BREDON HOMOLOGY OF PARTITION COMPLEXES

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Abstract. We prove that the Bredon homology or cohomology of the partition complex with fairly general coefficients is either trivial or computable in terms of constructions with the Steinberg module. The argument involves developing a theory of Bredon homology and cohomology approximation.

1. Introduction

Let $\mathcal{P}_n$ be the nerve of the poset of proper, nontrivial partitions of the set $\mathbf{n} = \{1, \ldots, n\}$, and $\mathcal{P}_n^\diamond$ its unreduced suspension, with the basepoint at the south pole. This space, with its natural action of the symmetric group Σ_n, arises in various contexts, and in particular it plays a role in the calculus of functors. We study the Bredon homology and cohomology of $\mathcal{P}_n^\diamond$.

For the moment, we focus on homology. Let G be a finite group and X a G-space (which for us means a simplicial G-set). One type of equivariant homology for X is the ordinary twisted homology of the Borel construction $X_{\mathrm{h}G}$ with coefficients in a G-module M. The *Bredon homology* of X is a finer invariant, which takes coefficients in an additive functor γ from finite G-sets to abelian groups. Our goal in this paper, in rough terms, is to sharpen the results of [1] about the mod p homology of the Borel construction on $\mathcal{P}_n^\diamond$ by proving similar results about the Bredon homology of $\mathcal{P}_n^\diamond$ with somewhat general coefficients.

This program depends on two types of new information. The first is an enhanced homology approximation theory (along the lines of [1], [6] and [7]), which shows that certain equivariant approximation maps induce isomorphisms, not just on the twisted homology of Borel constructions, but also on Bredon homology and Bredon cohomology with suitable coefficients. The second novelty is a detailed analysis of the fixed point sets of various subgroups of Σ_n acting on the partition

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complex. This allows us to apply the general Bredon homology approximation theory to $\mathcal{P}_n^\diamond$ and to extract an identifiable calculation.

We now introduce some terminology and then describe the results about $\mathcal{P}_n^\diamond$ in more detail. A *Mackey functor* for G is an additive functor Γ from a certain span-based category $\omega(G)$ of finite G-sets to abelian groups (Definition 3.2). The functor Γ gives rise to a pair of additive functors $(\gamma, \gamma^\natural)$ from finite G-sets to abelian groups, with γ covariant and $\gamma^\natural$ contravariant; hence γ can serve as a coefficient system for Bredon homology of G-spaces, and $\gamma^\natural$ as a coefficient system for Bredon cohomology. We denote the resulting homology and cohomology groups of a G-space X by $H_*^{Br}(X;\Gamma)$ and $H^*_{Br}(X;\Gamma)$ (Sections 2 and 3). A map that induces an isomorphism on this homology (resp. cohomology) is called a Γ_*-isomorphism (resp. Γ^*-isomorphism). A Γ_*^*-isomorphism induces an isomorphism on both.

Remark. Bredon homology or cohomology with coefficients in a Mackey functor is the same as ordinary equivariant $RO(G)$-graded homology or cohomology in the sense of [12].

Let p be a prime, and given a positive integer k, let B_k be the nerve of the poset of proper, nontrivial subgroups of the group $\Delta_k = (\mathbb{Z}/p\mathbb{Z})^k$. Let $B_k^\diamond$ be the unreduced suspension of B_k, with the south pole as basepoint. The complex B_k is the Tits building for $\mathrm{GL}_k = \mathrm{GL}_k(\mathbb{F}_p)$. If $n = p^k$, there is a pointed map $B_k^\diamond \longrightarrow \mathcal{P}_n^\diamond$ that comes from identifying the underlying set of Δ_k with $\mathbf{n} = \{1, \ldots, p^k\}$ and assigning to any subgroup V of Δ_k the partition of $\mathbf{n}$ given by the cosets of V in Δ_k.

Under the identification $\Delta_k \leftrightarrow \mathbf{n}$, the action of Δ_k on itself by left translation gives a group monomorphism $\Delta_k \to \Sigma_n$, which we use to identify Δ_k with a subgroup of Σ_n. The normalizer of Δ_k in Σ_n is then isomorphic to the affine group $\mathrm{Aff}_k \cong \mathrm{GL}_k \ltimes \Delta_k$, and the map $B_k^\diamond \to \mathcal{P}_n^\diamond$ extends to a Σ_n-equivariant map $\Sigma_{n+} \wedge_{\mathrm{Aff}_k} (E\,\mathrm{GL}_{k\,+} \wedge B_k^\diamond) \longrightarrow \mathcal{P}_n^\diamond$, which turns out to be a good enough approximation to $\mathcal{P}_n^\diamond$ to compute Bredon homology, as stated in the theorem below.

