Suggested Answers
Problem Set 2
ECON 60303

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1. Given $Y_{it} = \alpha + X_{it}\beta + u_i + v_t + \varepsilon_{it}$ but only two observations per block, generate the difference

$$\begin{aligned} Y_{i2} &= \alpha + X_{i2}\beta + u_i + v_2 + \varepsilon_{i2} \\ -Y_{i1} &= \alpha + X_{i1}\beta + u_i + v_1 + \varepsilon_{i1} \\ \hline \Delta Y_i &= \Delta X_i \beta + (v_2 - v_1) + \Delta\varepsilon_i \end{aligned}$$

Where $\theta = v_2 - v_1$ in the estimating equation $\Delta Y_i = \theta + \Delta X_i \beta + \Delta\varepsilon_i$ and this represents the growth in outcomes between years 1 and 2.

To figure out what $\Delta Y_i = \theta + \Delta X_i \beta + \lambda_i + \Delta\varepsilon_i$ is estimating, write the original model as

$$Y_{it} = \alpha + X_{it}\beta + u_i + v_t + \lambda_i time_t + \varepsilon_{it}$$

Where $time_t$ is a time trend that equals 1 in period 1, 2 in period 2, etc. The equation allows for a unique time trend within a particular state. The new model can be written as

$$Y_{it} = \alpha + X_{it}\beta + u_i + v_t + \lambda_i t + \varepsilon_{it}$$

And the first difference between any two adjacent years is

$$\begin{aligned} Y_{it} &= \alpha + X_{it}\beta + u_i + v_t + \lambda_i t + \varepsilon_{it} \\ Y_{it-1} &= \alpha + X_{it-1}\beta + u_i + v_{t-1} + \lambda_i (t-1) + \varepsilon_{it-1} \\ \hline \Delta Y_{it} &= \Delta X_{it}\beta + (v_t - v_{t-1}) + \lambda_i + \Delta\varepsilon_{it} \end{aligned}$$

And the estimating equation is $\Delta Y_{it} = \theta_t + \Delta X_{it}\beta + \lambda_i + \Delta\varepsilon_{it}$. In this equation, the year effects pick up growth rates between adjacent years and λ_i picks up the state-specific time trend.

2. Given the equation $y_{it} = \alpha + x_{it}\beta + u_i + \lambda_t + \varepsilon_{it}$

   - take the mean over all t_i for person i: $\bar{y}_i = \alpha + \bar{x}_i\beta + u_i + \bar{\lambda}_i + \bar{\varepsilon}_i$ where $\bar{\lambda}_i = \frac{1}{t_i}\sum_{t=1}^{t_i}\lambda_i$. Note that the means of the year fixed effects differ across y because each panel has different numbers of observations

   - take the mean over all n for year t: $\bar{y}_t = \alpha + \bar{x}_t\beta + \bar{u}_t + \lambda_t + \bar{\varepsilon}_t$ where $\bar{u}_t = \frac{1}{n_t}\sum_{i=1}^{n} u_i$ where n_t is the number of panels that are observed in time period t. Note that the means of the individual effects will vary by t because each year has different numbers of observations.

   - take the mean over observations: $\bar{y} = \alpha + \bar{x}\beta + \bar{u} + \bar{\lambda} + \bar{\varepsilon}$. Let M be the number of observations in the sample and hence $\bar{\lambda} = \frac{1}{m}\sum_{i=1}^{n}\sum_{t=1}^{t_i}\lambda_t$ and $\bar{\mu} = \frac{1}{m}\sum_{i=1}^{n}\sum_{t=1}^{t_i}\mu_i$

- Construct the difference $\tilde{y}_{it} = y_{it} - \bar{y}_i - \bar{y}_t + \bar{y}$. Because the observations vary by panel and the number of panels within a year vary as well, the differencing does not eliminate the means of the group and year effects and as a result, $\tilde{y}_{it} = (x_{it} - \bar{x}_i - \bar{x}_t + \bar{x})\beta - (\bar{\lambda}_i + \bar{\lambda}) - (\bar{\mu}_t + \bar{\mu}) + (\varepsilon_{it} - \bar{\varepsilon}_i - \bar{\varepsilon}_t + \bar{\varepsilon})$

