On Disturbance Propagation in Leader–Follower Systems with Limited Leader Information

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Abstract

This paper studies the problem of disturbance propagation in a string of vehicles aiming to proceed along a given trajectory while keeping a constant distance between each vehicle and its successor. It is assumed that each vehicle can control its position based on the spacing error with respect to the preceding vehicle in the string, as well as on coded information transmitted by the lead vehicle. Using information-theoretic techniques, this paper establishes a lower bound to the integral of the sensitivity function of spacing errors with respect to a stochastic disturbance acting on the lead vehicle. The derived bound depends on the open-loop poles and zeros of the vehicles’ dynamics as well as on the (possibly non-linear) controller used at each vehicle. The lower bound is shown to be tight for a specific class of systems and controllers.

Key words: Bode integral formula; Distributed control; Disturbance rejection; Information theory.

1 Introduction

There has recently been extensive interest in formation control problems, where the objective is to arrange a large number of autonomous agents in a predefined geometric structure while ensuring that the agents as a group achieve a predefined task [2, 11, 16, 18]. The simplest example is controlling a one dimensional string of vehicles that has to move along a given trajectory while keeping constant spacing between any two consecutive vehicles. This problem arises, for instance, in the design of automated highway systems [8, 9].

It is known that the control performance when strings of vehicles are being controlled in a distributed fashion depends on the amount of information available to each vehicle. In [18], it is shown that a predecessor–following control strategy, in which each vehicle has access to only its relative position with respect to the preceding vehicle, can cause error amplification along the string, eventually leading to string instability. This undesirable effect can be prevented under the predecessor–leader control strategy [20, 23, 29], wherein each vehicle is additionally given access to position information about the lead vehicle.

In this paper, we consider the formation control problem of a string of vehicles under the assumption that each vehicle has access to (i) its spacing error relative to the preceding vehicle, and (ii) limited information about the string leader, which is communicated from the leader to the other vehicles over capacity-limited side communication channels (see Fig. 1). The classical predecessor–following control strategy is recovered in the special case where the side channels have capacity zero, since no information can then be transmitted from the leader to the followers through such channels. On the other hand, the predecessor–leader control strategy, in which the followers have perfect and instantaneous leader position information, can be recovered by assuming that the side channels have infinite capacity. Given this setup, we study the sensitivity of the spacing errors along the string with respect to a stochastic disturbance acting on the lead vehicle and how this sensitivity changes as a function of the capacities of the side channels.

To deal with the stochasticity of the disturbance process and the presence of communication channels, we follow the information-theoretic approach of Martins et al. [14]. One of the most appealing features of this information-theoretic approach is that it permits the consideration of nonlinear controllers, unlike the traditional approach.
based on transfer functions which only applies to linear time invariant controllers. An additional advantage is that it allows us to directly relate the sensitivity to the Shannon capacities of the communication channels.

The main contribution of this paper is a lower bound on the integral of the sensitivity function of the spacing error at each vehicle with respect to the stochastic disturbance acting on the lead vehicle (see Theorem 1). This bound holds for a class of nonlinear controllers that includes, in particular, all linear time invariant (LTI) controllers. A specific class of systems and controllers for which the lower bound is tight is provided (see Proposition 1). When there is only one vehicle in the string, our bound recovers the disturbance propagation result in [14]. Assuming instead that all side channels have zero capacity, our result recovers a previous bound derived in [30] under the predecessor-following control strategy. Unlike the bounds presented in [25, 26], our results do not assume specific plant dynamics.

We remark that the performance metric used in this paper is complementary to the $H_{\infty}$-type metric considered in [1, 16, 18], which characterizes the worst case performance in the sense of measuring the peak magnitude of the frequency response of the transfer function from the disturbance to the spacing error. The metric under consideration, instead, reflects the average disturbance rejection performance over all frequencies. Similarly to Bode’s integral formula for single-input single-output (SISO) systems, our metric shows that there is a fundamental limit on how the error sensitivity can be shaped in frequency.

Next, we wish to mention a few additional related works in the literature. Barooah et al. [1] studied the problem of string stability when each vehicle has access to the spacing error with respect to the predecessor as well as the follower in the string, while [16] considered the case where each vehicle has access to the position of vehicles within a given communication range. Hao et al. [7] considered a general grid–like network structure and investigated the network stability as the number of nodes in the network grows as a function of the amount of information available to each node in the grid. Unlike any of these existing works, this paper considers the case where information is transmitted over finite capacity channels. We would also like to mention the work in [12] that considered the design of feedback gains to minimize in $H_2$ sense the influence of the stochastic disturbance on the performance output for a vehicle string. However, the dynamics and controller structure were constrained to obtain a tractable problem. Performance limitations similar to Bode’s integral formula have been derived for more general control settings [3, 19], including multidimensional [6], nonlinear [13, 21, 28] and time-varying systems [10].