The following is our main theorem and concerns the reduced Bredon homology of pointed Σ_n-spaces. The terms in the statement are defined in Definitions 3.6 and 8.3.

Theorem 1.1. *Suppose that Γ is a p-constrained Mackey functor for Σ_n and that Γ satisfies the centralizer condition and (if p is odd) the involution condition. If n is not a power of p, the groups $\tilde{H}_*^{Br}(\mathcal{P}_n^\diamond;\Gamma)$ and $\tilde{H}^*_{Br}(\mathcal{P}_n^\diamond;\Gamma)$ vanish. If $n = p^k$, then the map*

$$\Sigma_{n+} \wedge_{\mathrm{Aff}_k} (E\,\mathrm{GL}_{k\,+} \wedge B_k^\diamond) \longrightarrow \mathcal{P}_n^\diamond$$

is a Γ_*^*-isomorphism.

The proof is found in Section 9.

Remark. The condition that Γ is p-constrained (Definition 3.6) involves the transfer and is a little weaker than the assumption that Γ is a p-local cohomological Mackey functor (in the sense of [18]). The other two conditions restrict how the centralizers of certain elementary abelian p-subgroups $D \subset \Sigma_n$ can act on $\Gamma(\Sigma_n/D)$ (Definitions 8.1 and 8.2).

In the case $n = p^k$, Theorem 1.1 leads to an explicit calculation of the Bredon homology and cohomology of $\mathcal{P}_n^\diamond$. Let St_k denote $H_{k-1}(B_k; \mathbb{Z}_{(p)})$; this is a localized integral version of the Steinberg module for GL_k. The group GL_k acts on $\Gamma(\Sigma_n/\Delta_k)$ both from the right and from the left (via γ and $\gamma^\natural$). These two actions are easily seen to be obtained from one another via the antiautomorphism $g \mapsto g^{-1}$ (see Remark 3.1), and so here and in what follows we use this antiautomorphism to shift freely from left to right. Note that if Γ is p-constrained, then Γ takes values in $\mathbb{Z}_{(p)}$-modules (Lemma 3.8). Let R denote the ring $\mathbb{Z}_{(p)}[\mathrm{GL}_k]$.

Theorem 1.2. *In the setting of Theorem 1.1, suppose that $n = p^k$. Then $\tilde{H}_j^{\mathrm{Br}}(\mathcal{P}_n^\diamond; \Gamma)$ and $\tilde{H}^j_{\mathrm{Br}}(\mathcal{P}_n^\diamond; \Gamma)$ vanish unless $j = k$. Moreover, with $M = \Gamma(\Sigma_n/\Delta_k)$ there are isomorphisms*

$$H_k^{\mathrm{Br}}(\mathcal{P}_n^\diamond; \Gamma) \cong \mathrm{St}_k \otimes_R M$$
$$H^k_{\mathrm{Br}}(\mathcal{P}_n^\diamond; \Gamma) \cong \mathrm{Hom}_R(\mathrm{St}_k, M)\,.$$

As in [1], the nonvanishing groups above can be computed even more explicitly in terms of a Steinberg idempotent, but we do not pursue this here.

Example 1.3. Recall that Σ_n acts on the one-point compactification S^n of $\mathbb{R}^n$ by permuting coordinates, and hence on the j-fold smash product S^{nj}. Say that a functor F from spectra to spectra is *additive* if it respects equivalences and preserves finite coproducts up to equivalence. Given an additive F and a fixed integer j, there is a graded Mackey functor Γ_F for Σ_n which assigns to a finite Σ_n-set T the graded abelian group

$$\Gamma_F(T)_* = \pi_* F\left((\Sigma^\infty T_+ \wedge S^{nj})_{\tilde{h}\Sigma_n}\right)\,.$$

Let $L_{(p)}$ denote the functor on spectra given by localization at p. We make two assumptions:

- j is odd, or $p = 2$, and
- the natural map $F \to F \circ L_{(p)}$ is an equivalence.

Under these conditions the graded consituents of Γ_F satisfy the hypotheses of Theorem 1.1. This is discussed in detail in Section 10.

In the special case in which h is a p-local spectrum and $F(X) = h \wedge X$, Example 1.3 ties Theorems 1.1 and 1.2 to previous work. If X is a pointed Σ_n-space, then filtering X by its skeleta gives the "isotropy spectral sequences"

$$\tilde{H}_a^{Br}(X, (\Gamma_F)_b) \Rightarrow h_{b+a}\left(X \wedge \mathbf{S}^{jn}\right)_{\tilde{h}\Sigma_n}$$
$$\tilde{H}^a_{Br}(X; (\Gamma_F)_b) \Rightarrow h_{b-a}\,\mathrm{map}\left(X, \mathbf{S}^{jn}\right)_{\tilde{h}\Sigma_n}$$

where the second necessarily converges only if X has a finite number of Σ_n-cells. These spectral sequences can be used to obtain the main theorem of [1] from Theorem 1.1. In effect, [1] calculates the abutments of these spectral sequences for $X = \mathcal{P}_n^\diamond$ and $h = H\mathbb{Z}/p$, while Theorem 1.1 and Theorem 1.2 calculate the E^2-pages in a form that implies a collapse result.

