Feedback Passivation of Discrete-Time Systems Under Communication Constraints

Yingbo Zhao and Vijay Gupta

***Abstract*—Passivity is a desirable property of a dynamical system because it implies stability and is invariant under negative feedback and parallel interconnections. Feedback passivation is the process of making a nonpassive system passive through feedback control. In this technical note, we study the problem of feedback passivation when the controller has limited information about the state of the plant. Nonlinear plants that are linear in the control inputs are considered. The main result of the technical note is a certainty equivalence principle: any state feedback controller that ensures closed-loop input-strict passivity with index μ using the exact state of the plant will also ensure closed-loop stochastic quasi passivity using an estimate of the state, provided that the infinity norm of the estimation error process is bounded by some function of μ. A corollary is that for linear systems, although passivity is more strict than stability, feedback passivation does not place more constraints on the estimation error and hence does not demand more from the communication channel than mean square stabilization.**

***Index Terms*—Networked control, nonlinear system, passivity.**

I. INTRODUCTION

Passivity-based control is an important approach to the study of complex systems and nonlinear systems, especially for stabilization purposes [1]–[4]. Passivity is a desirable property for a number of reasons. First, Lyapunov function candidates can usually be obtained naturally from the storage functions induced by passivity [5], [6], which means that stability and stabilization problems can often be solved immediately once passivity is guaranteed [7]. Second, the fact that passivity is preserved under negative feedback and parallel interconnections makes it an attractive approach for the analysis and design of large-scale and composite systems [8].

A discrete-time system

$$x(k+1) = f\left(x(k), u(k)\right)$$
$$y(k) = g\left(x(k), u(k)\right) \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^q$ is the control input, $y(k) \in \mathbb{R}^q$ is the output and f, g are smooth mappings of appropriate dimensions, is said to be (locally) passive [9], [10], if there exists a positive definite function $V(x(k))$ such that for all $k \in \mathbb{Z}_{\geq 0}$ and all $(x(k), u(k))$ in a neighborhood of the origin $(\mathbf{0}_n, \mathbf{0}_q)$

$$V\left(x(k+1)\right) - V\left(x(k)\right) \leq y^T(k)u(k). \tag{2}$$

Manuscript received August 3, 2015; revised November 14, 2015; accepted December 24, 2015. Date of publication January 8, 2016; date of current version October 25, 2016. This work was supported in part by the National Science Foundation CCF-1117728. Recommended by Associate Editor J. Daafouz.

The authors are with the Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556 USA (e-mail: yzhao5@nd.edu; vgupta2@nd.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2016.2515848

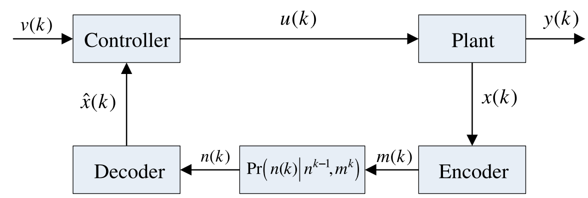

Fig. 1. Feedback passivation across a communication channel.

The system (1) is said to be state-strictly passive if there exists $c \in \mathbb{R}_{>0}$ such that

$$V\left(x(k+1)\right) - V\left(x(k)\right) \leq y^T(k)u(k) - c\left\|x(k)\right\|^2 \tag{3}$$

and input-strictly passive with index μ [11], [12], if there exists $\mu \in \mathbb{R}$ such that

$$V\left(x(k+1)\right) - V\left(x(k)\right) \leq y^T(k)u(k) - \mu\left\|u(k)\right\|^2. \tag{4}$$

The function $V(x(k))$ is called the storage function and can be interpreted as the energy stored in the system at time k. The right-hand side is called the supply rate, which characterizes the energy supplied to the system at time k. Thus, the implication of inequality (2) is that at every time step, the increase in the energy stored in a passive system is upper bounded by the energy supplied to this system.

The related problem of feedback passivation, where the objective is to transform a nonpassive system (plant) to a passive system by means of a suitable feedback controller has also been studied in the literature [7], [13]. For a plant in the form of (1), the aim is to find a regular state feedback controller $u(k) = \alpha(x(k), v(k))$, where $v(k) \in \mathbb{R}^q$ is an exogenous input, such that the closed-loop mapping from v to y is passive. If such a controller exists, then the plant is said to be feedback passive. The notion of feedback passivity is obviously more general than that of passivity since if the open-loop plant is already passive we can introduce the dummy controller $u(k) = v(k), \forall\, k \in \mathbb{Z}_{\geq 0}$. In other words, a passive system is trivially feedback passive.

Under the assumption that the controller knows the exact state, Byrnes *et al.* [7] showed that a continuous, input-affine, nonlinear system is feedback passive if and only if it has relative degree 1 and is weakly minimum phase. Similar necessary and sufficient conditions for feedback losslessness, which is a special case of feedback passivity, were obtained for discrete-time nonlinear affine-in-control systems by Byrnes and Lin [14], where relative degree 0 and lossless zero dynamics are required. Navarro *et al.* [13] considered feedback passivation of nonlinear discrete-time plants that are non-affine in the control input.

To the best of our knowledge, almost all existing results on feedback passivation assume that the controller has access to the exact state of the plant. In this technical note, we study feedback passivation when the exact state is not available at the controller. In particular, we assume that the plant transmits its state information to the controller across a communication channel as shown in Fig. 1.