This work generalizes Bode’s formula to a specific distributed control problem. As the literature on networked control systems demonstrates, even centralized control when information available to the controller is transmitted across communication channels displays a rich and non-obvious behavior. Although there has been some recent work on stabilizability of distributed systems (e.g., [15, 27]), performance guarantees in distributed systems remain hard to obtain.

The rest of the paper is organized as follows. The problem formulation is stated in Section 2. Section 3 is devoted to the main result of the paper, whose proof is provided in Section 4. Section 5 focuses on the case of linear controllers and provides some examples where the derived lower bound is tight. Section 6 concludes the paper.

Throughout the paper, we denote random variables using boldface letters. For any $k \leq j$ we use the notation $x_k = (x(k), x(k+1), \ldots, x(j))$ to denote a finite segment of a sequence $x(0), x(1), \ldots$ and we omit the subscript $k$ when it is equal to 0.

2 Problem formulation

Consider a string of $n+1$ coupled SISO systems as shown in Fig. 1. The dynamics of the $i$-th plant, $i = 0, \ldots, n$, is given by

$$
x_i(k) = A_i x_i(k-1) + B_i u_i(k-1),
$$

$$
y_i(k) = C_i x_i(k),
$$

where $x_i(k) \in \mathbb{R}^{n_i}$ is the system state, $y_i(k) \in \mathbb{R}$ is the system output, $u_i(k) \in \mathbb{R}$ is the control input, $(A_i, B_i)$ is controllable and $(A_i, C_i)$ is observable. The initial conditions $x_i(0)$’s are assumed to be continuous random variables. Vehicle 0 (also called the leader or the head vehicle) aims at tracking a reference command signal $r(k)$, while the objective of vehicles 1, 2, ..., $n$ (referred to as the followers) is to maintain a constant distance from their predecessor. Specifically, given a constant spacing $\delta$, the $i$-th vehicle ($i \geq 1$) regulates its output $y_i(k)$ to satisfy $y_i(k) - y_{i-1}(k) = \delta$ at every time step $k$. We define the spacing errors between successive vehicles as

$$
e_i(k) = y_{i-1}(k) - y_i(k) + \delta, \quad i = 1, \ldots, n.
$$

We assume that every vehicle with index $i \geq 2$ has access to the tracking error $e_i$ and, possibly, to information transmitted from the leader over a side communication channel. Formally, the side channel that carries information to the $i$-th vehicle consists of an input set $X_i$, an output set $Y_i$, and a family of transition probability mass functions $p(y_i|\bar{x}_i)$, one for each
\[ \dot{x}_i \in \hat{X}_i. \] We assume that the channel is memoryless \cite{5} and has Shannon capacity equal to \( C_i \) \cite{4}.

At time \( k \), encoder \( E_i \) maps \( y^k_i \) into a symbol \( \hat{x}_i(k) \in \hat{X}_i \) which is transmitted to the \( i \)-th vehicle over the \( i \)-th communication channel. The controller of the \( i \)-th vehicle uses the spacing error \( e^k_i \) and the channel output \( y^k_i \) to generate a control signal of the form

\[ u_i(k) = u_{i,k}(\hat{y}^k_i, e^k_i). \]  

(3)

Notice that the controller of the \( i \)-th vehicle is also the decoder for the communication channel across which the leader transmits information to the \( i \)-th vehicle. We assume that the control laws in (3) are such that the random processes describing the closed-loop dynamics have well defined continuous joint probability density functions and are asymptotically stationary processes. In addition, we make the following technical assumption.

**Assumption 1** For every \((\hat{y}^k_i, e^{k-1}_i) \in \mathbb{R}^{2k+1} \), the non-linear function \( u_{i,k}(\hat{y}^k_i, e^{k-1}_i, z) \) is a continuously differentiable function of the variable \( z \in \mathbb{R} \).

As we will see later, Assumption 1 is needed to relate the differential entropies of the random variables at the input and output of each plant.