In fact, for $F(X) = H\mathbb{Z}/p \wedge X$, the Bredon cohomology groups $\tilde{H}^*_{Br}(\mathcal{P}_n^\diamond; (\Gamma_F)_b)$ were first calculated in [2] by brute force, using detailed knowledge of the homology of symmetric groups. This paper gives a new, more conceptual approach to those calculations.

Intended applications. We are particularly interested in the graded Mackey functors Γ_F as in Example 1.3 when h is the p-completed sphere and $F(X) = h \wedge X$. As discussed in Section 10, in this case Theorem 1.1 leads to new proofs of some theorems of Kuhn and Behrens on the relationship between the Goodwillie tower of the identity and the symmetric power filtration of $H\mathbb{Z}$, as well as another approach to Kuhn's proof of the Whitehead Conjecture. We will pursue this in another paper.

Theorem 1.1 can also be applied to Γ_F when F is given by

$$F(X) = L_K\left(E \wedge X\right).$$

Here E is one of Morava's E-theories and L_K denotes localization with respect to the corresponding Morava K-theory. In this case Theorem 1.1 seems to offer an alternative approach to some recent calculations of Behrens and Rezk, and again we hope to discuss this elsewhere.

We devote the remainder of this introduction to explaining the idea of the proofs of Theorems 1.1 and 1.2. Let G be a finite group and suppose that Γ is a Mackey functor for G. Let $\mathcal{C}$ be a collection of subgroups of G. For a G-space X, let $X_{\mathcal{C}} \longrightarrow X$ be the *$\mathcal{C}$-approximation to X*; this is characterized up to homotopy by the fact that $X_{\mathcal{C}}$ has isotropy only in $\mathcal{C}$ and $X_{\mathcal{C}} \longrightarrow X$ induces an equivalence on K-fixed points for $K \in \mathcal{C}$ (Section 4). The relevance of $\mathcal{C}$-approximation is that the map $X_{\mathcal{C}} \longrightarrow X$ might be a Γ^*_*-equivalence without being an equivariant homotopy equivalence. Indeed, a first result along these lines is that if

Γ has transfers that are well-behaved at p ("Γ is p-constrained"), then the collection $\mathcal{A}$ of all p-subgroups of G is such a collection, as follows.

Proposition 4.6. *Let $\mathcal{A}$ be the collection of all p-subgroups of G, let X be a G-space, and let Γ be a p-constrained Mackey functor for G. Then $X_{\mathcal{A}} \to X$ is a Γ_*^*-isomorphism.*

Next we develop a criterion for reducing the size of the approximating collection of subgroups, so as to use only a strategically chosen subcollection of $\mathcal{A}$. Our principal result along these lines is Proposition 5.3 below. We need some notation to explain the criterion. Given a subgroup K of G and a p-subgroup D of K, let N be the normalizer of D in K, let W be the quotient N/D, and let L be the nerve of the poset of nontrivial p-subgroups of W. The group W acts on L (by conjugation), and there is a natural map

$$L_{hW} \to *_{hW} = BW\,. \tag{1.4}$$

If M is a W-module, we say that the group D is *M-prunable for K* if (1.4) induces isomorphisms on ordinary twisted homology and cohomology with coefficients in M (Definition 5.1).

Proposition 5.3. *Let Γ be a Mackey functor for G taking values in $\mathbb{Z}_{(p)}$-modules, and let $\mathcal{C}$ be a collection of p-subgroups of G that is closed under passage to p-supergroups. Suppose that for each $K \in \mathrm{Iso}(X)$ and each p-subgroup $D \subset K$ with $D \notin \mathcal{C}$, we know that D is $\Gamma(G/D)$-prunable in K. Then the map $X_{\mathcal{C}} \to X_{\mathcal{A}}$ is a Γ_*^*-isomorphism.*

Applying this proposition requires some algebraic input to know when a subgroup is prunable. We use the same notation as above.

Proposition 5.6. *Assume that M is a $\mathbb{Z}_{(p)}[W]$-module and that there exists an element of order p in W that acts trivially on M. Then D is M-prunable in K.*

Remark. The centralizer condition in Theorem 1.1 is present to allow for copious application of Proposition 5.6. The involution condition is there to guarantee that the relevant twisted homology and cohomology groups vanish identically whenever Proposition 5.6 cannot be applied.

Now let $G = \Sigma_n$ and consider the particular Σ_n-space $X = \mathcal{P}_n^\diamond$. It turns out that most p-subgroups of Σ_n actually have contractible fixed point sets, according to the following proposition.

