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Algebraic K-Theory Eventually Surjects onto Topological K-Theory

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We are going to prove the following theorem. Its hypotheses are satisfied by regular quasiprojective algebraic varieties over algebraically closed or finite fields of characteristic not ℓ, by localizations away from ℓ of rings of integers in number fields, and in fact by virtually all regular schemes over $\mathbb{Z}(\ell^{-1})$ that are not recognized as pathological in some way.

Theorem 1. *Let X be a regular scheme, quasiprojective over some noetherian ring of finite Krull dimension. Let $\ell > 3$ be a prime invertible in that ring, and suppose that all the residue fields of X have uniformly bounded etale cohomological dimension for ℓ-torsion sheaves. Then for any positive integer n and any p the natural map*

$$(1.0) \qquad \rho: K_p(X; \mathbb{Z}/\ell^n)(\beta_n^{-1}) \to K_p^{\text{et}}(X; \mathbb{Z}/\ell^n)$$

is a split surjection.

Here $K_*(X; \mathbb{Z}/\ell^n)(\beta_n^{-1})$ is mod ℓ^n algebraic K-theory of X localized by inverting a Bott element, as studied in [14]. $K^{\text{et}}(X; \mathbb{Z}/\ell^n)$ is the mod ℓ^n topological or etale K-theory of X, as studied in [5, 6], and [7]. The map ρ is constructed in these references.

Background. For X a smooth scheme over $\mathbb{C}$, $K_*^{\text{et}}(X; \mathbb{Z}/\ell^n)$ is isomorphic to the usual mod ℓ^n topological K-theory of X as a complex manifold ([6]). In this case, Theorem 1 essentially answers a question of Fulton, [9], §5. For many such X the map ρ: $K_0(X) \to K_0^{\text{et}}(X)$ cannot be a surjection, as non-trivial Hodge conditions on the Chern classes prevent some topological bundles from being stably algebraic. Nevertheless, Theorem 1 shows that the existence of these non-algebraic topological bundles is reflected in the *higher-dimensional* algebraic K-groups of X.

There are several conjectures in this area which are part of the Lichten-

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baum conjecture complex. In particular, it is thought that the split surjection ρ of (1.0) is an isomorphism, and that for large i the domain group $K_i(X;\mathbb{Z}/\ell^n)(\beta_n^{-1})$ of ρ is in fact isomorphic to $K_i(X;\mathbb{Z}/\ell^n)$. These conjectures are interesting because the range groups of ρ can usually be computed. (In fact the range groups are essentially the topological K-groups, with possibly twisted coefficients, of a certain space associated to X [5].) The conjectures are known to be true for certain specific varieties over the algebraic closure of a finite field, e.g., projective spaces minus planes [7], complete rational surfaces, reductive group schemes G and their homogeneous spaces G/P, P a parabolic subgroup [15].

Theorem 1 has an immediate corollary the following (much weaker) statement.

Corollary. *Let X be as in Theorem* 1, *and assume that $K_j^{\mathrm{et}}(X;\mathbb{Z}/\ell)$ is finite for all $j \geqq 0$. Then for any positive integer n there exists a J such that for all $j \geqq J$ the natural map.*

$$K_j(X;\mathbb{Z}/\ell^n) \to K_j^{\mathrm{et}}(X;\mathbb{Z}/\ell^n)$$

is surjective.

This is the statement referred to in the title of the paper. Its hypotheses are satisfied if X is regular and quasiprojective over an algebraically closed field of characteristic not ℓ.

Remark. That a localized version of algebraic K-theory might be better behaved than algebraic K-theory itself was suggested by Max Karoubi. Theorem 1 was proved for X over an algebraically closed field of finite characteristic in [14]; some cases were treated in [12]. Surjectivity of the map $K_j(X;\mathbb{Z}/\ell^n) \to K_j^{\mathrm{et}}(X;\mathbb{Z}/\ell^n)$ (no Bott element inverted) has been proved for some values of j and some low-dimensional X, cf. [5, 7, 13–15]. C. Soulé was a pioneer in this work.

Sketch of the Proof. The idea behind the proof of Theorem 1 is to construct an explicit splitting of the map ρ of (1.0) by using Snaith's approximation to algebraic K-theory [12]. We show by calculation that this succeeds strictly locally in the etale topology, i.e., when X is the spectrum of a strict local Hensel ring. The etale local-to-global principle of [14] for localized algebraic K-theory and its easier analogue for etale K-theory then yield the general result by a formal argument.

Note that in what follows we will usually make no notational distinction between a ring A and the corresponding affine scheme Spec A.

Bott Elements

The first step in the proof of Theorem 1 is to construct various Bott elements. The procedure is complicated by the need to find a Bott element in the mod ℓ^n algebraic K-theory of a ring R even if R does not contain the ℓ^n'th roots of unity. For a ring R, let $S_*(R;\mathbb{Z}/\ell^n)$ denote the mod ℓ^n stable homotopy groups of the ring spectrum $\Sigma^\infty(BGL_1(R)\amalg *)$. This construction was studied by Snaith in [11, 12].