3. a. When data is sorted by year then group. Let $Y_t = X_t\beta + V_t$ be the n observations for year t. In this case, $E[V_tV_t']$ is (n x n) and equals $\Omega_n = \sigma_\varepsilon^2 I_n + \sigma_v^2 i_t i_t'$ and then $E[VV']$ which is (nt x nt) is $I_t \otimes \Omega_n$.

b. When data is sorted by group then year. $Y_i = X_i\beta + V_i$ be the t observations for group i. In this case, $E[V_iV_i']$ is (t x t) and equals $\Omega_t = (\sigma_\varepsilon^2 + \sigma_v^2)I_t$. However, the 1st observation in panel 1 will be correlated with the 1st observation in panel 2, panel 3, etc. Therefore, $E[VV']$ can be generalized to read

$$E[VV'] = \begin{bmatrix}
\begin{pmatrix} \sigma_\varepsilon^2+\sigma_v^2 & 0 & \cdots & 0 \\ 0 & \sigma_\varepsilon^2+\sigma_v^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_\varepsilon^2+\sigma_v^2 \end{pmatrix} &
\begin{pmatrix} \sigma_v^2 & 0 & \cdots & 0 \\ 0 & \sigma_v^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_v^2 \end{pmatrix} & \cdots &
\begin{pmatrix} \sigma_v^2 & 0 & \cdots & 0 \\ 0 & \sigma_v^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_v^2 \end{pmatrix} \\
\begin{pmatrix} \sigma_v^2 & 0 & \cdots & 0 \\ 0 & \sigma_v^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_v^2 \end{pmatrix} &
\begin{pmatrix} \sigma_\varepsilon^2+\sigma_v^2 & 0 & \cdots & 0 \\ 0 & \sigma_\varepsilon^2+\sigma_v^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_\varepsilon^2+\sigma_v^2 \end{pmatrix} & \cdots &
\begin{pmatrix} \sigma_v^2 & 0 & \cdots & 0 \\ 0 & \sigma_v^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_v^2 \end{pmatrix} \\
\vdots & \vdots & \cdots & \vdots \\
\begin{pmatrix} \sigma_v^2 & 0 & \cdots & 0 \\ 0 & \sigma_v^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_v^2 \end{pmatrix} &
\begin{pmatrix} \sigma_v^2 & 0 & \cdots & 0 \\ 0 & \sigma_v^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_v^2 \end{pmatrix} & \cdots &
\begin{pmatrix} \sigma_\varepsilon^2+\sigma_v^2 & 0 & \cdots & 0 \\ 0 & \sigma_\varepsilon^2+\sigma_v^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_\varepsilon^2+\sigma_v^2 \end{pmatrix}
\end{bmatrix}$$

Which can be re-written as

$$E[\tilde{V}\tilde{V}'] = \sigma_\varepsilon^2 I_{nt} + (i_n i_n' \otimes \sigma_v^2 I_t).$$

5. a. Putting the results from class and the previous problem together, it is clear that

Therefore $E[VV'] = \sigma_\varepsilon^2 I_{nt} + (i_n i_n' \otimes \sigma_\lambda^2 I_t) + (I_n \otimes \sigma_u^2 i_t i_t') = \Omega$

b. $\hat{\beta} = (X'\Omega^{-1}X)^{-1}(X'\Omega^{-1}Y)$

6. Note that $\bar{y}^{00} = \alpha + \bar{\varepsilon}^{00}$ and $\bar{y}^{01} = \alpha + \theta + \bar{\varepsilon}^{01}$. Because $y_i = \alpha + G_i\gamma + A_i\theta + D_i\beta + \varepsilon_i$ then y_i^1 or only the observations for the treatment group become $y_i^1 = \alpha + \gamma + A_i\theta + A_i\beta + \varepsilon_i$ because $A_i = D_i$ in this sample. Let y_i^{10} be the observations for the treatment group before the intervention and y_i^{11} be the observations after the intervention. Then