Proposition 6.1. *Let $H \subseteq \Sigma_n$ be a p-group. If $(\mathcal{P}_n)^H$ is not contractible, then H is elementary abelian and acts freely on $\mathbf{n}$.*

Let $\mathcal{C}$ be the collection of p-subgroups of Σ_n obtained by removing from $\mathcal{A}$ all those elementary abelian p-subgroups that act freely on $\mathbf{n}$. Let $\mathcal{D}$ be the collection of elementary abelian p-subgroups of Σ_n that act freely and transitively on $\mathbf{n}$; hence $\mathcal{D}$ is empty unless $n = p^k$, in which case it consists exactly of the conjugates of Δ_k.

If Γ satisfies the hypotheses of Theorem 1.1, then the strategy of the proof is to use the above approximation machinery and an analysis of elementary abelian p-subgroups to establish that the map

$$(\mathcal{P}_n)_{\mathcal{C}\cup\mathcal{D}} \to (\mathcal{P}_n)_{\mathcal{A}}$$

is a Γ^*_*-equivalence. This proves Theorem 1.1 for $n \neq p^k$. Handling $n = p^k$ requires further analysis of the difference between $(\mathcal{P}_n)_{\mathcal{C}\cup\mathcal{D}}$ and $(\mathcal{P}_n)_{\mathcal{D}}$ (see Lemma 5.4 and equation (9.2)), and it turns out that this brings up the Tits building.

Organization and notation. Throughout the paper, G is a finite group and p is a prime. We use "space" to mean a simplicial set, and we often do not distinguish between a category and its nerve, trusting in context to clarify which is under discussion. When we refer to a G-space X, we mean a simplicial G-set. We choose a model for a free, contractible G-space EG, and if X is a G-space, we write X_{hG} for the unreduced Borel construction $(EG \times X)/G$. If X has a basepoint and the basepoint is fixed by the G-action, we write $X_{\tilde{h}G}$ for the reduced Borel construction $(EG_+ \wedge X)/G$.

We regard partitions of $\mathbf{n}$ as corresponding to equivalence classes of an equivalence relation $\sim$. If there is more than one partition being discussed, we write $x \sim_\lambda y$ to specify that the partition λ is being used.

In Sections 2 and 3, we give background on Bredon homology and cohomology and Mackey functors, and we state the key properties that we use and prove some basic results. Section 4 reviews approximation theory from [1] and proves Proposition 4.6, an initial approximation result for Bredon homology and cohomology. Section 5 discusses how to discard ("prune") subgroups from an approximating collection. Section 6 shows that most p-subgroups of Σ_n have contractible fixed point sets and identifies those that may not (Proposition 6.1), and then Section 7 studies centralizers of such groups with a view to acquiring the algebraic input for Proposition 5.6.

Section 8 establishes that the "prunability" hypothesis needed to use Proposition 5.3 holds in the cases of interest to us. This gives the data needed to prove Theorems 1.1 and 1.2 in Section 9. Finally, Section 10 looks at the coefficient functors of Example 1.3.

2. Bredon homology and cohomology

In this section, we give general background on G-spaces and on their Bredon homology and cohomology. For us, a G-space X is a simplicial set with a G-action. A G-map $f : X \to Y$ is a *G-equivalence* if it induces an equivalence $X^K \to Y^K$ of fixed point sets for each subgroup K of G. The map f is an *equivalence* if this condition simply holds for $K = \{e\}$, in other words, if f is an ordinary (weak) homotopy equivalence of spaces.

Given a G-space X, let $\mathrm{Iso}(X)$ denote the set of all subgroups of G that appear as isotropy subgroups of simplices of X. The following lemma gives an economical criterion for recognizing G-equivalences.

Lemma 2.1. [6, 4.1] *If $f : X \to Y$ is a map of G-spaces that induces equivalences $X^K \to Y^K$ for each $K \in \mathrm{Iso}(X) \cup \mathrm{Iso}(Y)$, then f is a G-equivalence.*

Let γ be an additive functor from G-sets to abelian groups, i.e., a functor taking coproducts of G-sets to sums of abelian groups. The Bredon chains on X with coefficients in γ are obtained by applying γ degreewise to X to obtain a simplicial abelian group and normalizing to obtain a chain complex. The Bredon homology of X with coefficients in γ, denoted $H_*^{G,\mathrm{Br}}(X;\gamma)$ or $H_*^{\mathrm{Br}}(X;\gamma)$, is the homology of this chain complex.

