Abstract. In this paper, we consider a distributed control design problem. Multiple agents (or subsystems) that are dynamically uncoupled need to be controlled to optimize a joint cost function. An interconnection graph specifies the topology according to which the agents can access information about each others’ state. We propose and partially analyze a new model for determining the influence of the topology of the interconnection graph on the performance achieved by the subsystems. We consider the classical LQR cost function and propose making one of the weight matrices to be topology-dependent to capture the extra cost incurred when more communication between the agents is allowed. We present results about optimal topologies for some models of the dependence of the weight matrix on the communication graph. We also give some results about the existence of “critical prices” at which adding supplementary edges becomes detrimental to closed-loop performance. One conclusion of the work is that if the communication between the agents comes at a cost, then adding communication edges may be harmful for the system performance.

1. Introduction. The question of optimal decentralized or structured control design for systems composed of interconnected subsystems has been widely studied at least since the seventies (see e.g., [4, 19]). The defining feature of these problems is that, while optimization of the cost function can demand that the control of one subsystem know the states of all the other subsystems, the topology (or information pattern) imposed by the specified system structure may not allow such interactions to happen. A lot is known about the information patterns for which an optimal structured control law exists [23] or has (or does not have) some desirable properties, such as being linear and satisfying a separation principle [19, 24], being computable in polynomial time in the dimension of the plant’s data [5, 18] and so on. Many synthesis methods are also available that can impose a specific topology on a controller (see, e.g., [14] and the references therein), result in control laws that can be proved to be decentralized a posteriori [2, 7] or give an approximation to the exact optimal structured controller (see, e.g., [10] and the references therein).

In all the works mentioned above, however, the controller’s interconnection topology is always (if sometimes implicitly) assumed to be known to the designer prior to synthesis, and optimization and/or design are to be performed among control laws with this particular structure. While this problem formulation is appropriate when decentralization is viewed as an external constraint (e.g., when controlling a system with a pre-existing interconnection topology such as a power distribution network), there are situations where the interconnection graph and the controller are both designed at the same time. As an example, while designing cooperative multi-agent systems, the choice of the architecture or the information pattern (leader-follower, fully decentralized or some other structure) may itself be a valid design choice, and thus, an integral part of the control design process. In such cases, it makes sense to try to find the minimal (with respect to some cost function) topology needed to achieve a particular control goal.

This structural optimization problem is complementary to other communication theoretic trade-offs arising in network/distributed controller co-design (as discussed,
e.g., in [15]). The optimization of the communication topology along with the control laws followed by the individual agents also falls within the framework of the theory of organizational efficiency and information cost introduced in Economics by Marschak & Radner in [16]. It is surprising that, while the tools and ideas of Team Decision Theory that were developed in [16] have been successfully applied to distributed control problems with given information structure [12], the very question of efficiency of decision architectures, that these authors were originally interested in, has received relatively little attention in the control literature. In particular, we are only aware of the following recent attempt at finding a minimal control interconnection structure. In [21], the authors have shown that, when constructing a distributed controller from a set of observer-based controllers using different and parallel observations, the star interconnection topology is minimal, in the sense that the resulting control design problem has the minimal number of free parameters needed to ensure closed-loop stability.

In this paper, we present a new model for studying the role of controller topology in distributed control problems, which is inspired by ideas from the field of Games over Networks in Economics [13, 20]. More precisely, we propose to modify the classical structured LQR cost function by making one of the weight matrices topology-dependent. This amounts to explicitly accounting for the energy used by the controller for communication and allows us to explore the trade-off between a controller’s performance and its communication network topology.

The paper is organized as follows. We begin by introducing our model and various notations in Section 2. As in most of the works on distributed control mentioned above, we assume the communication links to be ideal, if they exist. Then in Section 3, we consider particular models of the dependence of the weight matrices on topology and compute the optimal cost achieved by various topologies. In particular, we prove the somewhat surprising result that, under certain assumptions, the optimal control topology according to our criterion is the fully decentralized one. Then, in Section 4, we consider arbitrary graphs and arbitrary weight matrices. We provide some conditions for the existence of “critical prices” at which it becomes detrimental to add edges to a pre-existing controller topology. Finally, in Section 5, we illustrate some of the results by considering a simple example.

2. Problem Formulation.

2.1. Notation. For a matrix $M$, we will denote the $(i, j)$-th element by $[M]_{ij}$. Similarly for a column vector $v$, the $i$-th element is denoted as $v_i$. For matrices and vectors that have been defined block-wise we will abuse the notation and use $[M]_{ij}$ to mean the $(i, j)$-th block of $M$ and $[V]_i$ to mean the $i$-th block of $V$. The particular use will be clear from the context. For a matrix $M$ or a vector $V$, the complex conjugate of the matrix $M$ (resp. the vector $V$) will be denoted by $M^*$ (resp. $V^*$). The norm of a vector $V$ is denoted by the symbol $\|V\|$. For a matrix $M$, we denote the spectral radius of $M$ by the symbol $\lambda_{\text{max}}(M)$ and the maximum singular value by $\sigma_{\text{max}}(M)$. Given matrices $M_1, M_2, \cdots, M_n$, we will denote the block-diagonal matrix formed by placing the matrices $M_i$’s along the diagonal as $\text{diag}(M_i)$. The space of all symmetric positive definite $n \times n$ matrices is denoted by $\mathbb{S}^n$. For two matrices $M_1$ and $M_2$, we will say $M_1 \geq M_2$ if $M_1 - M_2$ is positive semi-definite.