Note that due to the presence of the communication channel, the system evolves in a stochastic manner and we must adopt a definition of passivity for stochastic systems. Such notions have been proposed in, e.g., [15]–[18]. We would also like to mention some related works on stochastic passivity. [19] considered the stochastic passivity of linear time-invariant (LTI) systems with state-dependent noise. The stochastic passivity for nonlinear Markovian jump systems has been studied in the continuous-time framework in [17] and in the discrete-time setting in [20]. Moreover, [21] proposed the notion of stochastic passivity for general nonlinear discrete-time systems with non-anticipative stochastic disturbances and studied the H_∞ control problem.

Also related is the work on feedback stabilization across communication channels. It has been shown in [22] that for the stabilization of a linear time-invariant system across a noiseless digital channel, if the control input u or the control law is known to the encoder and the decoder, then the estimation performance depends only on the quality of the communication channel and is independent of the controller. Other types of communication channels have been studied (see, e.g., [23]–[25]), and the encoder/decoder designs can be directly used in our framework.

The main result of this technical note is a certainty equivalence principle: any state feedback controller that ensures closed-loop input-strict passivity with index μ using the exact state of the plant will also ensure closed-loop stochastic passivity using an estimate of the state, provided that the infinity norm of the estimation error is bounded by some function of μ. A corollary of our result is that although passivity is, in some sense, a more strict condition than stability, for linear systems feedback passivation does not place more constraints on the estimation quality of the state than stabilization. This means, for instance, that if a linear plant can be stabilized in the mean squared sense across a communication channel, then it can also be feedback passivated across that channel. Further, for linear plants, we also provide a specific construction method of feedback passivating controllers.

Notations: Throughout this technical note, $\mathbb{E}[x]$ denotes the expectation of a random variable x. For a vector $x \in \mathbb{R}^n$, we denote the 2-norm $\|x(k)\| \triangleq \sqrt{x^T(k)x(k)}$. A matrix $A \in \mathbb{R}^{n\times n}$ is Schur stable if all its eigenvalues are inside the unit circle on the complex plane. For a real symmetric matrix A, we use $\bar{\lambda}(A)$ and $\underline{\lambda}(A)$ to denote its largest and smallest eigenvalues, respectively. For two positive semi-definite matrices A, B, $A \geq B$ means that $A - B$ is positive semi-definite. For symmetric block matrices, we use an asterisk $*$ to represent the term that can be readily induced by symmetry. The n-vector with all zero elements is denoted by $\mathbf{0}_n$ and the $m \times n$ matrix with all zero elements by $\mathbf{0}_{m\times n}$. The identity matrix of dimension $n \times n$ is denoted by I_n. For any $j \leq k \in \mathbb{Z}_{\geq 0}$, we use $\{x\}_j^k$ to denote the series $\{x(j), x(j+1), \ldots, x(k)\}$, and omit j if $j = 0$. We use $\mathrm{diag}(A_1, \ldots, A_n)$ to denote a block-diagonal matrix.

II. Problem Formulation

In this technical note, we consider the feedback passivation problem for the class of nonlinear plants that is linear in the control input with state space representation

$$\begin{aligned} x(k+1) &= f\left(x(k)\right) + Bu(k) \\ y(k) &= h\left(x(k)\right) + Du(k) \end{aligned} \tag{5}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^q$ is the control input, $y(k) \in \mathbb{R}^q$ is the output, f, h are smooth mappings and B, D are real matrices of appropriate dimensions.

Assume that the open-loop mapping from u to y is nonpassive. The objective is to find a controller that feedback passivates the plant (5), under the assumption that the state x is transmitted to the controller across a communication channel, as depicted in Fig. 1. The model for the communication channel is described as follows.

Definition 1: (Communication Channel Model): A discrete time communication channel $\mathcal{C}$ is defined to be a probabilistic system with an input alphabet $\mathbb{M}$, an output alphabet $\mathbb{N}$ and a probability transition matrix $\Pr(n(k)|\{n\}^{k-1}, \{m\}^k)$ that characterizes the probability of the decoder receiving the output symbol $n(k) \in \mathbb{N}$ given that the symbol $m(k) \in \mathbb{M}$ is sent at time k by the encoder. At time k, the encoder is a mapping $\mathbb{M}^k \times \mathbb{R}^{n(k+1)} \to \mathbb{M}$ given by $E_k(\{m\}^{k-1}, \{x\}^k) = m(k)$ and the decoder is a mapping $\mathbb{N}^{k+1} \to \mathbb{R}^n$ given by $D_k(\{n\}^k) = \hat{x}(k)$. These mappings are under the control of the designer. If $\Pr(n(k)|\{n\}^{k-1}, \{m\}^k) = \Pr(n(k)|m(k))$, then the channel $\mathcal{C}$ is said to be memoryless.

The controller has access to only the output of the decoder, denoted by $\hat{x}(k)$. Formally, the controller is given by

$$u(k) = \alpha\left(\hat{x}(k)\right) + Fv(k) \tag{6}$$

where $\alpha(\cdot)$ is a regular function and $v(k) \in \mathbb{R}^q$ is an exogenous input. The closed-loop system evolution is thus

$$\begin{aligned} x(k+1) &= f_\alpha\left(x(k)\right) + B\eta(k) + BFv(k) \\ y(k) &= h_\alpha\left(x(k)\right) + D\eta(k) + DFv(k) \end{aligned} \tag{7}$$

where $f_\alpha(\cdot) \triangleq f(\cdot) + B\alpha(\cdot)$, $h_\alpha(\cdot) \triangleq h(\cdot) + D\alpha(\cdot)$ and $\eta(k) \triangleq \alpha(\hat{x}(k)) - \alpha(x(k))$. All the discussions in this section are confined to a neighborhood near the origin $(x, v) = (\mathbf{0}_n, \mathbf{0}_q)$ denoted by $\mathbb{X} \times \mathbb{V}$. Moreover, we assume that the function $\alpha(\cdot)$ is locally Lipschitz, i.e., there exists $L_\alpha \in \mathbb{R}_{>0}$ such that $\forall\, x_1, x_2 \in \mathbb{X}$, it holds that $\|\alpha(x_1) - \alpha(x_2)\| \leq L_\alpha \|x_1 - x_2\|$. This assumption implies that $\eta(k) \leq L_\alpha \|e(k)\|$, where $e(k) \triangleq \hat{x}(k) - x(k)$ represents the estimation error.