Let $\gamma^\natural$ be a contravariant functor from G-sets to abelian groups that takes coproducts of G-sets to products of abelian groups. The Bredon cochains on X with coefficients in $\gamma^\natural$ are obtained by applying $\gamma^\natural$ degreewise to X to obtain a cosimplicial abelian group and normalizing to obtain a cochain complex. The Bredon cohomology $H^*_{G,\mathrm{Br}}\left(X;\gamma^\natural\right)$ or $H^*_{\mathrm{Br}}(X;\gamma^\natural)$ is the cohomology of this cochain complex.

Sometimes it is convenient to take the coefficient functor γ for Bredon homology to be an additive functor from finite G-sets to abelian groups, or even an arbitrary functor from transitive G-sets to abelian groups. Such a γ can be extended to an additive functor on all G-sets in a unique way. Similar remarks apply to $\gamma^\natural$.

Bredon homology has good formal properties.

Lemma 2.2. [7, 4.6] *If $f : X \to Y$ is a G-equivalence, then f induces isomorphisms on Bredon homology and cohomology (with any coefficients).*

Lemma 2.3. [7, 4.11] *A homotopy pushout square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

of G-spaces gives long exact sequences in Bredon homology and cohomology (with any coefficients).

Remark 2.4. A homotopy pushout square of G-spaces is a square that, upon taking fixed points $(-)^K$ for any subgroup $K \subset G$, becomes a homotopy pushout square of spaces.

Lastly, we recall that Bredon homology and cohomology have restriction of coefficients. If $\gamma, \gamma^\natural$ are as above and K is a subgroup of G, there are Bredon coefficient functors $\gamma|_K, \gamma^\natural|_K$ from K-sets to abelian groups given by

$$\gamma|_K(S) = \gamma(G \times_K S) \qquad \gamma^\natural|_K(S) = \gamma^\natural(G \times_K S)\,.$$

For any K-space Y we have

$$\begin{aligned} H_*^{K,\mathrm{Br}}(Y; \gamma|_K) &= H_*^{G,\mathrm{Br}}(G \times_K Y; \gamma) \\ H^*_{K,\mathrm{Br}}(Y; \gamma^\natural|_K) &= H^*_{G,\mathrm{Br}}(G \times_K Y; \gamma^\natural)\,. \end{aligned} \tag{2.5}$$

3. Mackey functors

For our work in Section 4, we need well-behaved Bredon homology and cohomology transfers for finite covers of G-spaces. Transfers allow us to construct a canonical initial approximation for Bredon homology or cohomology provided that the coefficients are p-constrained (Definition 3.6). To obtain well-behaved transfers, we use coefficient functors that extend to Mackey functors, and this section collects appropriate background.

Given two finite G-sets S, T, let $\mathrm{sp}(S,T)$ denote the set of isomorphism classes of finite G-sets over $S \times T$, or, alternatively, the set of equivalence classes of spans $S \leftarrow V \rightarrow T$, where two spans are equivalent if there exists a commutative diagram of finite G-sets

$$\begin{array}{ccccc} S & \longleftarrow & V & \longrightarrow & T \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle =} \\ S & \longleftarrow & V' & \longrightarrow & T \end{array}$$

There is a category $\omega(G)$ in which the objects are finite G-sets and the maps $S \to T$ are given by $\mathrm{sp}(S,T)$. Composition is given by fiber product: given two maps

$$S \leftarrow V \rightarrow T \qquad T \leftarrow W \rightarrow R$$

the composed map from S to R in $\omega(G)$ is the span

$$S \leftarrow V \times_T W \rightarrow R\,.$$

Remark 3.1. Disjoint union of G-sets represents both the categorical coproduct and the categorical product in $\omega(G)$. There is a functor from the category of finite G-sets to $\omega(G)$ which is the identity on objects and is given on morphisms by

$$(S \rightarrow T) \quad \mapsto \quad (S \leftarrow S \rightarrow T)\,,$$

as well as a similar functor from the opposite of the category of finite G-sets to $\omega(G)$ given on morphisms by

$$(S \rightarrow T) \quad \mapsto \quad (T \leftarrow S \rightarrow S)\,.$$

The following definition is due to Lindner [14]; see also Thevenaz-Webb [17] or Webb [18] for more information.

Definition 3.2. A *Mackey functor (for G)* is an additive functor from $\omega(G)$ to abelian groups.

Remark 3.3. The word "additive" signifies that the Mackey functor takes coproducts in $\omega(G)$ to coproducts of abelian groups (or equivalently, products in $\omega(G)$ to products of abelian groups).

It follows from Remark 3.1 that a Mackey functor Γ gives rise both to an additive covariant functor γ from finite G-sets to abelian groups, and to a contravariant functor $\gamma^\natural$ from finite G-sets to abelian groups that takes coproducts to products. In this situation, we denote the associated Bredon homology (resp. cohomology) by $H_*^{\mathrm{Br}}(-;\Gamma)$ (resp. $H^*_{\mathrm{Br}}(-;\Gamma)$).