In the sequel, for every \((\hat{y}^k_i, e^{k-1}_i) \in \mathbb{R}^{2k+1} \) we define

\[ u'_{i,k}(\hat{y}^k_i, e^{k}_i) \triangleq \frac{\partial}{\partial x} \left. u_{i,k}(\hat{y}^k_i, e^{k}_i, z) \right|_{z=e_i(k)}. \]  

(4)

We denote by \( v_i \geq 1 \) the relative degree of the \( i \)-th plant, i.e., the smallest integer such that

\[ D_i \triangleq C_i A_{i}^{v_i-1} B_i \neq 0, \]  

(5)

and by \( t_i \) the sum of the relative degrees up to plant \( i \), i.e.,

\[ t_i \triangleq v_0 + v_1 + \cdots + v_i. \]

By definition of relative degree, the control input \( u_i(k) \) does not affect the outputs \( y_i(k), \ldots, y_i(k + v_i - 1) \).

Thus, \( v_i \) can be interpreted as the input-output delay introduced by the \( i \)-th plant.

Suppose that the reference signal \( r(k) \) is perturbed by a stochastic disturbance process \( d \), as depicted in Fig. 1, such that the error signal at the input of the leader’s controller at time \( k \) is given by

\[ e_0(k) = r(k) + d(k) - y_0(k). \]  

(6)

Because of the coupling between consecutive vehicles, the disturbance acting on the leader propagates downstream and affects the state and hence the error signal, of all vehicles in the string.

The objective of this paper is to analyze the sensitivity of the spacing errors (2) and (6) with respect to the disturbance \( d \). The definition for the sensitivity function is borrowed from \cite{14}.

**Definition 1** A zero-mean stochastic process \( x \) is said to be asymptotically stationary if

\[ R_x(\tau) \triangleq \lim_{k \to \infty} \mathbb{E} \left[ x(k)x^T(k + \tau) \right] \]

exists for every integer \( \tau \). The power spectral density \( \Phi_x(\omega) \) of \( x \) is defined as the discrete-time Fourier transform of \( R_x(\tau) \).

The sensitivity function between two asymptotically stationary processes is defined as follows.

**Definition 2** The sensitivity function \( S_{x,y}(\omega) \) between two stationary stochastic processes \( x \) and \( y \) with power spectral densities \( \Phi_x(\omega) \) and \( \Phi_y(\omega) \), respectively, is defined as

\[ S_{x,y}(\omega) \triangleq \sqrt{\frac{\Phi_y(\omega)}{\Phi_x(\omega)}}. \]  

(7)

We refer the reader to \cite{14} for a discussion on the relationship between \( S_{x,y}(\omega) \) and the classical sensitivity function.
In this framework, we follow an information-theoretic approach and compute a lower bound on
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_1}(\omega) \, d\omega \]  
by evaluating the differential entropy rates of the spacing errors in the closed-loop systems, under the following assumptions.

**Assumption 2** The disturbance \( d \) is a zero-mean wide-sense stationary Gaussian process with power spectral density \( \Phi_d(\omega) \).

**Assumption 3** The initial conditions \( x_0(0), \ldots, x_n(0) \) (denoted as \( x_0^n(0) \)) form a Markov sequence of continuous random variables with finite differential entropy independent of the disturbance \( d \).

A consequence of Assumption 3 is that the conditional differential entropy \( h(x_i(0)|\hat{y}_i) \) is finite because \( \hat{y}_i \) is a function of \( x_i(0) \) and \( d \). Since \( \bar{I}(\hat{y}_i; x_i(0)) = h(x_i(0)) - \bar{h}(x_i(0)|\hat{y}_i) \), it follows that
\[ \bar{I}(\hat{y}_i; x_i(0)) \triangleq \lim_{k \to \infty} \frac{1}{k} \bar{I}(\hat{y}_{i-1}; x_i(0)) = 0. \]  

**Assumption 4** The controllers have been designed to ensure that all the closed-loop systems are mean-square stable, i.e., \( \sup_k \mathbb{E} \left[ x_i^T(k) x_i(k) \right] < \infty \), for all \( i = 0, \ldots, n \).

### 3 Main result

The main result of this brief paper is the following.

**Theorem 1** Let Assumptions 1-4 hold. Then, for every \( i = 0, \ldots, n \),
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_1}(\omega) \, d\omega \geq \sum_{\lambda \in \mathcal{P}_i} \log |\lambda| - \mathcal{C}_i + \sum_{l=0}^{i-1} \log |D_l| \]
\[ + U_i + \sum_{\beta \in \mathcal{Z}_i} \log |\beta| - \mathcal{C}_i \]
\[ + U_i + \sum_{\beta \in \mathcal{Z}_i} \log |\beta| - \mathcal{C}_i \],
\[ \text{(10)} \]
where \( \mathcal{P}_i \) and \( \mathcal{Z}_i \) denote the set of unstable open-loop poles and zeros of the \( i \)-th system, respectively, \( D_l \) is defined in (5), and
\[ U_i \triangleq \lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} \mathbb{E}(\log |u'_{i,j}(e_i)| | \hat{y}_{i-1}, x_i(0), \hat{y}_{i+1}). \]  
\[ \text{(11)} \]

\( A \) few remarks about the result are in order.