We propose the following definition of stochastic feedback quasi passivity based on the stochastic passivity in [21].

Definition 2: (Stochastic Feedback Quasi Passivity): System (1) is said to be *stochastically feedback quasi passive* if there exist a regular static feedback controller (6), a storage function $V(\cdot) : \mathbb{X} \to \mathbb{R}_{>0}$ and a scalar $b \in \mathbb{R}_{>0}$ such that $\forall\, k \in \mathbb{Z}_{\geq 0}$ and $\forall\, (x(0), \{v\}^k) \in \mathbb{X} \times \mathbb{V}^k$

$$\mathbb{E}\left[V\left(x(k+1)\right) - V\left(x(k)\right)\right] \leq \mathbb{E}\left[y^T(k)v(k) + b\left\|e(k)\right\|^2\right]. \tag{8}$$

Further if there exists $c \in \mathbb{R}_{>0}$ such that

$$\begin{aligned} &\mathbb{E}\left[V\left(x(k+1)\right) - V\left(x(k)\right)\right] \\ &\quad \leq \mathbb{E}\left[y^T(k)v(k) - c\left\|x(k)\right\|^2 + b\left\|e(k)\right\|^2\right] \end{aligned} \tag{9}$$

then system (1) is said to be *stochastically state-strictly feedback quasi passive*.

We use the term *quasi passivity* because $b\|e(k)\|^2$ on the right-hand side of inequalities (8) and (9) may cause the system to be nonpassive in the traditional sense. However, for deterministic systems with $\|e(j)\| \equiv 0$, the above definition recover the classical passivity.

Remark 1: (Classical Passivity is Recovered When the Controller has Access to the Exact State): For the special case when the controller has access to the exact state, we let $\mathbb{M} = \mathbb{N} = \mathbb{R}^n$, $\Pr(n(k)| \{n\}^{k-1}, \{m\}^k) = 1$ if $n(k) = m(k)$ and $\Pr(n(k)|\{n\}^{k-1}, \{m\}^k) = 0$ otherwise, to obtain $e(k) = \mathbf{0}_q, \forall\, k \in \mathbb{Z}_{\geq 0}$. In this case, the definitions in (8) and (9) recover the classical passivity definition.

We impose the following technical assumptions on $V(\cdot)$.

Assumption 1: The storage function $V(\cdot)$ is smooth and $V(f(x) + Bu)$ is quadratic in u.

Assumption 1 is common in the passivity literature, e.g., [14], [26], [27]. The function $V(\cdot)$ being smooth implies that all finite-order

TABLE I
COMPARISON OF DETERMINISTIC AND STOCHASTIC QUASI PASSIVITY

	(quasi) Passivity	State-strict (quasi) passivity
Deterministic system with $\|\|e(j)\|\| \equiv 0$	Lyapunov stable	Asymptotically stable
System (7) with $\sum_{j=0}^{\infty} \mathbb{E}[\|\|e(j)\|\|^2] < \infty$	Mean square stable	Mean square asymptotically stable
System (7) with $\sup_j \mathbb{E}[\|\|e(j)\|\|^2] < \infty$	Possibly unstable	Mean square stable

derivatives of $V(\cdot)$ are locally Lipschitz, which means that there exist $L_{\dot{V}}, L_{\ddot{V}} \in \mathbb{R}_{>0}$ such that $\forall\, x_1, x_2 \in \mathbb{X}$

$$\left\| \left.\frac{\partial V(z)}{\partial z}\right|_{z=x_1} - \left.\frac{\partial V(z)}{\partial z}\right|_{z=x_2} \right\| \le L_{\dot{V}} \|x_1 - x_2\|$$

$$\left\| \left.\frac{\partial^2 V(z)}{\partial z^2}\right|_{z=x_1} - \left.\frac{\partial^2 V(z)}{\partial z^2}\right|_{z=x_2} \right\| \le L_{\ddot{V}} \|x_1 - x_2\|. \tag{10}$$

Assumption 2: The storage function $V(\cdot)$ is locally Lipschitz of order 2, i.e., there exists $L_V, \check{L}_V \in \mathbb{R}_{\ge 0}$ such that $\forall\, x_1, x_2 \in \mathbb{X}$, $\check{L}_V \|x_1 - x_2\|^2 \le \|V(x_1) - V(x_2)\| \le L_V \|x_1 - x_2\|^2$.

There exists close relation between stochastic quasi passivity defined in this technical note and mean square stability, which is analogous to the relation between classical passivity and Lyapunov stability. This relation is summarized in Table I, where the results in the first row are standard and the other entries can be proved in a similar fashion as in [18]. In particular, the second row shows that if the estimation error of the state is bounded in l^2 norm, then stochastic quasi passivity implies mean square stability and state-strict stochastic quasi passivity implies mean square asymptotic stability. Furthermore, like classical passivity, the stochastic quasi passivity is invariant under both parallel and negative feedback interconnections.