Let $A=\mathbb{Z}(\ell^{-1},\zeta)$ where ζ is a primitive ℓ'th root of unity. The cyclic subgroup of $GL_1(A)$ generated by ζ can be identified by means of a Bockstein homomorphism with $\pi_2\,(BGL_1(A);\mathbb{Z}/\ell)$. Let b_1 be the image under the natural map $\pi_2(BGL_1(A);\mathbb{Z}/\ell)\to S_2(A;\mathbb{Z}/\ell)$ of the element corresponding to ζ.

Note that A is finite and etale over $\mathbb{Z}(\ell^{-1})$ with Galois group π cyclic of order $\ell-1$. Since π acts via ring automorphisms on A, it also acts via ring automorphisms on $S_*(A;\mathbb{Z}/\ell^n)\,(n\geqq 1)$. The action of π on the cyclic subgroup of $GL_1(A)$ generated by ζ gives a canonical isomorphism $\chi\colon\pi\to(\mathbb{Z}/\ell)^*$, and the construction of b_1 immediately implies that π also acts through χ on the cyclic subgroup of $S_2(A;\mathbb{Z}/\ell)$ generated by b_1. It follows that the $(\ell-1)$'st cup power β_1 of b_1 is *invariant* under the action of π.

Lemma 2. *For each $n>1$, the ℓ^{n-1} cup power of β_1 in $S_*(A;\mathbb{Z}/\ell)$ is the reduction mod ℓ of an element β_n in $S_*(A;\mathbb{Z}/\ell^n)$. Moreover, β_n can be chosen to be invariant under the action of π on $S_*(A;\mathbb{Z}/\ell^n)$.*

Proof. The first statement is proved by induction on n, with the help of the derivation properties of the Bockstein spectral sequence ([3]). The second statement follows easily from the fact that the order of π is prime to ℓ.

Pick a collection of β_n's as in Lemma 2, so that β_n is invariant under the action of π. There is a certain ambiguity here, but it is unimportant. Any two candidates β'_n,β''_n for β_n differ by an element which is divisible by ℓ. It follows that this difference element $\beta'_n-\beta''_n$ is nilpotent in the ring $S_*(A;\mathbb{Z}/\ell^n)$, so that localizing a module over $S_*(A;\mathbb{Z}/\ell^n)$ by inverting β'_n yields the same result as localizing the module by inverting β''_n.

There is a natural map $\gamma\colon S_*(A;\mathbb{Z}/\ell^n)\to K_*(A;\mathbb{Z}/\ell^n)$, [12]. We will denote $\gamma(\beta_n)$ also by β_n.

Lemma 3. *For each $n\geqq 1$, the element β_n in $K_*(A;\mathbb{Z}/\ell^n)$ is the image under the natural map of a unique element in $K_*(\mathbb{Z}(\ell^{-1}),\mathbb{Z}/\ell^n)$.*

Proof. Since the order of the Galois group π is prime to ℓ, a transfer argument shows that the natural map from $K_*(\mathbb{Z}(\ell^{-1});\mathbb{Z}/\ell^n)$ to the Galois invariant elements of $K_*(A;\mathbb{Z}/\ell^n)$ is an isomorphism.

We will continue to use β_n to denote the elements in $K_*(\mathbb{Z}(\ell^{-1});\mathbb{Z}/\ell^n)$ provided by Lemma 3. This concludes the construction of Bott elements. Note that if X is a scheme over $\mathbb{Z}(\ell^{-1})$ then $K_*(X;\mathbb{Z}/\ell^n)$ is an algebra over $K_*(\mathbb{Z}(\ell^{-1});\mathbb{Z}/\ell^n)$ and so it is possible to form the localized K-theory $K_*(X;\mathbb{Z}/\ell^n)(\beta_n^{-1})$. The groups $S_*(A;\mathbb{Z}/\ell^n)$, $K_*(X;\mathbb{Z}/\ell^n)$, $K_*^{\mathrm{et}}(X;\mathbb{Z}/\ell^n)$ are all mod ℓ^n stable homotopy groups of spectra, and are also the ordinary stable homotopy groups of spectra which have been reduced mod ℓ^n ([15, 6, 7, 11, 12, 14]). We note that those mod ℓ^n spectra are all ring spectra ($\ell\neq 2,3$); and as in [14] 3.1–3.2 we may localize them to form $K(X;\mathbb{Z}/\ell^n)(\beta_n^{-1})$ naturally for X over $\mathbb{Z}(\ell^{-1})$ and $S(R;\mathbb{Z}/\ell^n)(\beta_n^{-1})$ naturally for R over $Z(\ell^{-1},\zeta)$. Since we are dealing with the non-connective version of K^{et}, the localization map $K^{et}(X;\mathbb{Z}/\ell^n)\to K^{et}(X;\mathbb{Z}/\ell^n)(\beta_n^{-1})$ is a homotopy equivalence, and so we do not distinguish these spectra notationally.