1. Similar to the Bode integral formula for SISO systems, the right hand side (RHS) of (10) depends on the \( i \)-th plant’s unstable open-loop poles but is independent of the \( i \)-th controller \( U_i \). Therefore, (10) characterizes a fundamental limitation that holds for all control laws that stabilize the \( i \)-th vehicle in the mean-square sense.

2. In the special case where \( i = 0 \) (centralized control case), Theorem 1 recovers the result in [13, Equation (26)]. Moreover, for \( i = 2 \), the RHS of (10) becomes
\[ \sum_{l=0}^{1} \left( \log |D_l| + U_l + \sum_{\beta \in \mathcal{Z}_i} \log |\beta| \right) + \sum_{\lambda \in \mathcal{P}_2} \log |\lambda| - \mathcal{C}_2, \]
which is a generalization of [14, Theorem 4.5] to the string case.

3. In the special case where the control laws in (3) are chosen as
\[ u_i(k) = f(k, \hat{y}_i, e_i^{-1}) + F_i e_i(k), \]
for some \( f(k, \cdot) : \mathbb{R}^{2k+1} \to \mathbb{R} \) and some real scalar constant \( F_i \), then (11) simplifies to \( U_i = \log |F_i| \). Such a choice is satisfied, for instance, if the controller is linear and time-invariant.

### 4 Proof of Theorem 1

To prove Theorem 1 we proceed in three steps. First, we lower bound the integral of the log sensitivity function (8) by the difference of the differential entropy rates \( \bar{h}(d) \) and \( \bar{h}(e_i) \) of the disturbance and error processes as follows
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_1}(\omega) \, d\omega \]
\[ = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left( \log \Phi_{e_1}(\omega) - \log \Phi_d(\omega) \right) d\omega \]  
\[ \geq \frac{1}{4\pi} \int_{-\pi}^{\pi} \log \Phi_{e_1}(\omega) d\omega - \bar{h}(d) \]
\[ \text{(12)} \]
Here (12) follows from the definition of sensitivity function (7), (13) and (14) follow from Assumption 2, and equality in (14) holds if and only if \( e_i \) is a Gaussian process with power spectral density \( \Phi_e \).

The second step is to bound the differential entropy rate \( \bar{h}(e_i) \).

**Lemma 1** Let \( y_{-1} \triangleq d \) and \( \hat{y}_0, \hat{y}_1 \triangleq 0 \). Then, for every \( i = 0, \ldots, n \),
\[ \bar{h}(e_i) = \bar{h}(y_{i-1}|\hat{y}_i) + \bar{I}(e_i; \hat{y}_i) + \bar{I}(e_i; x_i(0)|\hat{y}_i). \]  
\[ \text{(15)} \]
PROOF. See Appendix A.

For the first term at the RHS of (15), we establish the following.

**Lemma 2** For every $i = 1, \ldots, n$,

$$
\tilde{h}(y_{i-1}|\hat{y}_i) \geq \sum_{l=0}^{i-1} (\eta_l - C_l)^+ - \tilde{I}(\hat{y}_i; y_{i-1}|x_{i-1}(0)) + \tilde{h}(d),
$$

(16)

where $\eta_l \triangleq \log |D_l| + U_i + \tilde{I}(y_i; x_i(0)|\hat{y}_{i+1})$.

PROOF. See Appendix B.

Lemma 2 states that the conditional differential entropy rate of $y_{i-1}$ given $\hat{y}_i$ is at least as large as a scaled version of the differential entropy rate of the disturbance, where the scaling factor depends on the loop gains of the $i$-th follower’s predecessors.

The last step is completed by the following lemma.

**Lemma 3** Let $y_{n+1} \triangleq 0$. For every $i = 0, \ldots, n$,

\begin{align*}
\tilde{I}(e_i; x_i(0)|\hat{y}_i) &\geq \sum_{\lambda \in \mathbb{P}_i} \log |\lambda|, \\
\tilde{I}(y_i; x_i(0)|\hat{y}_{i+1}) &\geq \sum_{\lambda \in \mathbb{Z}_i} \log |\lambda|.
\end{align*}

(17) (18)

PROOF. See Appendix C.