Our objective is to find a controller in the form of (6) such that the closed-loop mapping from v to y in (7) is stochastically quasi passive.

III. FEEDBACK INPUT STRICT PASSIVITY GUARANTEES STOCHASTIC QUASI PASSIVITY

In this section, we derive the main result of this technical note. We first provide a necessary and sufficient condition for feedback input-strict passivity using exact state information that will be used in the proof of the main result.

Lemma 1: (A Necessary and Sufficient Condition for Feedback Input-Strict Passivity): Let $\hat{x}(k) = x(k)$, $\forall\, k \in \mathbb{Z}_{\ge 0}$. The closed-loop system (7) is input-strictly passive with index μ if and only if there exist a storage function $V(\cdot)$ and functions $l(\cdot) : \mathbb{X} \to \mathbb{R}^q$, $m(\cdot) : \mathbb{X} \to \mathbb{R}^{q \times q}$ such that

$$\begin{aligned} &V\left(f_\alpha\left(x(k)\right)\right) - V\left(x(k)\right) = -l^T l \\ &\left.\frac{\partial V(z)}{\partial z}\right|_{z=f_\alpha(x(k))} BF = h_\alpha^T - 2l^T m \\ &\frac{1}{2} F^T B^T \left.\frac{\partial^2 V(z)}{\partial z^2}\right|_{z=f_\alpha(x(k))} BF \\ &\quad = \frac{1}{2}(DF + F^T D^T) - \mu I_q - m^T m. \end{aligned} \tag{11}$$

Proof: (necessity) By the definition of input-strict passivity in (4), there exists a storage function $V(\cdot)$ such that $\forall\, k \in \mathbb{Z}_{\ge 0}$ and $\forall\,(x(k), v(k)) \in \mathbb{X} \times \mathbb{V}$

$$\begin{aligned} \Phi(x,u,k) &\triangleq V\left(x(k+1)\right) - V\left(x(k)\right) - y^T(k) v(k) + \mu \left\|v(k)\right\|^2 \\ &\overset{(a)}{=} V\left(f_\alpha\left(x(k)\right)\right) - V\left(x(k)\right) \\ &\quad + \left(\left.\frac{\partial V(z)}{\partial z}\right|_{z=f_\alpha(x(k))} BF - h_\alpha^T\left(x(k)\right) \right) v(k) \\ &\quad + \frac{1}{2} v^T(k) \left(F^T B^T \left.\frac{\partial^2 V(z)}{\partial z^2}\right|_{z=f_\alpha(x(k))} BF \right. \\ &\qquad \left. + 2\mu I_q - (DF + F^T D^T) \right) v(k) \le 0 \end{aligned} \tag{12}$$

where (a) in (12) holds by the Taylor series and Assumption 1. Moreover, since $\Phi(x,u,k)$ is quadratic in $v(k)$, there exist functions $l(\cdot) : \mathbb{X} \to \mathbb{R}^q$, $m(\cdot) : \mathbb{X} \to \mathbb{R}^{q\times q}$ such that

$$\Phi(x,u,k) = -\left(l + mv(k)\right)^T \left(l + mv(k)\right). \tag{14}$$

We obtain (11) by comparing (13) and (12).

(sufficiency) The proof for sufficiency is immediate by using the definition of input-strict passivity in (4). ■

It is worth noticing that the functions $l(\cdot)$ and $m(\cdot)$ depend on the coefficients in the Taylor series and thus Lemma 1 only holds locally. Lemma 1 is analogous to some results in the literature like sufficient and necessary testing conditions for (state-strict) passivity of continuous nonlinear systems in [5] and [28]. We are now ready to state our main result.

Theorem 1: (Bounded State Estimation Error Allows Certainty Equivalent Controllers): Consider the nonlinear plant (5). Assume that using the exact state, there exists a controller such that the closed-loop is input-strictly passive with index μ. If the communication channel is such that

$$\sup_{i \in \mathbb{Z}_{\ge 0}} \mathbb{E}\left[\|\hat{x}(i) - x(i)\|\right] < \frac{2\mu}{L_{\ddot{V}} \|B\| L_\alpha \bar{\lambda}(F^T B^T B F)} \tag{14}$$

then the same controller renders the closed-loop system stochastically quasi passive using the estimate of state, $\hat{x}(k)$.

Proof: It holds that

$$\begin{aligned} &V\left(f_\alpha\left(x(k)\right) + B\eta(k) + BFv(k)\right) \\ &\quad = V\left(f_\alpha\left(x(k)\right) + B\eta(k)\right) \\ &\qquad + \left.\frac{\partial V(z)}{\partial z}\right|_{z=f_\alpha(x(k))+B\eta(k)} BFv(k) \\ &\qquad + \frac{1}{2} v^T(k) F^T B^T \left.\frac{\partial^2 V(z)}{\partial z^2}\right|_{z=f_\alpha(x(k))+B\eta(k)} BFv(k) \\ &\quad \overset{(a)}{\le} V\left(f_\alpha\left(x(k)\right)\right) + L_V \|B\|^2 L_\alpha^2 \left\|e(k)\right\|^2 \\ &\qquad + \left.\frac{\partial V(z)}{\partial z}\right|_{z=f_\alpha(x(k))} BFv(k) \\ &\qquad + L_{\dot{V}} \|B\| L_\alpha \left\|e(k)\right\| \cdot \|BF\| \cdot \|v(k)\| \\ &\qquad + \frac{1}{2} v^T(k) F^T B^T \\ &\qquad \times \left(\left.\frac{\partial^2 V(z)}{\partial z^2}\right|_{z=f_\alpha(x(k))} + L_{\ddot{V}} \|B\| L_\alpha \left\|e(k)\right\| \right) BFv(k) \end{aligned} \tag{15}$$

where (a) in (15) follows from Assumptions 1 and 2. Let

$$b_\alpha = \frac{\left(L_{\dot V}\|B\|L_\alpha\|BF\| + L_\alpha\|D\|\right)^2}{4\mu - 2L_{\ddot V}\|B\|L_\alpha\bar\lambda(F^TB^TBF)\sup_{i\in\mathbb{Z}_{\geq 0}}\mathbb{E}\left[\|e(i)\|\right]} + L_V\|B\|^2L_\alpha^2. \quad (16)$$