$y_i^1 - [(1-A_i)\bar{y}^{00} + A_i\bar{y}^{01}]$ can be written as $(1-A_i)(y_i^{10} - \bar{y}^{00}) + A_i(y_i^{11} - \bar{y}^{01})$. Let's take each of these terms separately.

$$(y_i^{10} - \bar{y}^{00}) = \alpha + \gamma + \varepsilon_i - \alpha - \bar{\varepsilon}^{00} = \gamma + (\varepsilon_i - \bar{\varepsilon}^{00})$$

$$(y_i^{11} - \bar{y}^{01}) = \alpha + \gamma + \theta + \beta + \varepsilon_i - \alpha - \theta - \bar{\varepsilon}^{01} = \gamma + \beta + \varepsilon_i - \bar{\varepsilon}^{01}$$

Therefore

$$y_i^1 - [(1-A_i)\bar{y}^{00} + A_i\bar{y}^{01}] = (1-A_i)(y_i^{10} - \bar{y}^{00}) + A_i(y_i^{11} - \bar{y}^{01}) = (1-A_i)[\gamma + (\varepsilon_i - \bar{\varepsilon}^{00})] + A_i[\gamma + \beta + \varepsilon_i - \bar{\varepsilon}^{01}]$$

Which can be reduced to read $y_i^1 = \gamma + A_i\beta + \varepsilon_i - (1-A_i)\bar{\varepsilon}^{00} - A_i\bar{\varepsilon}^{01} = \gamma + A_i\beta + u_i$

7. There are two things wrong with the numbers. We know that random effects are a weighted average of between and within group estimates so they random effect estimates should lie between the two values. In this case it does not. Second, random effect is efficient estimation and the variance should be smaller than the fixed-effect estimate. Again, it is not.

   The Hausman test is $\hat{q} = (\hat{\beta}_{fe} - \hat{\beta}_{re})'\left[Var(\hat{\beta}_{fe}) - \mathrm{var}(\hat{\beta}_{re})\right]^{-1}(\hat{\beta}_{fe} - \hat{\beta}_{re})$ but when the parameter is a scaler, the test statistic reduces to
   $\hat{q} = (\hat{\beta}_{fe} - \hat{\beta}_{re})^2 / \left[Var(\hat{\beta}_{fe}) - \mathrm{var}(\hat{\beta}_{re})\right] = (0.04813 - 0.0635)^2 / [(0.02429)^2 - (0.02137)^2] = 1.77$
   The p-value for a chi-square distribution with 1 degree of freedom evaluated at 1.77 is 0.18, so we cannot reject the null the random and fixed-effect estimates are the same.

Computer problem

A sample program (psid_measurement_error.do) that generates these results is on the web and the some key results from that are reported below. In the OLS model, the attenuation bias is estimated to be $\hat{\theta} = \frac{\hat{\sigma}_t^2}{\bar{\hat{\sigma}}_v^2 + \hat{\sigma}_t^2}$ which is exactly what you find. In the fixed-effects case (first difference with two observations in the panel only), with two observations, we showed in class that the attenuation bias is $\left[1 - \frac{\sigma_v^2}{\sigma_x^2(1-\rho) + \sigma_v^2}\right]$ were ρ is the correlation in tenure over time. To estimate $\hat{\rho}$ estimated a random effects model and calculated the fraction of variance that is due to within-panel correlation which is $\hat{\rho} = 0.89$

$$\hat{\sigma}_t^2 = (6.91)^2 = 47.75$$

$$\hat{\beta}_{w/out}^{ols} = 0.0238 \text{ and } \hat{\beta}_{w/out}^{fe} = 0.0081$$

$$\bar{\hat{\sigma}}_v^2 = \frac{1}{1000}\sum_{i=1}^{1000}\hat{\sigma}_v^2(i) = 1.999^2 = 4$$

$$\bar{\hat{\beta}}_{with}^{OLS} = \frac{1}{1000}\sum_{i=1}^{1000}\hat{\beta}_{with}^{OLS}(i) = 0.0219$$