It should be remarked that the inequalities in Lemma 3 coincide with the ones derived in [14, 17] in the special case of a single feedback loop where $d$ is independent of the plant’s initial condition. Therefore, in this case Lemma 3 provides a generalization of the results in [14, 17] to the case of a string where vehicles’ initial conditions are correlated.

Lemma 3 is reminiscent of the data rate theorem [24] for stabilization of unstable plants over a communication constrained feedback channel, which states that the information rate that needs to be supported by the feedback channel to ensure that the closed-loop system is stable must be large enough as measured by the unstable modes of the system.

In summary, combining (14) with the above three lemmas we obtain

\begin{align*}
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{\tilde{d}, \tilde{e}}(\omega) d\omega \\
\geq h(\epsilon_i) - \tilde{h}(d) \\
= \tilde{h}(y_{i-1}|\hat{y}_i) + \tilde{I}(e_i; \hat{y}_i) + \tilde{I}(e_i; x_i(0)|\hat{y}_i) - \tilde{h}(d) \\
\geq \sum_{l=0}^{i-1} (\eta_l - C_l)^+ - \tilde{I}(\hat{y}_i; y_{i-1}|x_{i-1}(0)) + \tilde{I}(e_i; x_i(0)|\hat{y}_i) \\
\geq \sum_{l=0}^{i-1} (\log |D_l| + U_i + \sum_{\lambda \in \mathbb{Z}_i} \log |\lambda| - C_l)^+ - \tilde{I}(\hat{y}_i; \hat{x}_i) \\
\geq \sum_{l=0}^{i-1} (\log |D_l| + U_i + \sum_{\lambda \in \mathbb{Z}_i} \log |\lambda| - C_l)^+ - C_i \\
\geq \sum_{\lambda \in \mathbb{P}_i} \log |\lambda|,
\end{align*}

which completes the proof. Here (a) follows from the data processing inequality and (b) holds by the definition of the channel capacity.

5 Some special cases

5.1 Linear controllers and scalar systems

Consider the special case of $n + 1$ scalar systems with dynamics given by

\begin{align*}
x_i(k + 1) &= a_i x_i(k) + u_i(k), \\
y_i(k) &= x_i(k),
\end{align*}

(19)

for some $|a_i| > 1$, $i = 0, \ldots, n$. Suppose that there is no inter-vehicular communication and the initial conditions $x_0(0), \ldots, x_n(0)$ are jointly Gaussian random variables independent of the disturbance process. Finally, suppose that the $i$-th control map is given by

\begin{align*}
u_i(k) &= a_i^{k+1} \left( E[x_i(0)|e_i^{k-1}] - E[x_i(0)|e_i^k] \right),
\end{align*}

(20)

with $E[x_i(0)|e_i^{k-1}] \triangleq 0$. The control in (20) is chosen so that the state $x_i(k)$ satisfies

\begin{align*}
a_i^{-k} x_i(k) &= x_i(0) - \sum_{j=0}^{k-1} a_i^{-(j+1)} u_i(j) \\
&= x_i(0) - E[x_i(0)|e_i^{k-1}]
\end{align*}

(21)

as can be seen by telescoping the series in the first equality. By the linearity of the Minimum Mean Square Error (MMSE) estimator and the orthogonality
principle, it then follows that $\mathbf{x}_i(k)$ is a Gaussian random variable independent of $\mathbf{e}_i^{k-1}$. In this case, the lower bound in Theorem 1 is attained with equality, as shown next.

**Proposition 1** For every $i \geq 0$, in the Gaussian scalar systems (19) with control laws given by (20) it holds that

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_i}(\omega) d\omega = \log |a_i| + \sum_{l=0}^{i-1} U_l.
$$

**PROOF.** First, observe that (14) is tight because $(\mathbf{e}_i^k, \mathbf{u}_i(k))$ are jointly Gaussian random variables for all $k$. Then, by setting $\hat{y}_i = 0$ and $D_i = 1$ in Lemma 1 and Lemma 2, it follows that

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_i}(\omega) d\omega = \sum_{l=0}^{i-1} (U_l + I(y_i; x_i(0))) + I(e_i; x_i(0)). \tag{22}
$$

Next, observe that (21) and the properties of differential entropy yield

$$
h(\mathbf{x}_i(0) | \mathbf{e}_i^{k-1}) = h(\mathbf{x}_i(0) - \mathbb{E}(\mathbf{x}_i(0) | \mathbf{e}_i^{k-1}) | \mathbf{e}_i^{k-1})
= h(\mathbf{x}_i(0) - \mathbb{E}(\mathbf{x}_i(0) | \mathbf{e}_i^{k-1} - 1))
= h(a_i^{-k} \mathbf{x}_i(k))
= h(\mathbf{x}_i(k)) - k \log |a_i|
$$

and thus

$$
I(e_i; x_i(0)) = \lim_{k \to \infty} \frac{1}{k} (h(\mathbf{x}_i(0)) - h(\mathbf{x}_i(0) | \mathbf{e}_i^{k-1})) = \log |a_i|, \tag{23}
$$

because the plant is by assumption second moment stable. Following similar steps, it can be easily verified that

$$
I(y_i; x_i(0)) = 0, \tag{24}
$$

so the claim follows by combining (22)–(24).