Notice that $b_\alpha > 0$ because of (14). It then follows that:

$$\begin{aligned}
&\mathbb{E}\left[V(x(k+1)) - V(x(k)) - y^T(k)v(k) - b_\alpha\|e(k)\|^2\right]\\
&\overset{(a)}{\leq} \mathbb{E}\Bigg[V(f_\alpha(x(k))) - V(x(k))\\
&\quad + \left(\left.\frac{\partial V(z)}{\partial z}\right|_{z=f_\alpha(x(k))} BF - h_\alpha^T(x(k))\right)v(k)\\
&\quad + \frac{1}{2}v^T(k)\left(F^TB^T\left.\frac{\partial^2 V(z)}{\partial z^2}\right|_{z=f_\alpha(x(k))} BF\right.\\
&\quad\quad \left. - (DF + F^TD^T)\right)v(k)\Bigg]\\
&\quad + \left(L_V\|B\|^2L_\alpha^2 - b_\alpha\right)\mathbb{E}\left[\|e(k)\|^2\right]\\
&\quad + \left(L_{\dot V}\|B\|L_\alpha\|BF\| + L_\alpha\|D\|\right)\mathbb{E}\left[\|e(k)\|\right]\cdot\|v(k)\|\\
&\quad + \frac{1}{2}L_{\ddot V}\|B\|L_\alpha\bar\lambda(F^TB^TBF)\mathbb{E}\left[\|e(k)\|\right]\cdot\|v(k)\|^2\\
&\overset{(b)}{\leq} \mathbb{E}\left[-(l + mv(k))^T(l + mv(k))\right]\\
&\quad + \left(L_V\|B\|^2L_\alpha^2 - b_\alpha\right)\left(\mathbb{E}\left[\|e(k)\|\right]\right)^2\\
&\quad + \left(L_{\dot V}\|B\|L_\alpha\|BF\| + L_\alpha\|D\|\right)\mathbb{E}\left[\|e(k)\|\right]\cdot\|v(k)\|\\
&\quad + \left(\frac{1}{2}L_{\ddot V}\|B\|L_\alpha\bar\lambda(F^TB^TBF)\mathbb{E}\left[\|e(k)\|\right] - \mu\right)\|v(k)\|^2
\end{aligned}$$

where (a) follows from (15), (b) follows from Lemma 1, and the RHS of (b) is negative due to the choice of b_α in (16). ■

Theorem 2: If the nonlinear plant (5) is state-strictly feedback quasi passive and the communication channel guarantees that

$$\sup_{i\in\mathbb{Z}_{\geq 0}}\mathbb{E}\left[\|\hat x(i) - x(i)\|\right] < \frac{2\mu}{L_{\ddot V}\|B\|L_\alpha\bar\lambda(F^TB^TBF)}$$

then there exists a controller such that the closed-loop system (7) is stochastically state-strictly quasi passive.

Proof: The proof is similar to the proof of Theorem 1 by using the supply rate $y^T(k)u(k) - c\|x(k)\|^2 - \mu\|v(k)\|^2$. ■

IV. Feedback Passivation of Linear Plants

In this section, we show that stronger results than Theorems 1 and 2 hold for linear plants. Specifically, the existence of the index μ is not necessary for a certainty equivalent controller to ensure stochastic quasi passivity. In the sequel, we specialize (5) to consider a linear time-invariant plant of the form

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k)\\ y(k) &= Cx(k) + Du(k). \end{aligned} \quad (17)$$

We assume that (A, B, C, D) is minimal, B is full rank and D is nonsingular. It has been shown in [13] that the matrix D being nonsingular is necessary for the existence of a feedback passivating controller even if the controller knows the exact state of the plant.

With the static controller $u(k) = K\hat x(k) + Fv(k)$, the closed-loop system is given by

$$\begin{aligned} x(k+1) &= \bar Ax(k) + \bar Bv(k) + BKe(k)\\ y(k) &= \bar Cx(k) + \bar Dv(k) + DKe(k) \end{aligned} \quad (18)$$

where $(\bar A, \bar B, \bar C, \bar D) \triangleq (A + BK, BF, C + DK, DF)$.

We now show that if the communication channel in the feedback loop ensures $\sup_{i\in\mathbb{Z}_{\geq 0}}\mathbb{E}[\|e(i)\|^2] < \infty$, then certainty equivalent controllers can be used to achieve stochastic quasi passivity, even if input-strict passivity is lacking. Moreover, we obtain a feedback passivating controller by solving a set of linear matrix inequalities (LMIs).

Lemma 2: For any scalar $\alpha > 0$ and matrices A, B with appropriate dimensions

$$A^TB + B^TA \leq \alpha A^TA + \alpha^{-1}B^TB. \quad (19)$$

Proof: The proof follows from the inequality $(\alpha^{1/2}A - \alpha^{-(1/2)}B)^T(\alpha^{1/2}A - \alpha^{-(1/2)}B) \geq 0$.