2.2. Structured Control Laws. Assume we are given $N$ linear, discrete time, time-invariant (LTI) subsystems (or agents) described by

$$x_i(k+1) = A_ix_i(k) + B_iu_i(k)$$

(2.1)
for all $i = 1 \ldots N$ and $k \geq 0$. At each time $k$, the state $x_i(k)$ and input $u_i(k)$ of each subsystem is an element of $\mathbb{R}^{n_i}$ and $\mathbb{R}^{m_i}$, respectively. In the sequel, we will write $n := \sum_{i=1}^{N} n_i$ and $m := \sum_{i=1}^{N} m_i$. The state of the entire system can be defined by stacking the states of all the sub-systems in a column vector, which we denote by $x(k)$:

$$x(k) = \begin{pmatrix} x_1(k) \\ \vdots \\ x_N(k) \end{pmatrix}.$$  

We can similarly define the column vector $u(k)$ by stacking all the individual control inputs $u_i(k)$’s. Each pair $(A_i, B_i)$ is assumed to be controllable which, in turn (see, e.g., [10]), implies that the full system, with matrices $A := \text{diag}_i(A_i), B := \text{diag}_i(B_i)$ is also controllable.

Let $\mathcal{G}_N$ be the set of all undirected graphs with $N$ vertices. We will think of each vertex of a graph $g \in \mathcal{G}_N$ as representing the subsystem (2.1) labeled with the same index $i$, where $1 \leq i \leq N$. To every graph $g \in \mathcal{G}_N$ is associated an edge set $E(g)$ defining the edges present in $g$ and an adjacency matrix $A(g)$ defined as

$$[A(g)]_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E(g) \\ 0 & \text{otherwise.} \end{cases}$$

All the graphs we will consider have self-loops i.e., that $(i, i) \in E(g)$ for all $i = 1 \ldots N$, $g \in \mathcal{G}_N$. If a graph $g_1$ is a subgraph of $g_2$, i.e., if $E(g_1) \subset E(g_2)$, we will write $g_1 \preceq g_2$. Clearly, relation “$\preceq$” defines a partial order on $\mathcal{G}_N$.

Each graph in $\mathcal{G}_N$ specifies a communication topology that can be used to construct specific control laws for subsystems (2.1). To this end, we introduce the space $\mathcal{K}_{m,n}(g)$ of structured matrices with structure imposed by $g$. A matrix $K$ in $\mathcal{K}_{m,n}(g)$ is defined block-wise, each block $K_{ij}$ being a $m_i \times n_j$ matrix such that $[K]_{ij} = 0$ whenever $A(g)_{ij} = 0$. The space $\mathcal{K}_{n,n}(g)$ is defined similarly, with blocks of size $n_i \times n_j$.

According to these definitions, a control law $u$ defined by

$$\begin{pmatrix} u_1(k) \\ \vdots \\ u_N(k) \end{pmatrix} = K(g) \begin{pmatrix} x_1(k) \\ \vdots \\ x_N(k) \end{pmatrix} \text{ for all } k \geq 0$$

for some feedback matrix gain $K(g) \in \mathcal{K}_{m,n}(g)$ is such that $u_i(k)$ involves the values of $x_j(k)$ for only those $j$ such that $(i, j) \in E(g)$.

Each edge $(i, j)$ in a graph $g$ can thus informally be thought of as a communication link allowing agents $i$ and $j$ to use each other’s state value when computing their control input. The whole graph specifies the information pattern or the communication topology. A control law satisfying (2.2) is said to have structure $g$. An unstructured control law is one that has structure corresponding to the complete graph. It should also be noted from (2.2) that we are interested only in static and linear control laws so that the matrix $K(g)$ is time-invariant. Thus, if present, a communication link is
assumed to be perfect in the sense that we ignore effects such as quantization issues, data dropouts and data delays longer than one time step.

A control law of the form (2.2) will be called stabilizing if, in closed-loop,

$$\lim_{k \to \infty} x(k) = 0, \quad \forall x_0 \text{ such that } \|x_0\| \leq 1.$$ 

This is equivalent to the matrix $A + BK(g)$ being Schur, i.e., having all its eigenvalues with modulus strictly less than one.

2.3. Cost Functions. For any given positive definite matrix $Q \in \mathbb{S}^n$, control law $u$, and initial condition $x_0$, we define the familiar quadratic cost

$$J(Q, x_0; u) := \sum_{k=0}^{\infty} \begin{bmatrix} x_1(k) \\ \vdots \\ x_N(k) \end{bmatrix}^* \begin{bmatrix} Q & \cdots & u_1(k) \\ \vdots & \ddots & \vdots \\ u_N(k) & \cdots & \begin{bmatrix} x_1(k) \\ \vdots \\ x_N(k) \end{bmatrix}^* \end{bmatrix}$$

subject to

$$x_i(k+1) = A_i x_i(k) + B_i u_i(k) \quad \forall k \geq 0$$

$$x_i(0) = [x_0]_i \quad \forall i = 1 \ldots N. \quad (2.3)$$

In order to study the influence of a control law's structure on this cost, it is preferable to eliminate the dependence on initial conditions and consider the worst-case cost