$$\bar{\hat{\beta}}_{with}^{fe} = \frac{1}{1000}\sum_{i=1}^{1000}\hat{\beta}_{with}^{fe}(i) = 0.00455$$

$$\hat{\theta} = \frac{\hat{\sigma}_t^2}{\bar{\hat{\sigma}}_v^2 + \hat{\sigma}_t^2} = \frac{47.75}{4 + 47.75} = 0.92$$

$$\frac{\bar{\hat{\beta}}_{with}^{OLS}}{\hat{\beta}_{w/out}^{ols}} = \frac{0.0219}{0.0238} = 0.92$$

$$\frac{\bar{\hat{\beta}}_{with}^{fe}}{\hat{\beta}_{w/out}^{fe}} = \frac{0.00455}{0.00809} = 0.563$$

$$\left[1 - \frac{\sigma_v^2}{\sigma_x^2(1-\rho) + \sigma_v^2}\right] = \left[1 - \frac{4}{47.75(1-0.89)+4}\right] = 0.568$$

```
. ***************************************;
. * part a --- get the variance of tenure;
. ***************************************;
. sum wagel tenure;

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
       wagel |      1578    2.330113    .5435628   .7558519   4.299593
      tenure |      1578      7.6109    6.913697          0         33

. ***************************************;
. * part h --- estimate random effects model;
. * get within panel correlation in tenure;
. * that is sigma u squared;
. ***************************************;
. iis id;

. sort id year;

. xtreg tenure;

Random-effects GLS regression                   Number of obs      =      1578
Group variable: id                              Number of groups   =       789

R-sq:  within  = 0.0000                         Obs per group: min =         2
       between = 0.0000                                        avg =       2.0
       overall = 0.0000                                        max =         2

Random effects u_i ~ Gaussian                   Wald chi2(0)       =      0.00
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =         .

------------------------------------------------------------------------------
      tenure |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       _cons |     7.6109   .2394447    31.79   0.000     7.141597    8.080203
-------------+----------------------------------------------------------------
     sigma_u |  6.5304204
     sigma_e |  2.2759307
         rho |   .8916942   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. ***************************************;
. * part b --- get OLS and FE estimates;
. ***************************************;
. * regression of ln(wagel) on tenure;
```

```
. reg wagel tenure;

      Source |       SS       df       MS              Number of obs =    1578
-------------+------------------------------           F(  1,  1576) =  158.85
       Model |  42.6637683     1  42.6637683           Prob > F      =  0.0000
    Residual |  423.277514  1576  .268577103           R-squared     =  0.0916
-------------+------------------------------           Adj R-squared =  0.0910
       Total |  465.941283  1577  .295460547           Root MSE      =  .51824

------------------------------------------------------------------------------
       wagel |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      tenure |   .0237905   .0018876    12.60   0.000      .020088    .0274929
       _cons |   2.149046    .019406   110.74   0.000     2.110982     2.18711
------------------------------------------------------------------------------

. * regression of ln(wagel) on tenure;
. areg wagel tenure, absorb(id);

Linear regression, absorbing indicators                Number of obs =    1578
                                                       F(  1,   788) =    4.91
                                                       Prob > F      =  0.0270
                                                       R-squared     =  0.9077
                                                       Adj R-squared =  0.8153
                                                       Root MSE      =   .2336

------------------------------------------------------------------------------
       wagel |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      tenure |   .0080944    .003654     2.22   0.027     .0009216    .0152671
       _cons |   2.268507   .0284253    79.81   0.000     2.212709    2.324306
-------------+----------------------------------------------------------------
          id |       F(788, 788) =      8.844   0.000          (789 categories)

. qui save `main' , replace ;

. use psid_measurementerror_results;

. sum;

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
        v2_m |      1000   -.0000245    .0506558   -.164754   .1637709
       v2_sd |      1000    1.999933    .0356311   1.892301    2.10275
    beta_ols |      1000    .0219413    .0005363   .0203601   .0238582
     beta_fe |      1000    .0045495    .0018241  -.0007662   .0108822
```