascade channel if we set the channel gain \( g_{1,3} = 0 \). The basic idea behind the coding scheme is to transmit the last \( n \) elements of the initial state \( S(0) \) to the controller. For pedagogical ease, we begin by describing the scheme for the case \( m = n = 1 \), so that only one number needs to be transmitted. For this special case, denote the matrix \( A \) in (1) by \( a \) (with \( a > 1 \)) and assume the matrix \( B \) is unity. We denote the coding scheme in this case by \( S(S(0), \hat{S}_3(k)) \), where \( S(0) \) is the initial condition, and \( \hat{S}_3(k) \) is the estimate of \( S(0) \) at time \( k \) at the controller.

**Description of the coding scheme for the case \( m = 1 \):**

*Distributed stochastic approximation:* The coding scheme that we propose is based on distributed stochastic approximation (Kushner and Yin 2003; Borkar 2008), which is a distributed method used to calculate the zero of a vector function \( f(T) : \mathbb{R}^m \to \mathbb{R}^m \) from noisy observations of the function. Denote the \( i \)-th component of the function by \( f_i \) and the current estimate of the zero to be \( \hat{T}(k) \). The method works as follows. Noisy observations of the form \( \hat{f}_i(\hat{T}(k)) = f_i(\hat{T}(k)) + N_i(k) \) are obtained and used to update each component of the estimate \( \hat{T}(k) \) according to the relation \( \hat{T}_i(k) = \hat{T}_i(k-1) - \beta_i(k)\hat{f}_i(\hat{T}(k)) \), where \( \hat{T}_i(k) \) is the \( i \)-th component of \( \hat{T}(k) \), \( \beta_i(k) \) is the step size, and the initial estimate \( \hat{T}_i(0) \) can be assumed to be any arbitrary finite value. Under certain technical constraints on the step size and the function \( f(\cdot) \), the estimate \( \hat{T}(k) \) converges (in the mean squared estimate sense) to the zero of the function \( f(\cdot) \) as \( k \to \infty \) (Borkar 2008).

In our coding scheme, the sensor, relay, and the controller implement a distributed stochastic approximation scheme to calculate the zeros of a function \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) whose \( i \)-th component is
given by

\[ f_i(S_1, S_2, S_3) = \begin{cases} 0 & \text{for } i = 1 \\ \sum_{j: g_{ij} \neq 0} \mu_i (S_i - S_j) & \text{for } i = 2, 3. \end{cases} \]  \tag{17}

Specifically, the variable \( S_1(k) \) is associated with the sensor node and is identically set to the value \( S(0) \). The variable \( S_2(k) \) is associated with the relay node while the variable \( S_3(k) \) is associated with the controller node, both of which calculate estimates \( \hat{S}_2(k) \) and \( \hat{S}_3(k) \) of these quantities at every time \( k \). The terms \( \mu_i \)'s are suitably designed to satisfy the transmission power constraints (4) and (5). Since the zeros of the function \( f \) are obtained at \( S_1 = S_2 = S_3 = S(0) \), the estimates \( \hat{S}_2(k) \) and \( \hat{S}_3(k) \) serve as an estimate of the initial state \( S(0) \) at the relay and the controller, respectively. By the properties of distributed stochastic approximation (Borkar 2008), both these quantities converge to \( S(0) \) as \( k \to \infty \) in the mean squared sense.

Define \( \epsilon_2(k) := \hat{S}_2(k) - S(0) \), \( \epsilon_3(k) := \hat{S}_3(k) - S(0) \), and \( \epsilon_{3,2}(k) := \hat{S}_3(k) - \hat{S}_2(k) \). Let \( \alpha_i(k) \) represent the covariance of \( \epsilon_i(k) \), \( i \in \{2\}, \{3\}, \{3, 2\} \). At every time \( k \), the sensor can calculate \( \epsilon_3(k) \), while the relay can calculate \( \epsilon_{3,2}(k) \). Since the relay and controller calculate linear MMSE estimates, the proposed coding scheme will ensure that the quantities \( \epsilon_3(k) \) and \( \epsilon_{3,2}(k) \) are jointly Gaussian with zero means, covariances \( \alpha_3(k) \) and \( \alpha_{3,2}(k) \) (respectively), and correlation coefficient \( \rho(k) \). The quantities \( \alpha_2(k) \), \( \alpha_3(k) \), \( \alpha_{3,2}(k) \) and \( \rho(k) \) are deterministic and can be calculated prior to transmission by all the nodes. Recursive expressions to calculate these quantities are provided later. In particular, note the initial conditions \( \alpha_{3,2}(0) = \alpha_3(0) + \alpha_2(0) \) and \( \rho(0) = \frac{\alpha_{3,2}(0)}{\sqrt{(\alpha_3(0) + \alpha_2(0)) \alpha_{3,2}(0)}} \). The coding scheme works as follows.