$$J(Q, u) := \sup_{\|x_0\| \leq 1} J(Q, x_0; u).$$

The minimum value of this worst-case cost over the choice of control laws is defined as

$$J^*(Q) := \inf_u J(Q, u).$$

It is well known that for all initial conditions $x_0$ and control laws $u$, $J(Q, x_0; u) \geq x_0^*Px_0$, where $P$ is the unique positive definite solution of the Riccati equation

$$P = A^*PA + Q - A^*PB(B^*PB + I)^{-1}B^*PA. \quad (2.4)$$

Further, the cost $x_0^*Px_0$ is achievable by the familiar LQR control. Thus, we immediately obtain,

$$J^*(Q) = \lambda_{\max}(P).$$

When a graph $g \in G_N$ imposes a structure on the allowed control laws, we can analogously define the best performance achievable by a structured control law as

$$J^*_g(Q) := \left\{ \begin{array}{ll} \inf_u J(Q, u) \\ \text{subject to } u \text{ has structure } g \end{array} \right. \quad (2.5)$$

It is clear that for any graph $g$, $J^*(Q) \leq J^*_g(Q)$. In keeping with the spirit of the previous notations, we will write $J_g(Q, u)$ instead of $J(Q, u)$ when the control law $u$ at hand has structure $g$. We will also sometimes abuse notation by writing $J_g(Q, K(g))$.
By definition, there exists equation (2.5) is attained, i.e., there exists an optimal control gain\(\bar{g}\) which means that, for\(l > l_0\)
and all\(x_0\) with norm less than one,
\[
x_0^\dagger K_1^\dagger K_1 x_0 \leq J_g(Q, x_0; K_1) \leq J_g(Q, K_1) \leq J_g(Q, u_0) < +\infty,
\]
which means that, for\(l > l_0\), \(K_1\) belongs to the closed ball (for the norm\(\sigma_{\max}(\cdot)\))
\[
\{ K \in \mathcal{K}_{m,n}(g) | \sigma_{\max}(K) \leq J_g(Q, u_0) \}.
\]
Since \(\mathcal{K}_{m,n}(g)\) is finite-dimensional, this ball is compact and thus \(\{K_1\}\) has a converging subsequence (which we still denote as \(\{K_1\}\)) with limit \(\bar{K}(g)\) in this ball. Finally, it is easy to show that function \(J_g(Q, \cdot)\) is continuous on its domain and, hence, that
\[
J_g^*(Q) = \lim_{n \to \infty} J_g(Q, K_n) = J_g(Q, \bar{K}(g)).
\]
\(\Box\)
2.4. Value of a Graph. We are now in a position to introduce the value of a graph. The terminology is borrowed from [13]. Let a mapping \( Q : G \rightarrow S^n \) be given. The value of graph \( g \) is defined as
\[
V(g) := J_g^*(Q(g)).
\] (2.7)

The motivation for introducing such graph-dependent weighting matrices and cost functions is the following. Assume that we are interested in finding a controller minimizing the cost \( J(Q_0, \cdot) \) for some positive definite matrix \( Q_0 \). If there is no restriction on the structure of the control law, the optimal controller’s interconnection topology will typically be a full graph. In practice, however, building and maintaining each of the graph’s communication edges has a cost which, if taken into account, may make this control law less attractive. It is to capture this trade-off between closed-loop performance and controller topology that we introduce an information cost [16] associated to every communication graph \( g \). Of course, there are many ways in which such a cost could be defined. Our choice of a graph-dependent weight matrix \( Q \) means that we are putting a price on the amount of energy used for communication, which fits naturally in the LQR framework. Several choices are possible for the map \( Q \), some of which are detailed below along with a possible physical interpretation.

Definition 2.3. We say that a weight map \( Q : G \rightarrow S^n \) is

- Edge separable without interference: if it satisfies
\[
Q(g) := Q_0 + \sum_{(i,j) \in E(g)} P_{ij},
\] (2.8)

for all graphs \( g \), with each matrix \( P_{ij} \geq 0 \) being partitioned according to the subsystems and having all blocks equal to zero except the \((i,i)^{th}\), \((i,j)^{th}\), \((j,i)^{th}\) and \((j,j)^{th}\) ones. In this case, a subsystem pays an energetic price for transmitting the value of its state to and another subsystem.

- Edge separable with externalities: if it is still given by (2.8), but with all the diagonal blocks of each matrix \( P_{ij} \) being possibly non-zero. Such information costs can model situations in which not only do the subsystems that are exchanging information pay a price, but also, the subsystems not directly involved in the communication agree to reduce their own transmission power, e.g., to reduce the effect of interference. Such a model is reasonable when subsystems are cooperating with each other. The situation when the cost of one agent can be influenced by the action of others is referred to as an “externality” in the Economics literature.

- Non-separable or “with interference”: if \( Q(g) \) can be a full matrix without any particular structure for every graph \( g \). This corresponds to the general case where the energetic cost paid for communication over every link depends on all the other links present in the graph. It is to capture this parasitic effect of edges on each other that we say that “there is interference”.