### 5.2 Side information across AWGN channels

Next, we consider a setup in which the classical transfer function approach leads to closed-form expressions for the sensitivity functions, that can be compared to the corresponding lower bounds given in Theorem 1. Throughout this section, we make the following assumptions:

- **d** is a white Gaussian process, i.e., $\Phi_d = 1$.
- The $i$-th vehicle has open-loop transfer function $P_i$ with relative degree $v_i$. The $i$-th controller consists of a channel decoder $Q_i$ and a stabilizing control $K_i$ acting on $\hat{y}_i$ and $e_i$, respectively, whose outputs are added and fed to the plant (see Fig. 2).

We denote by

$$
L_i \triangleq \frac{1}{1 + P_iK_i} \quad \text{and} \quad T_i \triangleq \frac{P_iK_i}{1 + P_iK_i}
$$

the sensitivity function and complementary sensitivity function, respectively, associated with the plant/controller pair $(P_i, K_i)$.

**Fig. 2.** Block diagram for vehicle $i$ ($Q_i$ present only if $i \geq 2$).

- The $i$-th channel, $i \geq 2$, is an additive white Gaussian noise (AWGN) channel with capacity

$$
C_i = \frac{1}{2} \log (1 + \sigma_i^2).
$$

Thus, $\hat{y}_i(k) = \hat{x}_i(k) + \tilde{z}_i(k)$, where $\tilde{z}_i(k)$ is a standard Gaussian random variable independent of $\hat{x}_i(k)$, and $\hat{x}_i$ is subject to the average power constraint $\sigma_i^2$.

- The $i$-th channel encoder, $i \geq 2$, has transfer function

$$
E_i = e^{-j\omega v_i T_0} - \sigma_i.
$$

Notice that in the time domain (25) yields $\hat{x}_i(k) = \sigma_i d(k - v_i)$. This means that the disturbance $d$ is transmitted uncoded from the leader to the followers.

- The $i$-th channel decoder, $i \geq 2$, has transfer function

$$
Q_i = P_i^{-1} e^{j\omega v_i R_i} \frac{\sigma_i}{\sigma_i^2 + 1},
$$

where $R_i = \frac{\sigma_i^2 + T_0}{\sigma_i^2 + 1} \ldots \frac{\sigma_i^2 + T_0}{\sigma_i^2 + 1} T_0$ denotes the chain of operators applied to $d$ as the disturbance propagates downstream up to the $i$-th follower. In other words, decoder $i$ computes the MMSE estimate $\mathbb{E}(d(k - v_i) | \hat{y}_i(k))$ and then filters it through $P_i^{-1} e^{j\omega v_i R_i}$ to cancel out the effect of $d(k - v_i)$ on the $i$-th plant.

Under these assumptions, the power spectral densities of the spacing errors can be derived explicitly. At the second follower we have $\Phi_{\mathbf{e}_2}(\omega) = (1 + \sigma_2^2)^{-1} |L_2 T_1 T_0|^2$ and thus

$$
S_{d,e_2}(\omega) = 2^{-\gamma_2} |L_2 T_1 T_0|. \tag{26}
$$

Then, by applying the classical Bode’s integral formula and the integral formula for the complementary
sensitivity function of linear time-invariant systems [22], we obtain
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{a,e_2}(\omega) d\omega = \frac{1}{z} \left( \log |D_t| + \sum_{\beta \in Z} \log |\beta| + U_t \right) + \sum_{\lambda \in P_3} \log |\lambda| - C_2.
\]

(27)

By comparing (10) with (27) we conclude that in this case the lower bound in Theorem 1 is tight. Observe from (26) that the sensitivity \( S_{a,e_2}(\omega) \) decreases exponentially with the capacity \( C_2 \) equally at all frequencies. Accordingly, the integral of the log sensitivity function reduces linearly as \( C_2 \) increases and thus approaches \(-\infty\) as \( C_2 \rightarrow \infty \). In this case, follower 2 can perfectly predict the disturbance \( d \).