\textit{Initialization}: At time step \( k = 0 \),
- The sensor observes \( S(0) \) and transmits the input \( X_1(0) \) given by
  \[ X_1(0) = \sqrt{\frac{P_1}{\sigma_{S(0)}^2}} S(0). \]  \tag{18}
- The relay node transmits nothing. It receives \( Y_2(0) = g_{1,2} X_1(0) + Z_2(0) \), and calculates an estimate of \( S(0) \) by scaling: \( \hat{S}_2(0) = \frac{1}{g_{1,2}} \sqrt{\frac{\sigma_{Y_2(0)}^2}{P_1}} Y_2(0) \). The estimation error \( \epsilon_2(0) \) zero-mean Gaussian with variance
  \[ \alpha_2(0) = \frac{\sigma_{S(0)}^2 \sigma_{2}^2}{g_{1,2}^2 P_1}. \]  \tag{19}
- The controller node 3 receives \( Y_3(0) = g_{1,3} X_1(0) + Z_3(0) \) and calculates an estimate of \( S(0) \) as
  \[ \hat{S}_3(0) = \frac{1}{g_{1,3}} \sqrt{\frac{\sigma_{Y_3(0)}^2}{P_1}} Y_3(0) \]. The estimation error \( \epsilon_3(0) \) is again zero-mean Gaussian with variance
  \[ \alpha_3(0) = \frac{\sigma_{S(0)}^2 \sigma_{3}^2}{g_{1,3}^2 P_1}. \]  \tag{20}

The controller calculates \( \hat{S}(0) \) from (15) and transmits both the control \( U(0) = K \hat{S}(0) \) and the estimate \( \hat{S}_3(0) \).

\textit{Update}: At every time step \( k \geq 1 \),
- Using the message transmitted by the controller at time \( k - 1 \), the sensor and the relay have access to \( \hat{S}_3(k - 1) \). The sensor calculates \( \epsilon_3(k - 1) \), while the relay calculates the quantity \( \epsilon_{3,2}(k - 1) \).
The sensor transmits \( X_1(k) = \sqrt{\frac{P_1}{\alpha_3(k-1)}} \varepsilon_3(k-1) \).

The relay transmits \( X_2(k) = \sqrt{\frac{P_2}{\alpha_{3,2}(k-1)}} \varepsilon_{3,2}(k-1) \). It also updates its estimate as follows. It first calculates

\[
Y_2'(k) = Y_2(k) - g_{1,2} \sqrt{\frac{P_1}{\alpha_3(k-1)}} (\hat{S}_3(k-1) - \hat{S}_2(k-1))
\]

\[
= g_{1,2} \sqrt{\frac{P_1}{\alpha_3(k-1)}} (\hat{S}_2(k-1) - S(0)) + Z_2(k). \tag{21}
\]

Using this quantity, the relay calculates the linear minimum mean squared error estimate of \( S(0) \) given \( Y_2'(k) \) and \( \hat{S}_2(k-1) \) as

\[
\hat{S}_2(k) = \hat{S}_2(k-1) - \beta_2(k) Y_2'(k)
\]

\[
\beta_2(k) = \frac{\mathbb{E}[Y_2'(k)e_2(k-1)]}{\mathbb{E}[Y_2'^2(k)]}. \tag{23}
\]

We provide an expression for calculation of \( \beta_2(k) \) later in (29).