Accounting for the communication cost through an increase in the value of the matrix \( Q \) implies that the communication cost varies as the square of the state value. This model may not be directly applicable to some multi-agent systems. If coding strategies are employed by the transmitter, the true communication cost may depend on metrics such as the bit rate, or, in turn, the entropy rate of the quantity being
transmitted. In fact, obtaining a measure of communication cost that is suitable for all applications of multi-agent systems and admits of a theoretical analysis is an important research direction. Our simple model allows us to use standard tools based on LQR design theory and derive initial results on this important problem.

We will also make use of the following two properties of a map $Q$.

**Definition 2.4.** We say that a map $Q : \mathcal{G}_N \to \mathbb{S}^n$ is

(i) **non-decreasing** if $\forall g, g' \in \mathcal{G}_N, \ g \leq g' \Rightarrow Q(g) \leq Q(g')$.

(ii) **structure-compatible** if $\forall g \in \mathcal{G}_N, \ Q(g) \in K_{n,n}(g)$.

Note that edge separability in the mapping $Q$ is not sufficient for structure compatibility, since the matrix $Q_0$ in (2.8) may not be block-diagonal.

Our goal, in the next sections, is to characterize the optimal (or, again in the terminology of [13], efficient) graph $g^*$ defined by

$$g^* := \arg\min_{g \in \mathcal{G}_N} V(g). \tag{2.9}$$

The structure imposed by $g^*$ corresponds to the minimal communication requirements (in the sense of the cost function (2.3)) needed to control the $N$ agents. Since there are only finitely many elements in $\mathcal{G}_N$, $g^*$ always exists provided the value function $V$ is proper.

One way to determine $g^*$ would be to compute the optimal structured controller and hence the value $V(g)$ of every graph $g \in \mathcal{G}_N$ and then perform a search. However, this approach is computationally extremely demanding for two reasons. First, as already noted in Section 1, computing even a single optimal structured controller is typically a hard, non-convex optimization problem, which may thus not be solvable in polynomial time. Second, even if we could compute the value of every graph in $\mathcal{G}_N$, the cardinality of this set, which is exponential in $N$, would render any brute force search approach hopeless for large systems.

In the remainder of this paper, we explore two alternative and complementary approaches for characterizing the efficient graph $g^*$. In Section 3, we show that, for particular families of maps $Q$, it is possible to compute $g^*$ explicitly. Next, we consider more general maps $Q$ and derive tractable necessary and sufficient conditions for two graphs $g$ and $g'$ to satisfy $V(g) \leq V(g')$. While these tests do not directly determine $g^*$, they can be used in a branch-and-bound-type algorithm to quickly generate bounds.

3. Cliques and efficient graph.

3.1. **Non-decreasing** $Q$. In this section, we focus on maps $Q$ which are compatible with the partial order “$\leq$” on the graphs in $\mathcal{G}_N$. This will allow us to give conditions to compare the value of two graphs independently of the system (2.1) and characterize the efficient graph rigorously. We start with the following simple result.

**Proposition 3.1.**

(i) If $g \leq g'$, then for all $Q > 0$, $J^*_g(Q) \leq J^*_g(Q)$.

(ii) If $Q \leq Q'$ then, for all $g \in \mathcal{G}_N$, $J^*_g(Q) \leq J^*_g(Q')$.

We thus obtain the following immediate corollary.

**Corollary 3.2.** If the map $Q$ is non-decreasing and $g \leq g'$, $J^*_g(Q(g)) \leq V(g')$.

**Proof.** Applying item (ii) of Proposition 3.1 to $Q = Q(g)$ and $Q' = Q(g')$, yields

$$J^*_{g'}(Q(g)) \leq J^*_g(Q(g')).$$
Fig. 3.1. Some examples of graphs. All but (b) are clique graphs.

Since by definition
\[ J^*_g(Q(g')) = V(g'), \]
we immediately obtain
\[ J^*_g(Q(g)) \leq V(g'). \] (3.1)

Before we go further and prove our first result on graph efficiency, we need to introduce the following concept borrowed from [11].

**Definition 3.3.** A graph \( g \in \mathcal{G}_N \) is said to be a clique graph if each of its connected component is a clique i.e., a complete subgraph.

Examples and counter-examples of clique graphs are given in Figure 1. As we will see, these graphs are useful because their value can be readily computed.

**Proposition 3.4.** Let \( g \in \mathcal{G}_N \) be a clique graph and the map \( Q \) be structure-compatible. Then \( V(g) = J^*(Q(g)) \). In particular, \( V(g) = \lambda_{\text{max}}(P(g)) \) where \( P(g) \) is the unique positive-definite solution of Riccati equation
\[ P(g) = A^*P(g)A + Q(g) - A^*P(g)B(B^*P(g)B + I)^{-1}B^*P(g)A. \] (3.2)

**Proof.** We have already mentioned in the previous section that the unstructured control law minimizing \( J(Q(g),.) \) and the corresponding optimal value are given by
the solution to Riccati equation (3.2). Since \( g \) is a clique graph, using a reordering of the vertices, the matrix \( A(g) \) can be converted to a block-diagonal form. The systems thus become decoupled and one can readily show that, under our assumptions, the optimal control law \( K(g) \), in fact, has structure \( g \) (see, e.g., the proof of Proposition 2.6 in [9].)