The relationship between the integral of the log sensitivity function and the channel capacity is more involved for \( i > 2 \) than (27). For \( i = 3 \), for instance, it can be verified that
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{a,e_3}(\omega)d\omega
= \sum_{l=0}^{2} \left( (\log |D_l| + U_l + \sum_{\beta \in Z_l} \log |\beta|) \right)
+ \sum_{\lambda \in P_3} \log |\lambda| + \frac{1}{4\pi} \int_{-\pi}^{\pi} \log \Omega(\omega)d\omega,
\]
where
\[
\Omega(\omega) = 2^{-2(C_2+C_3)}(|T_2|^2 - 1) + 2^{-2C_3} + 2^{-2C_2}(1 - 2^{-2C_2})(1 - 2^{-2C_3})|L_2|^2.
\]

Notice that \( \Omega(\omega) \) depends on \( C_3 \) as well as on \( L_2, T_2, \) and \( C_2 \).

Fig. 3 (a) and (b) compare (28) with the lower bound in (10) for different values of \( C_2 \) and \( C_3 \) in the special case where
\[
P_i(z) = \frac{1}{z - 2}, \quad K_i(z) = \frac{2(z + 0.52)}{z + 0.2}
\]
for all \( i \geq 0 \). The following conclusions can be made from Fig. 3:

1. It is not possible to arbitrarily reduce the integral of the log sensitivity function at follower 3 by increasing the capacity \( C_3 \) while keeping \( C_2 \) constant, or by increasing the capacity \( C_2 \) while keeping \( C_3 \) constant.
2. The bound (10) is tight for follower 3 in the special case where \( C_2 = 0 \) and is asymptotically tight in the limit as \( C_2 \rightarrow \infty \).

6 Conclusion

In this brief paper, we studied the problem of disturbance propagation in a one-dimensional string assuming a predecessor and leader following information flow structure. The leader information is transmitted to the other vehicles across finite capacity channels. Using information-theoretic techniques we derived a lower bound on the integral of the log sensitivity function of tracking error for every vehicle with respect to a stochastic disturbance acting on the string leader. A specific class of systems and controllers...
for which the lower bound is tight was provided. Numerical evaluations were provided to validate the insights provided by the theoretical result. The result can be interpreted as the extension of the Bode integral formula to a particular distributed control problem.

This work can be extended in two main directions. In vehicle string control, we can consider other information structures, specifically the ones in which follower information is made available. In distributed control, future work can include an extension of the presented result to multiple-input multiple-output systems.

References


A Proof of Lemma 1

It holds for $i = 0, \ldots, n$ and all $j \in \mathbb{N}$ that

$$h(y_{i-1}(j)|y_{i-1}^{j-1}, \hat{y}_i) - I(y_{i-1}(j); x_0(0)|y_{i-1}^{j-1}, \hat{y}_i) = h(y_{i-1}(j)|y_{i-1}^{j-1}, \hat{y}_i, x_0(0))$$

where (a) follows because $y_i^j$ is a function of $(y_{i-1}^{j-1}, \hat{y}_i, x_0(0))$, (b) follows from the fact that $e_i(j) = y_{i-1}(j) - y_i(j) - \delta$, and (c) follows because $y_i^j$ is a function of $(e_i^{j-1}, \hat{y}_i, x_i(0))$.

Moreover,

$$\lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} I(y_{i-1}(j); x_0(0)|y_{i-1}^{j-1}, \hat{y}_i) = \lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} I(\hat{y}_i; x_0(0)) - I(\hat{y}_i; x_i(0))$$

where (a) holds by the chain rule of mutual information, (b) holds because of the data processing inequality and (9), and the last equality follows from Assumption 3. This further implies that the LHS equals 0 because of the positivity of mutual information.

By summing both sides of (A.1) over $j$ from 0 to $k-1$, dividing by $k$, and taking the limit as $k \to \infty$, it is obtained by using (A.2) that

$$h(y_i(j)|y_{i-1}^{j-1}, \hat{y}_{i+1}) - I(y_i(j); x_i(0)|y_{i-1}^{j-1}, \hat{y}_{i+1})$$

and that $y_i^{j-1} \to d^{j-v_0-1} \to d(j - v_0)$ form a Markov chain.

By summing both sides of (B.1) over $j$, dividing by $k$, and taking the limit as $k \to \infty$, it is obvious that (16) holds for $i = 1$. Assume now that (16) holds up to index $i$. Then,

$$h(y_i(j)|y_{i-1}^{j-1}, \hat{y}_{i+1}) - I(y_i(j); x_i(0)|y_{i-1}^{j-1}, \hat{y}_{i+1})$$

and that $y_i^{j-1} \to d^{j-v_0-1} \to d(j - v_0)$ form a Markov chain.