Theorem 3: (Design for Feedback Stochastically Passivating Controllers): Consider the LTI plant (17) and assume that $\sup_{i\in\mathbb{Z}_{\geq 0}}\mathbb{E}[\|e(i)\|^2] < \infty$. If there exist matrices $\bar P > 0 \in \mathbb{R}^{n\times n}$, $\bar K \in \mathbb{R}^{q\times n}$ and $F \in \mathbb{R}^{q\times q}$ such that

$$\Pi_{11} = \begin{bmatrix} \bar P & * & *\\ C\bar P + D\bar K & F^TD^T + DF & *\\ A\bar P + B\bar K & BF & \bar P \end{bmatrix} > 0 \quad (20)$$

then the feedback controller

$$u(k) = \bar K\bar P^{-1}\hat x(k) + Fv(k) \quad (21)$$

renders the closed-loop system (18) stochastically quasi passive. Moreover, if it further holds that

$$\begin{bmatrix} \Pi_{11} & *\\ \Pi_{21} & 0.5c^{-1}I_n \end{bmatrix} > 0 \quad (22)$$

where $\Pi_{21} = [\bar P\ \mathbf{0}_{n\times(q+n)}]$, then the closed-loop is stochastically state-strictly quasi passive with c in (9).

Proof: By a congruence transformation with matrix $\text{diag}\{\bar P^{-1}, I_q, I_n\}$ and using Schur complement, (20) is equivalent to

$$\Phi \triangleq \begin{bmatrix} P - \bar A^TP\bar A & *\\ \bar C - \bar B^TP\bar A & \bar D^T + \bar D - \bar B^TP\bar B \end{bmatrix} > 0 \quad (23)$$

where $P = \bar P^{-1}$, $K = \bar K\bar P^{-1}$ and $(\bar A, \bar B, \bar C, \bar D)$ have been defined after (18). Define $V(x(k)) = (1/2)x^T(k)Px(k)$ and two scalars

$$\begin{aligned} \alpha_p &= \bar\lambda\left(\Phi^{-1}\text{diag}\{\bar A^TP\bar A, \bar B^TP\bar B + DD^T\}\right) > 0 \quad (24)\\ b &= 2^{-1}\bar\lambda\left(K^T\left((1 + 2\alpha_p)B^TPB + \alpha_pI_q\right)K\right) \geq 0. \quad (25) \end{aligned}$$

For all $k \in \mathbb{Z}_{\geq 0}$ and all $v(k) \in \mathbb{R}^q$, we obtain

$$\mathbb{E}\left[y^T(k)v(k) + b\|e(k)\|^2 + V(x(k)) - V(x(k+1))\right] \overset{(a)}{\geq} 2^{-1}\mathbb{E}\left[\varphi^T(k)\Psi\varphi(k)\right] \overset{(b)}{\geq} 0 \quad (26)$$

where $\varphi(k) = [x^T(k)\ v^T(k)\ e^T(k)]^T$

$$\Psi = \text{diag}\left\{\Phi - \alpha_p^{-1}\text{diag}\{\bar A^TP\bar A, \bar B^TP\bar B + DD^T\},\ 2bI_n - K^T\left((1 + 2\alpha_p)B^TPB + \alpha_pI_q\right)K\right\}$$

and (a) in (26) follows by Lemma 2, (b) in (26) follows by (24) and (25). Thus, the closed-loop system is stochastically quasi passive by definition (8).

Moreover, if the LMI (22) also holds, then by a congruence transformation with matrix $\mathrm{diag}\{\bar{P}^{-1}, I_n, I_q, I_n\}$ and the Schur complement lemma, we have

$$\Phi_c \triangleq \begin{bmatrix} P - \bar{A}^T P \bar{A} - 2c & * \\ \bar{C} - \bar{B}^T P \bar{A} & \bar{D}^T + \bar{D} - \bar{B}^T P \bar{B} \end{bmatrix} > 0. \quad (27)$$

Define

$$\alpha_c = \bar{\lambda}\left(\Phi_c^{-1} \mathrm{diag}\{\bar{A}^T P \bar{A}, \bar{B}^T P \bar{B} + DD^T\}\right) > 0 \quad (28)$$

$$b_c = 2^{-1}\bar{\lambda}\left(K^T\left((1+2\alpha_c)B^T P B + \alpha_c I_q\right)K\right) \geq 0. \quad (29)$$

For all $k \in \mathbb{Z}_{\geq 0}$ and all $v(k) \in \mathbb{R}^q$, we obtain

$$\mathbb{E}\left[y^T(k)v(k) - c\left\|x(k)\right\|^2 + b_c\left\|e(k)\right\|^2 + V(x(k)) - V(x(k+1))\right] \overset{(a)}{\geq} 2^{-1}\mathbb{E}\left[\varphi^T(k)\Psi_c\varphi(k)\right] \overset{(b)}{\geq} 0 \quad (30)$$

where

$$\Psi_c = \mathrm{diag}\left\{\Phi_c - \alpha_c^{-1}\mathrm{diag}\{\bar{A}^T P \bar{A}, \bar{B}^T P \bar{B} + DD^T\},\ 2b_c I_n - K^T\left((1+2\alpha_c)B^T P B + \alpha_c I_q\right)K\right\}$$

and (a) in (30) holds because of Lemma 2, (b) in (30) follows by (28) and (29). Thus, the closed-loop system is stochastically state-strictly quasi passive by definition (9). ■

Since the condition $\sup_{i\in\mathbb{Z}_{\geq 0}} \mathbb{E}[\|e(i)\|^2] < \infty$ is also necessary for feedback stabilization in the mean squared sense, Theorem 3 shows that feedback passivation does not require any higher quality of service from the communication channel than stabilization.