Proposition 3.4. *Suppose that* Γ *is a Mackey functor (for G) and that* $f : X \rightarrow Y$ *is a map of G-spaces that, as a map of spaces, is a finite covering map. Then there are naturally associated transfer maps*

$$\mathrm{tr}_*^f \colon H_*^{\mathrm{Br}}(Y;\Gamma) \rightarrow H_*^{\mathrm{Br}}(X;\Gamma) \qquad \mathrm{tr}_f^* \colon H^*_{\mathrm{Br}}(X;\Gamma) \rightarrow H^*_{\mathrm{Br}}(Y;\Gamma)\,.$$

Proof. For simplicity we assume that X and Y are of finite type, the extension to general X and Y being routine. The map tr_*^f is induced by a map from the Bredon chains on Y to those on X given in each degree by applying Γ to the $\omega(G)$-morphism represented by the diagram

$$Y_n \leftarrow X_n \rightarrow X_n\,.$$

The fact that this morphism commutes with boundary maps follows from the definition of composition in $\omega(G)$ and the fact that, by the covering map hypothesis, for each appropriate simplicial operator σ the square

$$\begin{array}{ccc} X_n & \xrightarrow{\sigma} & X_m \\ \downarrow & & \downarrow \\ Y_n & \xrightarrow{\sigma} & Y_m \end{array}$$

is a pullback diagram. A similar construction works to give tr^*_f.

□

Remark 3.5. Transfers are natural for fiber squares. For instance, if the diagram on the left below is a pullback diagram in which the vertical maps are finite covers, then the diagram on the right commutes.

$$\begin{array}{ccc} X & \xrightarrow{a} & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{b} & V \end{array} \qquad \begin{array}{ccc} H_*^{\mathrm{Br}}(X;\Gamma) & \xrightarrow{a_*} & H_*^{\mathrm{Br}}(U;\Gamma) \\ {\scriptstyle \mathrm{tr}^f_*}\uparrow & & \uparrow{\scriptstyle \mathrm{tr}^g_*} \\ H_*^{\mathrm{Br}}(Y;\Gamma) & \xrightarrow{b_*} & H_*^{\mathrm{Br}}(V;\Gamma) \end{array}$$

Definition 3.6. Suppose that Γ is a Mackey functor for G. We say that Γ is *p-constrained* if for any map $S \to T$ of finite G-sets with fibers of cardinality prime to p, the morphism $T \leftarrow S \to T$ in $\omega(G)$ induces an isomorphism on $\Gamma(T)$.

We have borrowed the word "p-constrained" from work of Libman[13], although our definition is different. The following proposition is crucial for us. It follows from a direct inspection of the relevant Bredon chain or cochain map.

Proposition 3.7. *Suppose that Γ is a p-constrained Mackey functor for G, and that $f : X \to Y$ is a finite covering map of G-spaces with fibers of cardinality prime to p. Then the self-maps $f_* \mathrm{tr}^f_*$ and $\mathrm{tr}^*_f f^*$ of $H_*^{\mathrm{Br}}(Y;\Gamma)$ and $H^*_{\mathrm{Br}}(Y;\Gamma)$, respectively, are isomorphisms.*

Finally, we will need our Bredon coefficients to be p-local, but it turns out that this follows from being p-constrained.

Lemma 3.8. *If Γ is a p-constrained Mackey functor for G, then Γ takes values in $\mathbb{Z}_{(p)}$-modules.*

Proof. Let S be a finite G-set, and m an integer prime to p. Since Γ is p-constrained, the self-map of $\Gamma(S)$ induced by the morphism $S \leftarrow m \cdot S \to S$ of $\omega(G)$ is an automorphism of $\Gamma(S)$. By additivity of Γ (Remark 3.3), this self-map is m times the identity map of $\Gamma(S)$. It follows that multiplication by m is an automorphism of $\Gamma(S)$ for any integer m prime to p, and so $\Gamma(S)$ is p-local.

□

4. Approximations

Our overall goal in this paper is a result about the Bredon homology of partition complexes. In this section we develop general tools for approximating a G-space X by other G-spaces whose Bredon homology or cohomology is easier to calculate. First, we recall the notion of approximation used in [1] and give a sufficient condition for this approximation to induce an isomorphism on Bredon homology and cohomology (Definition 4.1, Lemma 4.2). Second, we give an explicit result for an approximation that involves only one conjugacy class of subgroups (Lemma 4.4). Lastly, we observe that taking coefficients in a p-constrained Mackey functor gives a canonical jumping-off point for approximations, namely the approximation to X that is controlled by the collection of all p-subgroups of G (Proposition 4.6). In the next section, we build on this beginning and set up an inductive process to reduce the size of the controlling collection without changing the Bredon homology or cohomology.