In [12], Snaith describes a map of ring spectra, natural in the ring R

(1.2) $$\gamma: S(R;\mathbb{Z}/\ell^n)\to K(R,\mathbb{Z}/\ell^n).$$

This map is ultimately induced by the monomial matrix inclusions $\Sigma_n \int GL_1(R)\to GL_n(R)$ and it itself induces on homotopy groups the map which was called γ above. We may invert β_n on both sides of (1.2) if R is an algebra over $\mathbb{Z}(\gamma^{-1},\zeta)$.

In [5–7], Friedlander and also Dwyer describe a map for many X:

(1.3) $$\rho: K(X;\mathbb{Z}/\ell^n)\to K^{et}(X;\mathbb{Z}/\ell^n)$$

This is a natural map of ring spectra for X affine. In general, it is only well-defined and natural up to homotopy. For X over $\mathbb{Z}(\ell^{-1})$, we may naturally invert β_n on both sides of (1.3), although this procedure has no effect on the right-hand side. The resulting map of localized spectra induces on homotopy the map called ρ in Theorem 1.

The Strictly Local Case

The next step in proving Theorem 1 is to prove it strictly locally in the etale topology, i.e. for $X=\mathrm{Spec}(R)$, R a regular strictly local hensel ring.

Lemma 4. *Let R be a regular noetherian strict local hensel ring containing ℓ^{-1}. Then*

$$S_*(R;\mathbb{Z}/\ell)(b_1^{-1})\cong\mathbb{Z}/\ell[b_1,b_1^{-1}].$$

Pf: Under the hypotheses, R is an integral domain containing all ℓ-power roots of unity, and R^* is ℓ-divisible. Let $\mathbb{Q}/\mathbb{Z}_{(\ell)}=\mu$ be the subgroup of all ℓ-power roots of unity in R^*. Then the cokernel R^*/μ is uniquely ℓ-divisible, and $B(R^*/\mu)$ has the mod ℓ^n stable homotopy type of a point. Thus we have isomorphisms (1.4).

(1.4) $$\pi_*^s(B\mu\amalg *;\mathbb{Z}/\ell^n)\cong\pi_*^s(BR^*\amalg *;\mathbb{Z}/\ell^n)=S_*(R;\mathbb{Z}/\ell^n)$$

The canonical class in $H^2(\mathbb{Q}/\mathbb{Z}_{(\ell)},\mathbb{Z})$ yields a map $B\mu=B\mathbb{Q}/\mathbb{Z}_{(\ell)}\to K(\mathbb{Z},2)$ $=\mathbb{C}P^\infty$ into the Eilenberg-MacLane space which becomes an equivalence upon ℓ-adic completion. Thus we have isomorphisms (1.5).

(1.5) $$S_*(R;\mathbb{Z}/\ell^n)\cong\pi_*^s(B\mu\amalg *;\mathbb{Z}/\ell^n)\cong\pi_*^s(\mathbb{C}P^\infty\amalg *;\mathbb{Z}/\ell^n)$$

By [11], 9.1.1, $\pi_*^s(\mathbb{C}P^\infty\amalg *;\mathbb{Z}/\ell)(b_1^{-1})$ maps isomorphically to $\pi_*(BU)$, where BU denote the spectrum for topological K-theory. Thus yields the result.

Lemma 4 was first proved by Snaith in [12] II §2.

Proposition 5. *Let R be a regular noetherian strict local hensel ring containing ℓ^{-1}. Then the composite $\rho\gamma: S(R;\mathbb{Z}/\ell^n)(\beta_n^{-1})\to K^{et}(\mathrm{Spec}(R);\mathbb{Z}/\ell^n)(\beta_n^{-1})$ is a weak homotopy equivalence.*

Pf: It suffices to show $\rho\gamma$ induces an isomorphism on π_*, $S_*(R;\mathbb{Z}/\ell^n)(\beta_n^{-1}) \cong K_*^{\text{et}}(R;\mathbb{Z}/\ell^n)(\beta_n^{-1})$. By the usual Bockstein argument, it suffices to do the case $n=1$, (e.g., [14] (3.7)-(3.10) and ambient discussion). Then by Lemma 4, both rings are abstractly isomorphic to $\mathbb{Z}/\ell[x, x^{-1}]$. Working through the definitions of γ and ρ in [5] and [12] one checks that b_1 goes to a generator in $K_2^{\text{et}}(R;\mathbb{Z}/\ell)$.