Now, note that, by definition of \( K(g) \),

\[
J(Q, x_0; u) \geq J(Q, x_0; K(g)),
\]

for all \( x_0 \) and arbitrary control laws \( u \). Hence, for all \( u \),

\[
J(Q(g), u) \geq J(Q(g), K(g))
\]

and, in particular, considering control laws with structure \( g \),

\[
V(g) = J^*(Q(g)) \geq J(Q(g), K(g)).
\]

The fact that \( K(g) \) has structure \( g \) then completes the proof. □

Proposition 3.4 and its proof are reminiscent of the results of [2], where it is shown that, for spatially invariant system, the optimal controller is itself spatially invariant. Here, the optimal controller has the same structure as the clique graph, when the map \( Q \) is structure-compatible. This property allows us to compare the value of a clique graph to that of its supergraphs.

**Theorem 3.5.** Let \( Q \) be non-decreasing and structure-compatible and \( g \) be a clique graph. Then \( V(g) \leq V(g') \) for all \( g' \in G_N \) such that \( g \not\leq g' \). In particular, the graph \( g^* \) characterized by \( E(g^*) = \{(i, i) : 1 \leq i \leq N\} \) is efficient, as defined in (2.9). In other words, the minimal control topology (for cost (2.3)) is fully decentralized.

**Proof.** By Proposition 3.4, since \( g \) is a clique graph and \( Q \) is structure-compatible, \( V(g) = J^*(Q(g)) \). Also, since \( Q \) is non-decreasing, we can use relation (3.1) to write \( J^*_g(Q(g)) \leq V(g') \). Finally, by definition of the various minimization problems, we have

\[
V(g) = J^*(Q(g)) \leq J^*_g(Q(g)) \leq V(g').
\]

That the fully decentralized topology is minimal follows from the fact that \( g^* \), as defined in the theorem, is a clique graph and, clearly, that \( g^* \not\leq g \) for all \( g \in G_N \). □

The result of Theorem 3.5 illustrates that, if a cost is charged for communication, cooperation can sometimes be detrimental. Note that from [10] we know that for the purposes of stabilizability and controllability, all communication topologies are identical. However, different information patterns yield different performance and this result states that more communication links (and hence less constraints on the structure of the control law) may not automatically translate to better performance in the case of structure compatibility.

### 3.2. Merely structure-compatible \( Q \)

The previous section has shown that it is possible to compare clique graphs to their supergraphs when map \( Q \) is structure compatible and non-decreasing, and that adding edges to a clique graph is then always detrimental to the value. These results hold independently of the subsystems (2.1) under consideration.

In this section, we remove the non-decreasing assumption on map \( Q \) and derive conditions for comparing clique graphs to arbitrary graphs. In the reminder of this section, we will assume that either
\( g \) is a clique graph and map \( Q \) is structure-compatible, or that \( g \) is the complete graph, i.e., \( E(g) = \{(i,j), 1 \leq i \leq j \leq N\} \), and map \( Q \) is arbitrary.

**Theorem 3.6.** Let \( \gamma = \lambda_{\max}(P(g)) \), where \( P(g) \) is the solution of Riccati equation (2.4) with \( Q = Q(g) \). A graph \( g' \) satisfies \( V(g') \leq \gamma = V(g) \) if and only if \( \lambda^* \leq \gamma \), where \( \lambda^* \) is defined as

\[
\lambda^* := \begin{cases} 
\arg\min_{P', K'} \lambda_{\max}(P') \\
\text{subject to} \quad P' \geq (A + BK')^*P'(A + BK') + Q(g') + K'^*K' \\
K' \in \mathcal{K}_{m,n}(g') 
\end{cases} \tag{3.3}
\]

*Proof.* Assume \( \lambda^* \leq \gamma \). Then there exists \( P' > 0 \) and \( K' \in \mathcal{K}_{m,n}(g') \) such that (3.3) holds and \( \lambda_{\max}(P') \leq \gamma \). Also, we see that along any closed-loop trajectory with initial condition \( x_0 \),

\[
x^*(k)P'x(k) - x^*(k+1)P'x(k+1) \geq x^*(k)Q(g')x(k) + u^*(k)u(k).
\]

Then, summing over \( k \), we obtain

\[
x_0^*P'x_0 \geq \sum_{k=0}^{\infty} (x^*(k)Q(g')x(k) + u^*(k)u(k)) = J_{g'}(Q(g'), x_0; u), \tag{3.4}
\]

for the control law \( u \) satisfying \( u(k) = K'x(k) \). Taking the supremum over \( x_0 \) over the unit ball yields

\[
\gamma \geq J(Q(g'), K') \geq V(g').
\]

Reciprocally, assume \( \lambda^* \leq \gamma \). Then there exists a stabilizing control law with structure \( g' \). By virtue of Proposition 2.2, this implies that there exists \( K' \) such that \( V(g') = J(Q(g'), K') \). The control law corresponding to \( K' \) must itself be stabilizing and thus, according to Proposition 2.1,

\[
V(g') = \lambda_{\max}(\hat{P}'),
\]

where \( \hat{P}' \) is the positive definite solution of Lyapunov equation (2.6) with \( Q = Q(g') \).