At time \( k \geq 1 \), the controller calculates the linear MMSE of \( S(0) \) given \( Y_3(k) \) and \( \hat{S}_3(k-1) \)

\[
\hat{S}_3(k) = \hat{S}_3(k-1) - \beta_3(k) Y_3(k)
\]

\[
\beta_3(k) = \frac{\mathbb{E}[Y_3(k)e_3(k-1)]}{\mathbb{E}[Y_3^2(k)]}. \tag{25}
\]

We provide an expression for calculation of \( \beta_3(k) \) in (32). Finally, the controller calculates the control input \( U(k) = K \hat{S}(k) \) using equation (15) and transmits it.

Calculation of the Variances: We now provide recursive expressions for the calculation of \( \alpha_2(k) \), \( \alpha_3(k) \), \( \alpha_{3,2}(k) \), \( \rho(k) \), \( \beta_2(k) \) and \( \beta_3(k) \) as used in the coding scheme presented above. The following recursions do not depend on the data, and can be executed by any node.

\textbf{Variance} \( \alpha_2(k) \) of the error at the relay node: Since \( \alpha_2(k) = \mathbb{E}[\varepsilon_2^2(k)] \), equations (22), and (23) yield \( \alpha_2(k) = \alpha_2(k-1) - \frac{\mathbb{E}^2[Y_2'(k)e_2(k-1)]}{\mathbb{E}[Y_2'^2(k)]]} \) with the initial condition in equation (19).

\[
\mathbb{E}[Y_2'^2(k)] = g_{1,2}^2 \frac{P_1}{\alpha_3(k-1)} \alpha_2(k-1) + \sigma_2^2 \tag{26}
\]

\[
\mathbb{E}[Y_2'(k)e_2(k-1)] = g_{1,2} \sqrt{\frac{P_1}{\alpha_3(k-1)}} \alpha_2(k-1), \tag{27}
\]

so that we can write \( \alpha_2(k) = \alpha_2(k-1)r(k-1) \), where

\[
r(k-1) = \left( \frac{\sigma_2^2}{g_{1,2}^2 \frac{P_1}{\alpha_3(k-1)} \alpha_2(k-1) + \sigma_2^2} \right). \tag{28}
\]

It can also be verified that \( \beta_2(k) \) can be calculated as

\[
\beta_2(k) = \frac{g_{1,2} \sqrt{\frac{P_1}{\alpha_3(k-1)}} \alpha_2(k-1)}{g_{1,2}^2 \frac{P_1}{\alpha_3(k-1)} \alpha_2(k-1) + \sigma_2^2}. \tag{29}
\]
Variance $\alpha_3(k)$ of the error at the controller node: In a similar manner, the variance $\alpha_3(k) = \mathbb{E}[e_3^2(k)]$ can be calculated from equations (24), and (25) to be

$$\alpha_3(k) = \alpha_3(k-1)q(k-1)$$

$$q(k-1) = \frac{g^2_{3,3}P_2(1-\rho^2(k-1))+\sigma^2_3}{g^2_{1,3}P_1+g^2_{2,3}P_2+2g_{1,3}g_{2,3}\sqrt{P_1P_2}\rho(k-1)+\sigma^2_3}$$

and the initial condition in equation (20). It can be verified that $\beta_3(k)$ can be calculated as

$$\beta_3(k) = \frac{\sqrt{\alpha_3(k-1)}(g_{1,3}\sqrt{P_1}+g_{2,3}\sqrt{P_2}\rho(k-1))}{g^2_{1,3}P_1+g^2_{2,3}P_2+2g_{1,3}g_{2,3}\sqrt{P_1P_2}\rho(k-1)+\sigma^2_3}.$$  

Variance $\alpha_{3,2}(k)$ and the correlation coefficient $\rho(k)$: We express $\mathbb{E}[e_{3,2}(k)e_2(k)] = \gamma(k) - \alpha_2(k)$ where $\gamma(k) := \mathbb{E}[e_3(k)e_2(k)]$. Now, equations (22) and (24) yield

$$\gamma(k) = \left((k-1) - \beta_2(k)\mathbb{E}[e_3(k-1)Y'_2(k)]\right) - \left(\beta_3(k)\mathbb{E}[e_2(k-1)Y_3(k)] - \beta_3(k)\beta_2(k)\mathbb{E}[Y'_2(k)Y_3(k)]\right),$$