This may be done as follows. Note that maps between strict local hensel rings induce isomorphisms on $S_*(\ ;\mathbb{Z}/\ell^n)(b_n^{-1})$ and on $K_*^{\text{et}}(\ ;\mathbb{Z}/\ell^n)$. Naturality of $\rho\gamma$ with respect to the map to the residue field $R\to R/m$ reduces the problem to the case where R is a separably closed field. The usual Witt vector lifting then reduces the problem to characteristic 0, hence to the special case $R=\bar{\mathbb{Q}}$, or even $R=\mathbb{C}$. For $R=\mathbb{C}$, $\rho\gamma$ is induced by the obvious maps $BGL_1(\mathbb{C})\to\mathbb{C}P^\infty\to BU$. Then Snaith's theorem ([11] 9.1.1) yields the result, as $BGL_1(\mathbb{C})$ and $K(\mathbb{Z},2)=\mathbb{C}P^\infty$ are ℓ-adically equivalent as in Lemma 4.

Descent for Algebraic K-Theory

We next use the cohomological descent machinery of Thomason [14] to reduce Theorem 1 to this strictly local case.

Consider the restricted etale site of X. This is the category of affine schemes etale over X. The Grothendieck topology has as covers the faithfully flat, possibly infinite families of etale maps. The maps γ and ρ are natural transformations of presheaves of spectra on this site.

We will apply the hypercohomology spectrum functor $\mathbb{H}(X;\ )$. Recall from [14] its construction. Let F be a presheaf of spectra on the site of X. For $\mathfrak{U}=\{U_i\to X|i\in I\}$ a cover, let $F_u^\bullet$ be the Čech cosimplicial spectrum (1.6).

$$F_u^\bullet = \prod_{i_0\in I} F(U_{i_0}) \rightrightarrows \prod_{(i_0,i_1)\in I^2} F(U_{i_0}\times_X U_{i_1}) \cdots \tag{1.6}$$

Let $\operatorname{colim}_{\mathfrak{U}} F_{\mathfrak{U}}^\bullet$ be the direct limit over all Godement covers $\mathfrak{U}$. Then $\mathbb{H}(X;F)$ is the homotopy limit over Δ of the cosimplicial spectrum $\operatorname{colim}_{\mathfrak{U}} F_{\mathfrak{U}}^\bullet$. It is a spectrum, and there is a natural augmentation (1.7).

$$\eta\colon F(x)\to\mathbb{H}(X;F) \tag{1.7}$$

Lemma 6. *For X a separated scheme, the Godement covers $\mathfrak{U}$ in the restricted etale site form a cofinal direct subsystem of the Godement covers in the full etale site. Thus* $\operatorname{colim}_{\mathfrak{U}} F_{\mathfrak{U}}$ *and* $\mathbb{H}(X;F)$ *formed in the two sites are naturally isomorphic.*
Pf: Clear.

The maps ρ and γ are defined as maps of presheaves of spectra only on the restricted etale site. On the other hand, it is only the full site which is natural in X. Lemma 6 resolves the difficulties caused by this, and allows us to form $\mathbb{H}(X;\rho)$ and $\mathbb{H}(X;\gamma)$ natural even in non-affine maps of X.

Theorem 7 (*c.f.* [14]). *Let X be a regular scheme, quasiprojective over some noetherian ring of finite Krull dimension. Let $\ell>3$ be a prime invertible in that ring. Suppose all residue fields of X have uniformly bounded etale cohomological* *dimension for ℓ-torsion sheaves. Then the natural augmentation* (1.8) *is a weak equivalence.*

$$\eta(x)\colon K(X;\mathbb{Z}/\ell^n)(\beta_n^{-1}) \xrightarrow{\sim} \mathbb{H}(X;K(\ ;\mathbb{Z}/\ell^n)(\beta_n^{-1})) \tag{1.8}$$

Corollary 8. Under the hypotheses of Theorem 7, there is a strongly convergent spectral sequence (1.9).

$$E_2^{p,q} = H_{\mathrm{et}}^p(X;K_q(\ ;\mathbb{Z}/\ell^n)(\beta_n^{-1})) \Rightarrow K_{q-p}(X;\mathbb{Z}/\ell^n)(\beta_n^{-1}) \tag{1.9}$$

Pf: For X over a ring with primitive ℓ^nth roots of unity, this is Thomason's main theorem, [14] 3.4. As remarked in [14], now that $K(\ ;\mathbb{Z}/\ell^n)(\beta_n^{-1})$ has been defined for more general X, it is easy to extend the result.

First, arguing via the Bockstein exact sequence as in [14] 3.4, one reduces to the case $n=1$.