Since \( (\hat{P}', \hat{K}') \) is feasible for the optimization problem on the right hand side of (3.3), this problem has the same optimal value as the problem \( (P) \)

\[
(P) \left\{ \begin{array}{l} 
\inf_{P', K'} \lambda_{\max}(P') \\
\text{subject to} \quad P' \geq (A + BK')^*P'(A + BK') + Q(g') + K'^*K' \\
K' \in \mathcal{K}_{m,n}(g') \\
\lambda_{\max}(P') \leq \lambda_{\max}(\hat{P}')
\end{array} \right.
\]

The feasible set of \( (P) \) is compact since it is clearly closed and, if \( (P, K) \) is a feasible for \( (P) \), both \( \lambda_{\max}(P) \) and \( \sigma_{\max}(K) \) are bounded by \( \lambda_{\max}\hat{P}' \). Hence the infimum in \( (P) \) is attained, i.e., there exists \( (P_0, K_0) \) such that the optimal value of problem \( (P) \) is \( \lambda_{\max}(P_0) \). Then, clearly, \( \lambda^* = \lambda_{\max}(P_0) \), which means that the optimal value is also attained in the problem on the right hand side of (3.3) and, by definition of \( P_0 \),

\[
\lambda^* = \lambda_{\max}(P_0) \leq \lambda_{\max}(\hat{P}') = V(g') \leq \gamma.
\]

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Condition (3.3), although both necessary and sufficient, is not practical for comparing the value of a graph to that of a clique graph, since problem (3.3) is not easily solved numerically. This is because it involves a bilinear constraint on variables $P'$ and $K'$, and the change of variables traditionally used to convexify static state feedback synthesis problems is inoperative here because of the structural constraint on $K'$. Some approaches have been proposed recently for obtaining merely sufficient but convex synthesis conditions in a similar context, from restricting oneself to a diagonal matrix $P'$ [1] to introducing additional variables [6]. While these methods could be applied to the present problem as well to obtain a computable upper-bound for $\lambda^*$ and, in turn, a sufficient condition for $g'$ having lower value than $g$, we give different sets of convex sufficient and necessary conditions for comparing the values of graphs.

**Theorem 3.7** (Sufficient Condition). Let $P(g)$ be the unique positive definite solution of Riccati equation (3.2) with $Q = Q(g)$. If there exists a matrix $K \in \mathcal{K}_{m,n}(g')$ such that the following Linear Matrix Inequality (LMI) in $K$ is satisfied

\[
\begin{bmatrix}
-P(g) + Q(g') & (A + BK)^* & K^* \\
(A + BK) & P(g)^{-1} & 0 \\
K & 0 & I
\end{bmatrix} < 0,
\]

then $V(g) \geq V(g')$.

**Proof.** If matrix $K$ is feasible for LMI (3.5), the pair $(P(g), K)$ is feasible for problem (3.3) since, according to the Schur complement formula,

\[-P(g) + Q(g') + (A + BK)^*P(g)(A + BK) + K^*K < 0.
\]

As a result, $\lambda^* \leq \lambda_{\text{max}}(P(g)) = \gamma$ and, according to Theorem 3.6, $V(g') \leq V(g)$. □

**Theorem 3.8** (Necessary Condition). If $V(g) \geq V(g')$, then there exists a matrix $K \in \mathcal{K}_{m,n}(g')$ and a matrix $X > 0$ such that the following LMI (in $K$ and $X$) is satisfied

\[
\begin{bmatrix}
-\gamma I + Q(g') & (A + BK)^* & K^* \\
(A + BK) & X & 0 \\
K & 0 & I
\end{bmatrix} \leq 0,
\]

where $\gamma$ is defined as in Theorem 3.6.

**Proof.** By Theorem 3.6, if $V(g) \geq V(g')$, there exists $(P', K')$ feasible for problem (3.3) with $\lambda_{\text{max}}(P') \leq \gamma$. Hence, there exist $P'$ and $K'$ such that $P' \leq \gamma I$ and $P' \geq (A + BK')^*P'(A + BK') + Q(g') + K'^*K'$ and, as a result,

\[
\gamma I \geq (A + BK')^*P'(A + BK') + Q(g') + K'^*K'.
\]

Using Schur complement and letting $X = P'$ shows that the LMI of Theorem 3.8 is feasible. □

4. Comparing arbitrary graphs. In this section, we extend some of the tools presented so far and compare the values of arbitrary graphs, for any given map $Q$. Because it is already difficult to compute the value of an arbitrary graph (as against the case of clique graphs and structure-compatible maps), we settle for tractable sufficient conditions on the weighting matrices $Q(g)$ and $Q(g')$, which allow to compare graphs $g$ and $g'$. More precisely, we ask what (tractably testable) properties of a map $Q$ are sufficient to guarantee that a graph has a smaller value than another one.
Theorem 4.1 (Sufficient Condition). Consider two graphs \(g\) and \(g'\). If there exist \(K \in K_{m,n}(g')\) and a matrix \(P > 0\) such that

\[
P = (A + BK)^*P(A + BK) + Q(g') + K^*K \tag{4.1}
\]
\[
Q(g) \geq Q(g') + (K + S^{-1}B^*PA)^*S(K + S^{-1}B^*PA) \tag{4.2}
\]
\[
S = B^*PB + I, \tag{4.3}
\]

then \(V(g) > V(g')\).