Using the expression of $Y'_2(k)$ from (21), that of $\beta_2(k)$ from (23), and the relations in (26) and (27), we obtain

$$\gamma(k-1) - \beta_2(k)\mathbb{E}[e_3(k-1)Y'_2(k)] = r(k-1)\gamma(k-1),$$

where $r(k-1)$ is defined in (28). In a similar manner, we can write

$$\mathbb{E}[e_2(k-1)Y_3(k)] = g_{1,3}\sqrt{\frac{P_1}{\alpha_3(k-1)}}\gamma(k-1) + g_{2,3}\sqrt{\frac{P_2}{\alpha_3(k-1)}}(\gamma(k-1) - \alpha_2(k-1)),$$

$$\mathbb{E}[Y'_2(k)Y_3(k)] = g_{1,3}g_{1,2}\sqrt{\frac{P_1}{\alpha_3(k-1)}}\gamma(k-1) + g_{1,2}g_{2,3}\sqrt{\frac{P_1P_2}{\alpha_3(k-1)\alpha_{3,2}(k-1)}}(\gamma(k-1) - \alpha_2(k-1)).$$

Substituting these simplifications in (33) yields a recursive relation for $\gamma(k)$, with the initial value $\gamma(0) = 0$. Finally, given $\alpha_2(k), \alpha_3(k)$, and $\gamma(k)$, we can calculate $\alpha_{3,2}(k)$ and $\rho(k)$ using $\alpha_{3,2}(k) = \alpha_3(k) + \alpha_2(k) - 2\gamma(k)$, and $\rho(k) = \frac{\alpha_3(k)-\gamma(k)}{\sqrt{\alpha_3(k)\alpha_{3,2}(k)}}$.

Description of the coding scheme for the case of arbitrary $m$: We now present the encoder and decoder design for the general case when the process in (1) has an arbitrary dimension $m$. As stated before, the basic approach for a vector process is to transmit the last $n$ elements of the initial state $S(0)$ to the controller. To achieve this aim, $n$ coding schemes of the type proposed above are used in parallel (one for each element of $S(0)$). Denote the relative frequency with which the scheme for the $j$-th element is executed by $n_j$, $j = m+1, \cdots, n$. Since $A$ is in the Jordan form, we can associate with every component $S^j(0)$ of $S(0)$, an eigenvalue $\lambda_j$ of $A$ that governs the rate with which it increases. We set $n_j = \log(\lambda_j)/\sum_{j=m+1}^{n} \log \lambda_j$. More formally, we design a periodic schedule with period $T$ such that in every period, the sensor, relay, and controller implement for $n_jT$ time steps, the coding scheme $S(S^j(0), S_3^j(k))$, with $S^j(k) = e_j^T S(0)$ and $S_3^j(k) = e_j^T S_3(k)$. The controller calculates the control input as follows. It maintains an estimate $\hat{S}_3(k)$ of the initial state $S(0)$. At each time $k$, it performs the following actions:

- If the transmission of the $j$-th component has occurred at time $k$, update $\hat{S}_3(k)$ as $\hat{S}_3(k) = \hat{S}_3(k-1) - e_j^T S_3(k-1)e_j + S_3^j(k)e_j$, with the initial condition $\hat{S}_3(-1) = 0$. 
• Calculate \( \hat{S}(k) \) using the relation (15).
• Transmit control \( U(k) = K \hat{S}(k) \).

**Proof of Theorem 3.2:** We need to prove that with the coding scheme used above, the conditions (13) and (14) in Theorem 3.5 are satisfied. Firstly, note that since \( \epsilon_3(0) = \mathbb{E}[\epsilon_{3,2}(0)] = \mathbb{E}[\epsilon_2(0)] = 0 \) and all updates in the coding scheme are linear, it is straightforward to see that equation (13) is satisfied. To show that (14) is also satisfied, denote \( M(k) = A^k \mathbb{E}[\epsilon_3(k) \epsilon_3^T(k)](A^T)^k \). Since \( M(k) \) is a positive semi-definite matrix, we need only show that the diagonal elements of \( M(k) \) converge to 0 as \( k \to \infty \). Since \( A_k \) is Schur-stable, the first \( m-n \) diagonal elements of \( M(k) \) clearly approach 0 as \( k \to \infty \). For the remaining \( n \) elements, we use the fact that \( A \) is in the Jordan form. Consider a Jordan sub-block of \( A \) governed by an eigenvalue \( \lambda_j \) which lies outside the unit circle. Let the size of the sub-block be \( m_1 \), corresponding to the components \( j, j+1, \ldots, j+m_1-1 \) of the state vector. First consider the \( (j+m_1-1, j+m_1-1) \)-th component of \( M(k) \). From equation (30) and the fact that every component of the error vector is updated with relative frequency \( n_j \) we see that as \( k \to \infty \), this component is equal to