Theorem 4: (Sufficient Condition for the Existence of a Feedback Passivating Controller): If the matrix $\hat{A} \triangleq A - BD^{-1}C$ is Schur stable, then the feedback controller

$$u(k) = -\left(D^{-1}C + 2\delta(B^T PB)^{-1}B^T P\hat{A}\right)x(k) + 2\delta(B^T PB)^{-1}D^T v(k), \quad 0 < \delta < 1 \quad (31)$$

makes the closed-loop system (18) stochastically quasi passive.

Proof: The proof is completed by direct substitution of (31) into the system dynamics (17). ■

V. A Numerical Example

In this section, a numerical example is provided to illustrate the feedback passivating controller design approach proposed in Theorem 3. Consider the scalar plant

$$x(k+1) = \frac{4}{3}x(k) + \frac{3}{8}u(k)$$

$$y(k) = x(k) + \frac{1}{2}u(k)$$

which is obviously nonpassive since it is unstable. Let $c = 0.5$ in the stochastic state-strict passivity inequality (9). By solving the LMI (22), we obtain the feedback controller $u(k) = -2.56\hat{x}(k) + 0.7912v(k)$. Moreover, we compute using (29) that b in (9) equals 16.21.

First consider the case when $e(k) = \mathbf{0}_q$, $\forall k \in \mathbb{Z}_{\geq 0}$. We compare the increase in the storage function with the supply rate $y^T(k)v(k) - c\|x(k)\|^2$ in Fig. 2. We see that the increase in storage is always less than or equal to the supply rate, which means that the designed controller guarantees the state-strict quasi passivity of the closed-loop system.

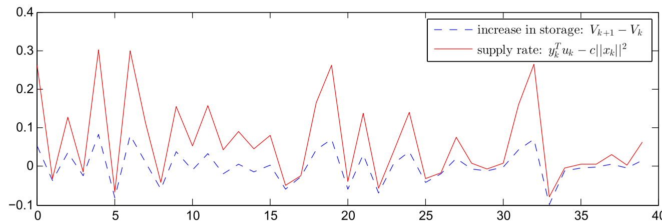

Fig. 2. Comparison between the increase of the storage function and the supply rate when the exact state is available at the controller (X axis: time; Y axis: storage increase versus supply rate).

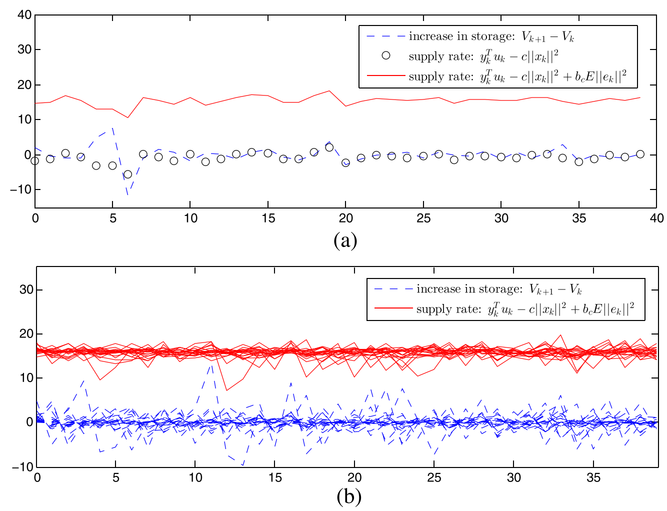

Fig. 3. Comparison between the increase of the storage function and the supply rate when only an estimate of the state is available at the controller (X axis: time; Y axis: storage increase vs supply rate).

Then, assume that the estimation error process $e = \{e(1), e(2), \ldots\}$ is zero-mean white Gaussian with unit covariance and the controller remains the same. This happens when the feedback channel $\mathcal{C}$ is AWGN and the encoder and decoder mappings are simply $m(k) = x(k)$, $n(k) = m(k) + e(k)$, $\hat{x}(k) = n(k)$.

In Fig. 3(a), the increase in the storage function is compared with two supply rates: the supply rate in (3) defining standard state-strict quasi passivity and the one proposed in (9) for stochastic state-strict quasi passivity. It shows that the standard supply rate $y^T(k)v(k) - c\|x(k)\|^2$ is not an upper bound of the storage increase while the new supply rate $y^T(k)v(k) - c\|x(k)\|^2 + b\mathbb{E}[\|e(k)\|^2]$ is. In Fig. 3(b), simulation results for 20 repeated tests are plotted.

VI. Conclusion

In this technical note, we considered the feedback passivation problem for discrete-time systems when the state of the plant is transmitted to the controller across a communication channel. By generalizing the standard passivity definition, a stochastic quasi passivity concept was introduced. We showed that any state feedback controller that ensures closed-loop input-strict passivity with index μ using the exact state of the nonlinear plant can guarantee stochastic quasi passivity using an estimate of the state, provided that the infinity norm of the estimation error is bounded by some function of μ and the system dynamics matrices. For linear plants, we obtained stronger results and the input-strict passivity is not necessary. The effectiveness of the controller design approach was illustrated through an example.

This work can be extended in several directions. More general dissipativity concepts such as QSR dissipativity can be considered, which include passivity as a special case. Further, generalization of the results to more general classes of nonlinear systems is also of interest.

REFERENCES

[1] H. Gao, T. Chen, and T. Chai, "Passivity and passification for networked control systems," *SIAM J. Control Optimiz.*, vol. 46, no. 4, pp. 1299–1322, 2007.