Proof. Assume \(P > 0\) satisfies (4.1). Then, since \(Q(g') + K^*K > 0\), by the properties of a discrete algebraic Lyapunov equation [8], the matrix \((A + BK)\) is Schur. Thus,

\[
\lim_{k \to \infty} x(k) = 0,
\]

for the closed loop system

\[x(k + 1) = (A + BK)x(k),\]

starting from any initial condition \(x_0\). Also, proceeding as in the proof of Theorem 3.6 we see that along any closed-loop trajectory with initial condition \(x_0\),

\[
x_0^*Px_0 = J_{g'}(Q(g'), x_0; u), \tag{4.4}
\]

for the control law \(u\) satisfying \(u(k) = Kx(k)\). Now from (4.1) we see that \(P\) satisfies

\[
P = A^*PA - A^*PBS^{-1}B^*PA + Q(g') + (K + S^{-1}B^*PA)^*S(K + S^{-1}B^*PA),
\]

where \(S = B^*PB + I\). This Riccati equation is identical to the one obtained using LQ control theory if we were to look for an unstructured control law that minimizes a cost function of the form (2.3), but with the weighting matrix

\[Q = Q(g') + (K + S^{-1}B^*PA)^*S(K + S^{-1}B^*PA).
\]

Thus, for all initial conditions \(x_0\) and control laws \(v\),

\[
x_0^*Px_0 \leq J(Q(g') + (K + S^{-1}B^*PA)^*S(K + S^{-1}B^*PA), x_0; v)
\]

and, maximizing over \(x_0\),

\[J_{g'}(Q(g'), u) \leq J(Q(g') + (K + S^{-1}B^*PA)^*S(K + S^{-1}B^*PA), v),
\]

for all control laws \(v\). But, from (4.2), we obtain that

\[J(Q(g') + (K + S^{-1}B^*PA)^*S(K + S^{-1}B^*PA)) \leq J(Q(g), v)
\]

and, thus, that \(J_{g'}(Q(g'), u) \leq J(Q(g), v)\) for all \(v\). In particular, this implies that

\[V(g') \leq J_{g'}(Q(g'), u) \leq V(g).
\]
network-builder) and that a topology, $g'$, has been agreed on and implemented. Then, the price designer can ensure that there is no incentive to build a new edge by choosing a stabilizing control gain $K \in \mathcal{K}_{m,n}(g')$, solving Lyapunov equation (4.1) and picking $Q$ such that (4.2) holds. On the other hand, the network-builder is only given the map $Q$. If it wants to use conditions (4.1) to determine whether it is to its advantage to add new edges to a pre-existing control topology (or, more aptly, to find a certificate that it is detrimental to do so) it has to solve equations (4.1) for both $P$ and $K$. Even after using the Schur complement formula on inequality (4.2) and rewriting (4.1) as two matrix inequalities, this is still a hard task to perform, since one then has to solve a set of bilinear matrix inequalities which, in general, is NP-hard [22].

Another type of sufficient conditions can be obtained for comparing arbitrary graphs by building on the ideas of Section 3. In particular, we can state the following

**Theorem 4.2.** Let $g, g'$ be two graphs and $Q$ be any weight map. If there exists a diagonal matrix $\Lambda$ such that $Q(g) < \Lambda < Q(g')$, then $V(g) < V(g')$.

**Proof.** Let $e$ be the fully decentralized graph, i.e. $E(e) = \{(i,i) : 1 \leq i \leq N\}$ and define $H(g) := \Lambda$ for all $g$. Map $H$ is clearly structure-compatible (and non-decreasing) and we can compute the value of any graph $h$ in $\mathcal{G}_N$ by using this weighting map instead of $Q$. We will denote this value by $V_H(h)$. We can now proceed in two steps:

(a) Using Proposition 3.1, we see that

$$V(g) = J^*_g(Q(g)) \leq J^*_g(Q(g)) \leq J^*_g(\Lambda) = V_H(e).$$

(b) Besides, according to Theorem 3.5 $V_H(e) \leq V_H(g')$ since $H$ is structure-compatible. But $H(g') = \Lambda \leq Q(g')$ and so

$$J^*_g(H(g')) \leq J^*_g(Q(g')).$$

Hence $V(g) \leq V_H(e) \leq V_H(g') \leq V(g')$.

In fact, the arguments of the previous proof are still valid if matrix $\Lambda$, instead of being diagonal, has the structure of a clique graph. We can thus state the following

**Corollary 4.3.** Let $h$ be a clique subgraph of both graphs $g$ and $g'$. If there exists a matrix $M \in \mathcal{K}_{m,n}(h)$ such that $Q(g) < M < Q(g')$, then $V(g) < V(g')$.

5. **Communication Topology Design for Multi-Vehicle Systems: An Example.** In this section, we illustrate our results by applying them to the problem of determining the most economical information pattern required to keep multiple vehicles in a geometric formation, while taking into account the cost of communication.