\[
\lim_{k \to \infty} \lambda_j^{2k} \mathbb{E}[(\epsilon_j^{j+m_1-1}(k))^2] = \lim_{k \to \infty} \lambda_j^{2k} n_j \mathbb{E}[(\epsilon_j^{j+m_1-1}(k-T))^2],
\]

where \( \epsilon_j^{j}(k) \) is the \( i \)-th component of \( \epsilon_3(k) \). The ratio test then yields that a sufficient condition for this component to converge to 0 as \( k \to \infty \) is that the condition

\[
\log |\lambda_j| < \lim \inf_{k \to \infty} \frac{n_j}{2} \log \frac{1}{q(k)},
\]

is satisfied. Now consider the \( (j+m_1-2, j+m_1-2) \)-th component of \( M(k) \). Because of the Jordan structure of \( A \), this element evolves as \( \lambda_j^{2k} \mathbb{E}[(\epsilon_j^{j+m_1-2}(k))^2] + 2k \mathbb{E}[(\epsilon_j^{j+m_1-2}(k) \epsilon_j^{j+m_1-1}(k)] + k^2 \mathbb{E}[(\epsilon_j^{j+m_1-1}(k))^2] \). If (35) is satisfied, the first and the third terms of this expression converge to 0 as \( k \to \infty \). Consequently, by Cauchy-Schwarz inequality, the second term also converges to 0. Thus, the entire \( (j+m_1-2, j+m_1-2) \)-th component of \( M(k) \) converges to 0 as \( k \to \infty \) if (35) is satisfied. A similar argument yields that all diagonal elements of \( M(k) \) corresponding to this Jordan block will converge to 0. Since the Jordan block that was considered is arbitrary, (14) holds if (35) is satisfied for every unstable eigenvalue of \( A \). The result now follows since (7) guarantees (35) for every unstable eigenvalue due to the definition of \( n_j \).

**Constraint C1:** The constraints on the transmission power (equations (4) and (5)) are satisfied by construction. We can show that the control cost constraint \( C_1 \) is also satisfied.

**Proposition 3.6.** If \( \hat{S}_3(k) \) is a linear MMSE estimate of \( S(0) \), and the process (1) is stabilized in the mean squared sense, then the controller proposed in Theorem 3.5 satisfies the constraint \( C_1 \).

**Proof.** First note that if \( \hat{S}_3(k) \) is an MMSE estimate of \( S(0) \), then \( \hat{S}(k) \) in (15) is an MMSE estimate of \( S(k) \). This implies that \( \mathbb{E}[S(k) S^T(k)] = \mathbb{E}[\hat{S}(k) \hat{S}^T(k)] + \mathbb{E}[\hat{\delta}(k) \hat{\delta}^T(k)] \). Thus,

\[
\sum_{k=0}^{\infty} \mathbb{E}[U^T(k) U(k)] = \sum_{k=0}^{\infty} \mathbb{E}[\hat{S}^T(k) K^T K \hat{S}(k)] = \sum_{k=0}^{\infty} \mathbb{E}[\text{tr}(\hat{S}(k) \hat{S}^T(k) K^T K)],
\]

which is finite if \( \sum_{k=0}^{\infty} \mathbb{E}[\text{tr}(\hat{S}(k) \hat{S}^T(k))] \) is finite. Now,