[2] N. Kottenstette and P. J. Antsaklis, "Wireless digital control of continuous passive plants over token ring networks," *Int. J. Robust Nonlin. Control*, vol. 19, no. 18, pp. 2016–2039, 2009.

[3] J. Zhao and D. Hill, "Dissipativity theory for switched systems," *IEEE Trans. Autom. Control*, vol. 53, no. 4, pp. 941–953, Apr. 2008.

[4] M. J. McCourt and P. J. Antsaklis, "Control design for switched systems using passivity indices," in *Proc. Amer. Control Conf.*, 2010, pp. 2499–2504.

[5] D. Hill and P. Moylan, "The stability of nonlinear dissipative systems," *IEEE Trans. Autom. Control*, vol. AC-21, no. 5, pp. 708–711, 1976.

[6] D. Hill and P. Moylan, "Connections between finite-gain and asymptotic stability," *IEEE Trans. Autom. Control*, vol. AC-25, no. 5, pp. 931–936, 1980.

[7] C. I. Byrnes, A. Isidori, and J. C. Willems, "Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems," *IEEE Trans. Autom. Control*, vol. 36, no. 11, pp. 1228–1240, Nov. 1991.

[8] M. Vidyasagar, "New passivity-type criteria for large-scale interconnected systems," *IEEE Trans. Autom. Control*, vol. 24, no. 4, pp. 575–579, Apr. 1979.

[9] J. C. Willems, "Dissipative dynamical systems part I: General theory," *Arch. Ration. Mechan. Anal.*, vol. 45, no. 5, pp. 321–351, 1972.

[10] B. Brogliato, O. Egeland, R. Lozano, and B. Maschke, *Dissipative Systems Analysis and Control: Theory and Applications*. London, U.K.: Springer London, 2007.

[11] J. Bao and P. L. Lee, *Process Control: The Passive Systems Approach*. New York: Springer, 2007.

[12] W. Lin, "Feedback stabilization of general nonlinear control systems: A passive system approach," *Syst. Control Lett.*, vol. 25, no. 1, pp. 41–52, 1995.

[13] E. M. Navarro-López and E. Fossas-Colet, "Feedback passivity of nonlinear discrete-time systems with direct input-output link," *Automatica*, vol. 40, no. 8, pp. 1423–1428, 2004.

[14] C. I. Byrnes and W. Lin, "Losslessness, feedback equivalence, and the global stabilization of discrete-time nonlinear systems," *IEEE Trans. Autom. Control*, vol. 39, no. 1, pp. 83–98, Jan. 1994.

[15] P. Florchinger, "A passive system approach to feedback stabilization of nonlinear control stochastic systems," *SIAM J. Control Optimiz.*, vol. 37, no. 6, pp. 1848–1864, 1999.

[16] M. Aliyu, "Dissipative analysis and stability of nonlinear stochastic state-delayed systems," *Nonlin. Dynam. Syst. Theory*, vol. 4, no. 3, pp. 243–256, 2004.

[17] Z. Lin, J. Liu, W. Zhang, and Y. Niu, "Stabilization of interconnected nonlinear stochastic Markovian jump systems via dissipativity approach," *Automatica*, vol. 47, no. 12, pp. 2796–2800, 2011.

[18] Y. Zhao and V. Gupta, "Feedback stabilization of Bernoulli jump nonlinear systems: A passivity-based approach," vol. 60, no. 8, pp. 2254–2259, 2015.

[19] I. Yaesh and U. Shaked, "Stochastic passivity and its application in adaptive control," *IEEE Trans. Autom. Control*, vol. 54, no. 1, pp. 136–142, Jan. 2009.

[20] Y. Wang, V. Gupta, and P. J. Antsaklis, "Stochastic passivity of discrete-time Markovian jump nonlinear systems," in *Proc. Amer. Control Conf.*, 2013.

[21] N. Berman and U. Shaked, "H_∞ control for discrete-time nonlinear stochastic systems," *IEEE Trans. Autom. Control*, vol. 51, no. 6, pp. 1041–1046, Jun. 2006.

[22] S. Tatikonda and S. Mitter, "Control under communication constraints," *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1056–1068, Jul. 2004.

[23] D. Liberzon and J. Hespanha, "Stabilization of nonlinear systems with limited information feedback," *IEEE Trans. Autom. Control*, vol. 50, no. 6, pp. 910–915, Jun. 2005.

[24] G. Nair, F. Fagnani, S. Zampieri, and R. Evans, "Feedback control under data rate constraints: An overview," *Proc. IEEE*, vol. 95, no. 1, pp. 108–137, Jan. 2007.

[25] P. Minero, M. Franceschetti, S. Dey, and G. Nair, "Data rate theorem for stabilization over time-varying feedback channels," *IEEE Trans. Autom. Control*, vol. 54, no. 2, pp. 243, Feb. 2009.

[26] E. M. Navarro-López, H. Sira-Ramirez, and E. Fossas-Colet, "Dissipativity, passivity and feedback passivity in the nonlinear discrete-time setting," in *Proc. 15th IFAC World Congr.*, Barcelona, Spain, 2002.

[27] I. Mizumoto, S. Ohdaira, and Z. Iwai, "Output feedback strict passivity of discrete-time nonlinear systems and adaptive control system design with a pfc," *Automatica*, vol. 46, no. 9, pp. 1503–1509, 2010.

[28] Z. Jiang and D. J. Hill, "Passivity and disturbance attenuation via output feedback for uncertain nonlinear systems," *IEEE Trans. Autom. Control*, vol. 43, no. 7, pp. 992–997, Jul. 1998.