Consider the system pictured in Figure 5.1 and composed of three independent vehicles modeled using the double integrator model, with the motion of the $i$-th vehicle described by the equation

$$\begin{pmatrix} \delta y_i(k+1) \\ \dot{\theta}_i(k+1) \end{pmatrix} = \begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta y_i(k) \\ \dot{\theta}_i(k) \end{pmatrix} + \begin{pmatrix} 0.005 \\ 0.1 \end{pmatrix} \omega_i(k).$$

We want these vehicles to evolve in formation so that

- angle $\theta_i(k)$ and deviation $\delta y_i(k)$ remain small for all $i$, at all instant $k$,
- vehicle number 3 is always at the middle point between vehicle 1 and 2, when starting in such a position.
With such requirements, it makes sense to design control laws \( \{ \omega_i \}_{i=1,...,3} \) for subsystems (5.1) so that the cost function

\[
\sum_{k=0}^{\infty} \left( \delta y_1^2(k) + \delta y_2^2(k) + \delta y_3^2(k) \right) + \left( \theta_1^2(k) + \theta_2^2(k) + \theta_3^2(k) \right) \\
+ \left( \delta y_3(k) - \frac{\delta y_1(k) + \delta y_2(k)}{2} \right)^2 + \left( \omega_1^2(k) + \omega_2^2(k) + \omega_3^2(k) \right)
\]

is minimized in closed-loop.

This amounts to solving an optimal control problem of the form (2.5) with matrix \( Q_0 \) given by

\[
Q_0 = \begin{pmatrix}
\begin{array}{cccccc}
\frac{5}{4} & 0 & \frac{1}{2} & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{7}{4} & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & -1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\end{pmatrix}.
\]

In (5.2), matrix \( Q_0 \) is partitioned conformably to the subsystems, with the first and second coordinate of subsystem \( i \)'s state being \( \delta y_i \) and \( \theta_i \), respectively.

Note that without any communication cost, the optimal controller for the matrix \( Q_0 \) is full, which means that the optimal graph is a full graph. In order to determine an efficient communication graph for this problem, we must define a communication cost in the form of a mapping \( Q : G_3 \rightarrow S^6 \). We choose to adopt an edge-separable
map with externalities, such that

\[ Q(g) = Q_0 + \sum_{(i,j) \in E(g)} P_{ij} \text{ for all } g, \]

with each positive definite matrix \( P_{ij} \) having all its block equal to zero except for the diagonal and the \((i, j)\) and \((j, i)\)-ones. Note that the map \( Q \) so obtained is non-decreasing but not structure compatible, since \( Q_0 \) is not block-diagonal. We want to determine price matrices such that the graphs pictured in Figure 5.1(b) satisfy \( V(g) \leq V(f) \) and \( V(h) \leq V(g) \).

We start by choosing

\[
P_{12} = \begin{pmatrix}
1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and use Theorem 4.1 to find a price matrix \( P_{13} \) such that \( V(h) \leq V(g) \). Picking the stabilizing controller

\[
K = \begin{pmatrix}
-1 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}
\]

with structure \( h \), one can show that equations (4.1) are satisfied for some matrix \( P > 0 \) if we take

\[
P_{13} = Q(g) - Q(h) = \begin{pmatrix}
10 & 0 & 0 & 0 & -0.5 & 0 \\
0 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 10 & 1 & 0 & 0 \\
0 & 0 & 1 & 10 & 0 & 0 \\
-0.5 & 0 & 0 & 0 & 10 & 0 \\
0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}.
\]

To find a matrix \( P_{23} \) such that \( V(f) \geq V(g) \), we can use Theorem 4.2 and solve the two LMIs

\[
Q_0 + P_{12} + P_{13} < \Lambda \\
\Lambda < Q_0 + P_{12} + P_{13} + P_{23}
\]

in the structured variables \( \Lambda \) and \( P_{23} \). Using SeDuMi, we find that the following pair is feasible

\[
P_{23} = \begin{pmatrix}
78.797 & 0 & 0 & 0 & 0 \\
0 & 78.797 & 0 & 0 & 0 \\
0 & 0 & 78.797 & 13.1328 & 0 \\
0 & 0 & 0 & 78.797 & 13.1328 \\
0 & 0 & 13.1328 & 0 & 78.797 \\
0 & 0 & 0 & 13.1328 & 0
\end{pmatrix} \ ; \ \Lambda = 39.3985 \ I_6.
\]

By adopting the price matrices given above, the price-designer can thus ensure that graph \( h \) will be chosen by the network-builder.
6. Conclusions and Future Work. In this paper, we proposed and partially analyzed a new model for determining the influence of the structure of a controller on closed-loop performance in distributed control design problems. For a plant composed of dynamically uncoupled subsystems, we proposed making one of the weight matrices of the classical LQR cost function to be topology-dependent to capture the cost of allowing sub-systems to communicate with each other. For some models of such dependencies, we investigated the existence and properties of an optimal structured controller. For arbitrary models, we presented some sufficient conditions for the existence of critical prices at which adding supplementary edges becomes detrimental to closed-loop performance.

This paper is only a first attempt at studying the influence of interconnection topology on performance in distributed control design problems. The problem is hard because the underlying problem of determining optimal structured controllers is hard. The chief virtue of our approach is allowing us to circumvent this problem and being able to give rigorous statements about optimal topologies and comparing topologies to each other. Our results thus complement the more heuristic claims of [3] and [21]. Even within the realm of this restricted model, many unanswered questions remain. For example, it would be nice to be able to compare the values of any two graphs and not only of those that are comparable for the partial order “≪”. Likewise, one may ask whether it is possible to derive bounds similar to those of Theorem 4.1 that are tractable and/or explicit. Another avenue of future research would be to make the communication cost model more physically motivated, by considering metrics such as entropy rate. Finally, the effect of imperfections in the communication links also needs to be studied.

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REFERENCES