Now for X over $\mathrm{Spec}(\mathbb{Z}(\ell^{-1}))$, let X' be the pull-back of X to $\mathbb{Z}(\ell^{-1},\zeta)$, where ζ is a primitive ℓth root of one. Then $X'\to X$ is finite etale with Galois group $(\mathbb{Z}/\ell)^* = \mathbb{Z}/\ell^{-1}$. The map $\eta(X')\colon K(X;\mathbb{Z}/\ell)(\beta_1^{-1}) \to \mathbb{H}(X';K(\ ;\mathbb{Z}/\ell)(\beta_1^{-1}))$ is a weak equivalence as X' is over $\mathbb{Z}(\ell^{-1},\zeta)$. The maps $\eta(X)$ and $\eta(X')$ are compatible under the transfer map from X' to X by [14] 2.14–2.20. As the order of $(\mathbb{Z}/\ell)^*$ is prime to ℓ, $K_*(X;\mathbb{Z}/\ell)(\beta_1^{-1})$ and $\pi_*\mathbb{H}(X;K(\ ;\mathbb{Z}/\ell)(\beta_1^{-1}))$ are identified to the Galois invariant elements in $K_*(X;\mathbb{Z}/\ell)(\beta_1^{-1})$ and $\pi_*\mathbb{H}(X';K(\ ;\mathbb{Z}/\ell)(\beta_1^{-1}))$ by the usual transfer argument. The map $\pi_*\eta(X)$ is induced by $\pi_*\eta(X')$ under this identification. Thus $\pi_*\eta(X)$ is an isomorphism and $\eta(X)$ is a weak equivalence, as $\pi_*\eta(X')$ is a Galois equivariant isomorphism.

The spectral sequence follows as in [14].

Descent for Etale K-Theory

Theorem 9. *Let X be a noetherian separated scheme over $\mathbb{Z}(\ell^{-1})$. Suppose X has finite Krull dimension, and all its residue fields have uniformly bounded etale cohomological dimension for ℓ-torsion sheaves. Then the natural augmentation map* (1.10)

$$\eta(X)\colon K^{\mathrm{et}}(X;\mathbb{Z}/\ell^n) \to \mathbb{H}(X;K^{\mathrm{et}}(\ ;\mathbb{Z}/\ell^n)) \tag{1.10}$$

is a weak equivalence.

Pf: Recall from [5] that for a simplicial scheme $U_\bullet$, $K^{\mathrm{et}}(U_\bullet;\mathbb{Z}/\ell^n)$ is the β_n^{-1} localization of the mod ℓ^n reduction of a spectrum constructed from a Γ-space. This Γ-space is $\coprod \mathrm{fib}(p_m)$ constructed from the connected components of the fibres of (1.11).

$$\begin{aligned} p_m\colon &\operatorname{holim}\mathrm{Hom}((U_\bullet)_{\mathrm{et}},\#\mathbb{Z}/\ell_\infty^\bullet(BGL_{m,R})_{\mathrm{et}}) \to \\ &\operatorname{holim}\mathrm{Hom}((U_\bullet)_{\mathrm{et}},\#(\mathrm{Spec}(R))_{\mathrm{et}}) \end{aligned} \tag{1.11}$$

over the point determined by the structure map $U_\bullet\to\mathrm{Spec}(R)$. The symbol $\#$ indicates the tower of coskeleta. Here $R=\mathbb{Z}(\ell^{-1})$. The holims are taken over the indexing category for the Hom shapes. $\mathbb{Z}/\ell^{\bullet}_{\infty}$ is the Bousfield-Kan fibrewise completion functor [2]. Consult Dwyer-Friedlander [5] for details. Let $F(U.)$ denote the connective spectrum obtained from this Γ-space, and let $F\{k\}(U.)$ denote the spectrum obtained from the analogous Γ-space constructed from fibres of $p_m\{k\}$. Here $p_m\{k\}$ is obtained by replacing $\#\mathbb{Z}/\ell^{\bullet}_{\infty}(BGL_{m,R})_{\text{et}}$ and $\#(\text{Spec}(R))_{\text{et}}$ in (1.11) by their k-th coskeleta.

To prove the theorem, we exhibit the following diagram of weak equivalences, (1.12).

(1.12)
$$\begin{array}{ccc} F(X) & \xrightarrow{\quad\quad\quad\quad\quad\eta\quad\quad\quad\quad\quad} & \mathbb{H}(X;F) \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\psi} \\ \operatorname*{holim}_k F\{k\}(X) \xrightarrow{\varepsilon} \operatorname*{holim}_k \operatorname*{colim}_{\mathfrak{U}} \operatorname*{holim}_{\Delta} F\{k\}_{\mathfrak{U}} & \xrightarrow{\theta} & \operatorname*{holim}_k \operatorname*{holim}_{\Delta} \operatorname*{colim}_{\mathfrak{U}} F\{k\}_{\mathfrak{U}} \end{array}$$

The indicated maps are all the canonical ones. The point is to show that $\varphi, \varepsilon, \theta$, and ψ are weak equivalences, or at least induce isomorphisms on π_q for q not small. Then η will be a weak equivalence after inverting β_n.