\[
\sum_{k=0}^{\infty} \mathbb{E}[\text{tr}(\hat{S}(k) \hat{S}^T(k))] = \sum_{k=0}^{\infty} \mathbb{E}[\text{tr}(S(k) S^T(k))] + \sum_{k=0}^{\infty} \mathbb{E}[\text{tr}(\hat{\delta}(k) \hat{\delta}^T(k))].
\]
If the process (1) is mean squared stabilized, the first summation is finite. The second summation can be written as

\[
\sum_{k=0}^{\infty} \mathbb{E}[\text{tr}(\delta(k)\delta^T(k))] = \sum_{k=0}^{\infty} \mathbb{E}[\text{tr}(A^k\epsilon_3(k)\epsilon_3^T(k)(A^T)^k)]
\]

\[
= \sum_{k=0}^{\infty} \lambda_j^{2k} \mathbb{E}[(\epsilon_j^2(k))^2].
\]

If the condition in (7) is satisfied, the above term is finite. Thus, the constraint \( C_1 \) is satisfied.

Remark 1. The case when state is partially observed: The above development assumed that the sensor observed the state of the process directly. We can further consider the case that the sensor only observes the system output \( y(k) = CS(k) \), where \( C \in \mathbb{R}^{1 \times m} \) at each step \( k \).

Since all eigenvalues of \( A \) are assumed to be unstable, to ensure stabilizability, we need the further assumption that \((A, C)\) is observable. The coding and controller design presented above can be used with the following change. We let the system evolve open loop for the first \( m \) steps. Since the system is observable, the sensor can obtain the initial state \( S(0) \) (and hence the state \( S(m) \)) using the output sequence \( y(0), y(1), \ldots, y(m-1) \). Now, the coding and controller design presented above can be used. In other words, the encoder and the controller design can be viewed as starting from step \( m \), when the sensor starts to send the value \( S(m) \). Therefore, the proposed stabilizing algorithm still works in this case and identical stabilization conditions are obtained.

4. Numerical Results

Consider a scalar process of the form (1) with the controller placed at a (spatial) distance of 2 units from the sensor. The relay is placed at the midpoint of the line joining the sensor and controller. The system parameters are considered to be \( \sigma^2_{S(0)} = \sigma^2_{2} = \sigma^2_{3} = 1 \) and \( \eta = 2 \). Under these constraints, the limit in (7) can be evaluated to be 0.2410 for the relay channel. Under the same constraints, the limit for the cascade channel is 0.1995. If we consider that there is no relay node and use the power \( P_1 + P_2 = 1 \) at the sensor node, the sufficient condition for mean square stability from Theorem 6 is \( \log(a) < 0.1610 \). This shows that using a relay node may be beneficial even if the total power constraint on transmitting nodes remains the same, i.e., even if the power used by the relay node is at the expense of the power used by the sensor node. To illustrate this further, we plot the limit in (7) as a function of the power \( P_1 \) and distance of the relay from the plant in Figure 3. The system parameters are the same as above. As shown, for any given position of the relay, we can always find a power distribution for which the stability region is enhanced.

5. Conclusions

In this paper, we derived sufficient conditions for stabilizability of a linear time invariant open loop unstable plant in the mean squared sense over an AWGN relay channel and a cascade of two point-to-point AWGN channels. We proposed a coding scheme to ensure a sufficiently fast rate of convergence of the estimate of the initial state at the controller to the correct value. The scheme can also be interpreted as a distributed version of the Schalkwijk-Kailath coding scheme for point-to-point channels with both the sensor and the relay transmitting innovations with respect to the estimate at the controller at every step. We analyzed the stability region of the closed loop system with the proposed scheme under average transmission power constraints for
both the sensor and the relay node. Interestingly, the stability region may be increased by using a relay node even if the total transmission power remains the same.

In this paper, we have assumed that the relay is full-duplex (i.e., the relay receives and transmits simultaneously). However, the analysis here can easily be extended to a half-duplex relay, where the relay receives and transmits at alternate time steps. Future work in this direction can consider the effect of process noise in (1). Another direction of work can be to utilize the distributed approach used in this paper to consider the problem of stabilizing a plant over more general networks. Finally, considering non-linear relaying strategies might lead to more relaxed constraints on plant parameters.

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6. Biographies

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Figure 7. J. Nicholas Laneman

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