Recall that a *collection* of subgroups of G is a set of subgroups that is closed under conjugation. If $\mathcal{C}$ is such a collection, a G-space X is said to have *$\mathcal{C}$-isotropy* if $\mathrm{Iso}(X) \subseteq \mathcal{C}$. A G-map $X \to Y$ is a *$\mathcal{C}$-equivalence* if $f^K : X^K \to Y^K$ is an equivalence for each $K \in \mathcal{C}$.

Definition 4.1. Given a G-space X, a *$\mathcal{C}$-approximation* to X is a $\mathcal{C}$-equivalence $X_{\mathcal{C}} \to X$ such that $X_{\mathcal{C}}$ has $\mathcal{C}$-isotropy.

Remark. From an abstract point of view, a $\mathcal{C}$-approximation to X is a cellularization of X in the sense of Farjoun [8] with respect to the set of G-spaces $\{G/K \mid K \in \mathcal{C}\}$. In [6, 4.8] there is a functorial construction of $\mathcal{C}$-approximations which makes it clear that if X is of finite type, then $X_{\mathcal{C}}$ is too. It follows from Lemma 2.1 that $\mathcal{C}$-approximations are unique up to a canonical zigzag of G-equivalences.

It is easy to see that $\mathcal{C}$-approximation commutes with the formation of homotopy pushouts of G-spaces. The Mayer-Vietoris property (Lemma 2.3) thus implies that in order to determine whether $X_{\mathcal{C}} \to X$ induces an isomorphism on Bredon homology or cohomology, it is enough to check this condition for the orbits used in building X.

Lemma 4.2. [1, 3.2 and 3.3] *Suppose that X is a G-space. If, for all $K \in \mathrm{Iso}(X)$, the map $(G/K)_{\mathcal{C}} \to G/K$ gives an isomorphism on Bredon homology with coefficients in γ (resp. cohomology with coefficients in $\gamma^\natural$), then $X_{\mathcal{C}} \to X$ gives such an isomorphism as well.*

If K is a subgroup of G, let $\mathcal{C} \downarrow K$ denote the collection of subgroups of K given by

$$\mathcal{C} \downarrow K = \{H \mid H \in \mathcal{C} \text{ and } H \subseteq K\}\,.$$

The following elementary lemma, applied with $Y = *$, is useful in applying Lemma 4.2.

Lemma 4.3. [1, 2.12] *Let K be a subgroup of G and Y a K-space. Then there is a canonical G-equivalence $G \times_K (Y_{\mathcal{C}\downarrow K}) \simeq (G \times_K Y)_{\mathcal{C}}$.*

In the case that the collection contains only one conjugacy class of subgroups of G, it is possible to give an explicit approximation to a G-space X, with associated homology and cohomology calculations. Let D be a subgroup of G, and let $\mathcal{D}$ be the collection consisting of all conjugates of D in G. We write N for the normalizer of D in G, and W for the quotient N/D. Note that N acts on the free contractible W-space EW, and also on the fixed point set X^D.

Lemma 4.4. *With the above notation, let $F = X^D$. Then for any G-space X, the natural map*

$$Y := G \times_N (EW \times F) \to X$$

is a $\mathcal{D}$-approximation $X_{\mathcal{D}} \to X$. For any Bredon coefficient systems γ, $\gamma^\natural$, there are natural isomorphisms

$$\begin{aligned} H_*^{\mathrm{Br}}(X_{\mathcal{D}};\gamma) &\cong H_*(F_{hW};\gamma(G/D)) \\ H^*_{\mathrm{Br}}(X_{\mathcal{D}};\gamma^\natural) &\cong H^*(F_{hW};\gamma^\natural(G/D)) \end{aligned} \tag{4.5}$$

The homology/cohomology groups on the right side of (4.5) are the ordinary local coefficient homology or cohomology groups associated to the natural action of W on the coefficients.

Proof of Lemma 4.4. The verification that $Y \simeq X_{\mathcal{D}}$ proceeds by checking orbit types ($\mathrm{Iso}(Y) \subseteq \mathcal{D}$) and fixed point sets ($Y^D$ is $EW \times F \simeq X^D$). The homology calculation, for instance, then follows from the fact that on the category of G-sets S that are unions of copies of G/D, there are natural isomorphisms $\gamma(S) \cong \mathbb{Z}[S^D] \otimes_{\mathbb{Z}[W]} \gamma(G/D)$.

□

To close this section, we establish a canonical starting point for approximation calculations when the Bredon coefficient system is p-constrained. Let $\mathcal{A}$ be the collection of all p-subgroups of G. The following proposition is essentially [7, 6.4].