The hypotheses on X imply that X and each component $U_{i_0} \times \ldots \times U_{i_n}$ of the nerve of a cover $\mathfrak{U}$ have bounded etale cohomological dimension ([1] X 4.1, [4]). The construction of an Atiyah-Hirzebruch type spectral sequence in [5] generalizes to yield a strongly converging fringed spectral sequence for $\pi_* F\{k\}(X)$ and $\pi_* F(X)$. It follows that $F\{k\}(X)$ and $F\{k\}(U_{i_0} \times \ldots \times U_{i_n})$ have only finitely many non-zero homotopy groups for each k, and $\pi_i F\{k\}(U_{i_0} \times \ldots \times U_{i_n}) = \pi_i F(U_{i_0} \times \ldots \times U_{i_n})$ for $k \gg i$, uniformly for $U_{i_0} \times \ldots \times U_{i_n}$. Consequently, ψ is a weak equivalence as $\pi_* \psi$ is an isomorphism by the Milnor sequence and the Mittag-Leffler condition (c.f. [2] IX § 3). Similarly, φ is a weak equivalence; in this case almost by definition. Compare the argument in [14] 1.44.

To show θ is a weak equivalence, it suffices to check that for each k

(1.13)
$$\theta\colon \operatorname*{colim}_{\mathfrak{U}} \operatorname*{holim}_{\Delta} F\{k\}_{\mathfrak{U}} \to \operatorname*{holim}_{\Delta} \operatorname*{colim}_{\mathfrak{U}} F\{k\}_{\mathfrak{U}}$$

is a weak equivalence. As $\pi_n F\{k\}(\;) = 0$ for $n \gg k$ uniformly, this is [14] Lemma 1.18.

To show ε is a weak equivalence, it suffices to show for each k and $\mathfrak{U}$ that (1.14)

(1.14)
$$\varepsilon(\mathfrak{U})\colon F\{k\}(X) \to \operatorname*{holim}_{\mathfrak{U}} F\{k\}_{\mathfrak{U}}$$

is a weak equivalence. Let the simplicial scheme $U. = N_X(\mathfrak{U})$ be the nerve of the cover $\mathfrak{U}$. There is a weak equivalence of cosimplicial spectra (1.15).

(1.15)
$$(n \to F\{k\}(U_n)) \simeq F\{k\}_{\mathfrak{U}}$$

Remember $F\{k\}(\;)$ is defined on simplicial schemes such as $U.$. As $U. \to X$ induces a weak equivalence of etale topological types ([8]), the natural map $F\{k\}(X) \to F\{k\}(U.)$ is a weak equivalence. Thus we need only show (1.16) is a weak equivalence.

(1.16)
$$F\{k\}(U.) \to \operatorname*{holim}_{\Delta} (n \to F\{k\}(U_n))$$

As in [5, 7], group completion of the Γ-space $\coprod \mathrm{fib}(p\{k\})$ doesn't change the homotopy type of the 0th space $\operatorname{colim}_m \mathrm{fib}(p_m\{k\})$. This is quite different from what happens in algebraic K-theory. It allows us to commute group completion and taking holim. (Note that $\operatorname{holim}_\Delta(n \to G(U_n))$ commutes with direct limits of G such that $\cos k_k G = G$. To see this, first replace $U.$ by a refining and refined noetherian hypercover to see $\check{H}^*(U.;\ )$ commutes with direct limits. Then appeal to the strongly converging spectral sequence analogous to [14] 1.17.) At least this is true modulo problems with π_0 resulting from the fringe effect in the spectral sequence. But $\pi_q \varepsilon$ will be an isomorphism for $q > 0$ and so η will become a weak equivalence after inverting β_n. Thus it suffices to find a weak equivalence (1.17):

$$\mathrm{fib}(p_m\{k\}) \to \operatorname{holim}_\Delta (n \to \mathrm{fib}(q_{mn}\{k\})) \tag{1.17}$$

Here $q_{mn}\{k\}$ is the map (1.18).

$$\begin{aligned} q_{m,n}\{k\}\colon &\operatorname{holim} \operatorname{Hom}((U_n)_{\mathrm{et}}, \#\mathbb{Z}/\ell^{\cdot}_{\infty}(BGL_{m,R}) \to \\ &\operatorname{holim} \operatorname{Hom}((U_n)_{\mathrm{et}}, \#(\operatorname{Spec} R)_{\mathrm{et}}). \end{aligned} \tag{1.18}$$

Because $(U.)_{\mathrm{et}} = \operatorname{diag}\{V..(i)\}_{i\in I}$, with $\{V_m,.(i)\}_{i\in I} = (U_m)_{\mathrm{et}}$ by [8], the required weak equivalence is provided by the following lemma.