B Proof of Lemma 2

We proceed by induction in $i$. For $i = 1$ and all $j \in \mathbb{N}$,

$$h(y_0(j)|y_0^{j-1}, \hat{y}_1) - I(y_0(j); x_0(0)|y_0^{j-1}, \hat{y}_1) = h(y_0(j)|y_0^{j-1}, x_0(0), \hat{y}_1)$$

where (a) holds because $y_0(j)$ is equal to $D_{0u_0,j-v_0}(e_0^{j-v_0})$ plus a function of $x_0(0)$ and $e_0^{j-v_0} = r^{j-v_0-1} + d^{j-v_0-1} - y_0^{j-v_0-1}$, and because the 0-th plant is observable and thus $x_0(0)$ is a function of $(y_0^{j-v_0-1}, d^{j-v_0-1})$. Step (b) follows from fact that

$$e_0(j - v_0) = r(j - v_0) + d(j - v_0) - y_0(j - v_0)$$

and that $y_0^{j-1} \to d^{j-v_0-1} \to d(j - v_0)$ form a Markov chain.

where (a) holds because the mapping between $y_{i-1}^{j-1} \leftrightarrow y_i^j$ is bijective given $(x_i(0), \hat{y}_i)$.
Furthermore,
\[
\eta_i + \hat{I}(\hat{y}_i; y_i|x_i(0)) - \bar{I}(\hat{y}_i; y_{i-1}|x_{i-1}(0)) \\
= \eta_i + h(y_i|x_i(0), \hat{y}_i) - \bar{h}(y_{i-1}|x_{i-1}(0), \hat{y}_i) \\
= \bar{h}(y_{i-1}|x_{i-1}(0)) + \bar{h}(y_{i-1}|x_{i-1}(0), \hat{y}_i) \\
\geq \bar{h}(y_{i-1}) - \bar{h}(y_{i-1}|x_{i-1}(0)) \\
\geq \eta_i + \sum_{\lambda \in Z} \log |\lambda|
\]
where \((a)\) follows from \([17, Proposition\ 12]\) and \((9)\).

By the chain rule of mutual information, we obtain
\[
\eta_i + \hat{I}(\hat{y}_i; y_i|x_i(0)) - \bar{I}(\hat{y}_i; y_{i-1}|x_{i-1}(0)) \geq (\eta_i - C_i)^{+}.
\]
Substituting the above inequality into \((B.3)\), we obtain that \((16)\) holds for index \(i+1\), which completes the proof.

### C Proof of Lemma 3

#### C.1 Proof of \((17)\)

By the chain rule of mutual information,
\[
I(e_i^{k-1}; x_i(0)|\hat{y}_i) = I(e_i^{k-1}; \hat{y}_i; x_i(0)) - I(\hat{y}_i; x_i(0)) \\
\geq I(u_i^{k-1}; x_i(0)) - I(\hat{y}_i; x_i(0))
\]
where \((a)\) follows from the data processing inequality.

Therefore,
\[
\hat{I}(e_i; x_i(0)|\hat{y}_i) = \lim_{k \to \infty} \frac{I(e_i^{k-1}; x_i(0)|\hat{y}_i)}{k} \\
\geq \hat{I}(u_i; x_i(0)) - I(\hat{y}_i; x_i(0)) \\
\geq \sum_{\lambda \in P} \log |\lambda|
\]
where \((b)\) follows from \([14, Lemma\ 4.1]\) and \((9)\).

#### C.2 Proof of \((18)\)

By the chain rule of mutual information,
\[
I(y_i^{k-1}; x_{i-1}(0)|\hat{y}_i) = I(y_i^{k-1}; \hat{y}_i; x_{i-1}(0)) - I(\hat{y}_i; x_i(0)) \\
\geq I(y_{i-1}^{k-1}; x_{i-1}(0)) - I(\hat{y}_i; x_i(0))
\]

Therefore,
\[
\hat{I}(y_{i-1}; x_{i-1}(0)|\hat{y}_i) = \lim_{k \to \infty} \frac{I(y_i^{k-1}; x_{i-1}(0)|\hat{y}_i)}{k} \\
\geq I(y_{i-1}; x_{i-1}(0)) - I(\hat{y}_i; x_i(0)) \\
\geq \sum_{\lambda \in Z_{i-1}} \log |\lambda|
\]
where \((a)\) follows from \([17, Proposition\ 12]\) and \((9)\).