Lemma 10. *Let $\{V..(i)\}_{i\in I}$ be a probisimplicial set and $\{W.(j)\}_{j\in J}$ be a prosimplicial set such that each $W.(j)$ is a Kan complex which is isomorphic to a finite coskeleton of itself via the canonical map. Then there is a natural weak equivalence* (1.19).

$$\begin{aligned} &\operatorname{holim}_J \operatorname{colim}_I \operatorname{Hom}(\operatorname{diag} V..(i), W.(j)) \xrightarrow{\ \simeq\ } \\ &\xrightarrow{\ \simeq\ } \operatorname{holim}_{m\in\Delta} \operatorname{holim}_J \operatorname{colim}_I \operatorname{Hom}(V_{m.}(i), W.(j)) \end{aligned} \tag{1.19}$$

Pf: This results from the sequence of canonical isomorphisms and weak equivalences ([2]) (1.20).

$$\begin{aligned} &\operatorname{holim}_J \operatorname{colim}_I \operatorname{Hom}(\operatorname{diag} V..(i), W.(j)) \\ &\cong \operatorname{holim}_J \operatorname{colim}_I \operatorname{Tot}_m \operatorname{Hom}(V_{m.}(i), W.(j)) \\ &\cong \operatorname{holim}_J \operatorname{Tot}_m(\operatorname{colim}_I \operatorname{Hom}(V_{m.}(i), W.(j)) \\ &\simeq \operatorname{holim}_J \operatorname{holim}_{m\in\Delta} \operatorname{colim}_I \operatorname{Hom}(V_{m.}(i), W.(j)) \\ &= \operatorname{holim}_{m\in\Delta} \operatorname{holim}_J \operatorname{colim}_I \operatorname{Hom}(V_{m.}(i), W.(j)) \end{aligned} \tag{1.20}$$

The first isomorphism is elementary, as in the proof of [2] XII 4.3. The third weak equivalence and the fourth isomorphism are [2] XI 4.4 and XI 4.3 respectively: the fibrancy conditions are easy to check. The second isomorphism is a bit trickier. We show the inducing map is an isomorphism for each fixed $j \in J$. Fix j and pick N such that $W(j) = \cos k_N W(j)$. Then for all Y,

$$\begin{aligned} \operatorname{Hom}(Y, W(j)) = \operatorname{Hom}(Y, \cos k_N W(j)) &\cong \cos k_N \operatorname{Hom}(Y, \cos k_N W(j)) \\ &\cong \cos k_N \operatorname{Hom}(Y, W(j)). \end{aligned}$$

Thus it suffices to show if $Z(i)$ is a direct system of cosimplicial sets such that for each $p \in \Delta$ and $i \in I$, $Z(i)^p = \cos k_N Z(i)^p$, then there is an isomorphism (1.21).

$$\operatorname*{colim}_I \operatorname{Tot}(p \to Z(i)^p) \cong \operatorname{Tot}(p \to \operatorname*{colim}_I Z(i)^p) \tag{1.21}$$

For we take $Z(i)^p = \operatorname{Hom}(X_{p.}(i), W(j))$.

A q simplex of $\operatorname{Tot}(Z(i))$ is a family of maps (1.22)

$$f(p)\colon \Delta(p) \times \Delta(q) \to Z(i)^p \tag{1.22}$$

which form a natural transformation of functors as p varies over Δ. As $Z(i)^p = \cos k_N Z(i)^p$, families (1.22) correspond to families (1.23) natural in p.

$$\tilde{f}(p)\colon s\,k^N(\Delta(p) \times \Delta(q)) \to Z(i)^p \tag{1.23}$$

But one calculates that a natural family of $\tilde{f}(p)$ is uniquely determined by its restriction to the subcategory of $p \leqq N$. Conversely any natural family of $\tilde{f}(p)$ defined for $p \leqq N$ extends to a natural family (1.23). The set of natural transformations on the finite subcategory of $p \leqq N$ is a finite end, and so computed by finite limits. Thus it commutes with direct limits like $\operatorname*{colim}_I$. Tracing back along the bijections, one gets (1.21) for q-simplices. This proves the lemma.

Lemma 11. *Let ℓ, n, X be as in Theorem 1. Then if X is affine, the diagram* (1.24) *strictly commutes. In those cases where $\rho(x)$ is defined by Jouanolou's device* ([5–7, 10]) (1.24) *commutes up to homotopy.*

$$\begin{array}{ccc} \mathbb{H}(X; K(\ ; \mathbb{Z}/\ell^n)(\beta_n^{-1})) & \xrightarrow{\mathbb{H}(X;\rho)} & \mathbb{H}(X; K^{\mathrm{et}}(\ ; \mathbb{Z}/\ell^n)) \\ \eta \Big\uparrow \wr & & \eta \Big\uparrow \wr \\ K(X; \mathbb{Z}/\ell^n)(\beta_n^{-1})) & \xrightarrow[\rho(x)]{} & K^{\mathrm{et}}(X; \mathbb{Z}/\ell^n)(\beta_n^{-1}) \end{array} \tag{1.24}$$

Pf: For X affine, this is just the naturality of the augmentations η. If Jouanolou's device applies to X, it follows as the above diagram is mapped via weak equivalences to the diagram for the affine resolution $\tilde{X}$ of X.

Note as the augmentations $\eta(X)$ are weak equivalences by Theorem 7 and 9, we may use (1.24) to define $\rho(X)$ up to homotopy for general X.

Proof of Theorem 1

We now prove Theorem 1 globally for X over $\mathbb{Z}(\ell^{-1}, \zeta)$.

Proposition 12. *Let ℓ, n, X be as in Theorem 1. Suppose also that X is over $\mathbb{Z}(\ell^{-1}, \zeta)$. Then the conclusion of Theorem 1 holds for X; i.e., $\rho(X)$ of* (1.0) *is a* *split surjection. In fact, $\rho(X)$ of (1.25) splits up to homotopy.*

$$\rho(X)\colon K(X;\mathbb{Z}/\ell^n)(\beta_n^{-1}) \xrightarrow{\rho} K^{\mathrm{et}}(X;\mathbb{Z}/\ell^n) \tag{1.25}$$

Pf: As X is over $\mathbb{Z}(\ell^{-1},\zeta)$, there is a sequence of natural transformations of presheaves of spectra on X, (1.26)

$$\begin{array}{ccccc} S(\ ;\mathbb{Z}/\ell^n)(\beta_n^{-1}) & \xrightarrow{\gamma} & K(\ ;\mathbb{Z}/\ell^n)(\beta_n^{-1}) & \to & K^{\mathrm{et}}(\ ;\mathbb{Z}/\ell^n) \\ & & \Big|\wr & & \\ & & K(\ ;\mathbb{Z}/\ell^n)(\beta_n^{-1}) & & \end{array} \tag{1.26}$$

Consider the diagram (1.27).

$$\begin{array}{ccccc} \mathbb{H}(X;S(\ ;\mathbb{Z}/\ell^n)(\beta_n^{-1})) & \xrightarrow{\mathbb{H}(X;\gamma)} & \mathbb{H}(X;K(\ ;\mathbb{Z}/\ell^n)(\beta_n^{-1})) & \xrightarrow{\mathbb{H}(X;\rho)} & \mathbb{H}(X;K^{\mathrm{et}}(\ ;\mathbb{Z}/\ell^n)) \\ & & \eta\uparrow\wr & & \eta\uparrow\wr \\ & & K(X;\mathbb{Z}/\ell^n)(\beta_n^{-1}) & \xrightarrow{\rho(X)} & K^{\mathrm{et}}(X;\mathbb{Z}/\ell^n) \end{array} \tag{1.27}$$

Contemplation reveals it suffices to show $\mathbb{H}(X;\rho\gamma)$ is a weak equivalence; i.e. that $\pi_*\mathbb{H}(X;\rho\gamma)$ is an isomorphism. By the strongly converging spectral sequences of [14] 1.31, it will be enough to show the $\rho\gamma$ of (1.26) induces an isomorphism of the sheafifications of π_*. We check this at each stalk. By [14] 1.25, this amounts to showing (1.26) is a weak equivalence when evaluated at each strict local henselization of X. But this was done in Proposition 5. (Note $K^{\mathrm{et}}(\ ;\mathbb{Z}/\ell^n)$ is "continuous" in the sense of [14] 1.25 because of the strongly converging Atiyah-Hirzebruch spectral sequence [5] and the "continuity" of etale cohomology ([1], VII§5); $S(\ ;\mathbb{Z}/\ell^n)$ is obviously "continuous".)

It remains to prove Theorem 1 for X over $\mathbb{Z}(\ell^{-1})$. Without ζ, we cannot for m $S(\ ;\mathbb{Z}/\ell^n)$ (β_n^{-1}) as a presheaf (i.e., naturally) on the etale site of X. We dodge this as follows.

Let X' be the pullback of X and $\mathbb{Z}(\ell^{-1},\zeta)$ over $\mathbb{Z}(\ell^{-1})$. As before, $p\colon X'\to X$ is finite etale with Galois group $(\mathbb{Z}/\ell)^*$. $K_*(Z;\mathbb{Z}/\ell^n)$ (β_n^{-1}) and $K_*^{\mathrm{et}}(X;\mathbb{Z}/\ell^n)$ are the $(\mathbb{Z}/\ell)^*$ invariant elements of $K_*(X';\mathbb{Z}/\ell^n)$ (β_b^{-1}) and $K_*^{\mathrm{et}}(X';\mathbb{Z}/\ell^n)$ respectively. The first follows from a transfer argument, the second from the descent spectral sequence for K_*^{et} of the cover $X'\to X$ (Theorem 9, [14, 1.42]). The desired result now follows easily from the fact that the order of $(\mathbb{Z}/\ell)^*$ is prime to ℓ, and the fact that a retract of a split epimorphism is a split epimorphism.

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