Proposition 4.6. *Let $\mathcal{A}$ be the collection of all p-subgroups of G, let X be a G-space, and let Γ be a p-constrained Mackey functor for G. Then $X_{\mathcal{A}} \to X$ is a Γ_*^*-isomorphism.*

Proof. Let P be a Sylow p-subgroup of G, and consider the commutative diagram of G-spaces

$$\begin{array}{ccc} G \times_P (X_{\mathcal{A}}) & \longrightarrow & G \times_P X \\ \downarrow & & \downarrow \\ X_{\mathcal{A}} & \longrightarrow & X \end{array}$$

This is a pullback diagram in which the vertical maps are covering maps with fibers of cardinality prime to p. According to Remark 3.5 and Proposition 3.7, applying Bredon homology or cohomology with Γ-coefficients produces a commutative diagram in which the lower arrow is a retract of the upper one. On the other hand, by (2.5) the map on $H_*^{G,\mathrm{Br}}$ or $H^*_{G,\mathrm{Br}}$ with Γ-coefficients induced by the upper arrow is the same as map on $H_*^{P,\mathrm{Br}}$ or $H^*_{P,\mathrm{Br}}$ with $\Gamma|_P$ coefficients induced by $X_{\mathcal{A}} \to X$. But the map $X_{\mathcal{A}} \to X$ is a P-equivalence, as is verified by checking fixed point sets, so the upper arrow induces an isomorphism. The proposition follows from the fact that the retract of an isomorphism is an isomorphism. □

5. Approximations controlled by smaller collections

In Section 4, we set up an approximation to a G-space X by a collection of subgroups of G, and we considered the approximation $X_{\mathcal{A}} \to X$ given by the collection $\mathcal{A}$ of all p-subgroups of G. If Γ is a p-constrained Mackey functor, then $X_{\mathcal{A}} \to X$ is a Γ_*^*-isomorphism (Proposition 4.6). If $\mathcal{C} \subseteq \mathcal{A}$ is a subcollection, then there is a natural factoring $X_{\mathcal{C}} \to X_{\mathcal{A}} \to X$, and we can ask if $X_{\mathcal{C}} \to X_{\mathcal{A}}$ is a Γ_*^*-isomorphism, in which case the map $X_{\mathcal{C}} \to X$ will be a Γ_*^* isomorphism as well. Our main result along these lines is Proposition 5.3 below, which is an essential ingredient in the proofs of Theorems 1.1 and 1.2.

We rely on Lemmas 4.2 and 4.3 to determine if approximation by a collection $\mathcal{C}$ correctly calculates Bredon homology and cohomology, so we use the following notation throughout the section. Given a finite group G and a G-space X, let K be an isotropy subgroup of X. Let

D be a p-subgroup of K, let N be the normalizer of D in K, and let W be the quotient N/D. Let L be the nerve of the poset of non-identity p-subgroups of W, and let M be a p-local W-module. Our goal is to establish a condition that allows D (and its conjugates) to be pruned from an approximating collection without changing the Bredon homology or cohomology.

Definition 5.1. In the above setting, we say the subgroup D is *M-prunable in K* if the natural maps

$$\begin{aligned} H_*(L_{hW}; M) &\to H_*(BW; M) \\ H^*(L_{hW}; M) &\leftarrow H^*(BW; M) \end{aligned} \tag{5.2}$$

are isomorphisms.

The terminology is meant to suggest the likelihood that D can be discarded from an approximating collection, as indicated in the following proposition, which is the main result of this section.

Proposition 5.3. *Let Γ be a Mackey functor for G taking values in $\mathbb{Z}_{(p)}$-modules, and let $\mathcal{C}$ be a collection of p-subgroups of G that is closed under passage to p-supergroups. Suppose that for each $K \in \mathrm{Iso}(X)$ and each p-subgroup $D \subset K$ with $D \notin \mathcal{C}$, we know that D is $\Gamma(G/D)$-prunable in K. Then the map $X_{\mathcal{C}} \to X_{\mathcal{A}}$ is a Γ^*_*-isomorphism.*

The proof will appear after a few lemmas. If $\mathcal{C}$ is a collection of subgroups of G, a subcollection $\mathcal{D} \subset \mathcal{C}$ is *initial* if whenever $D \in \mathcal{D}$ and $C \in \mathcal{C}$ with $C \subset D$, then $C \in \mathcal{D}$. Note that the union of two collections is a collection.

Lemma 5.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be disjoint collections of subgroups of G such that $\mathcal{D}$ is initial in $\mathcal{C} \cup \mathcal{D}$, and let $Y \to Z$ be a map of G-spaces that induces an equivalence $Y_{\mathcal{C}} \to Z_{\mathcal{C}}$. Then there is a homotopy pushout diagram of G-spaces*

$$\begin{array}{ccc} Y_{\mathcal{D}} & \longrightarrow & Y_{\mathcal{C}\cup\mathcal{D}} \\ \downarrow & & \downarrow \\ Z_{\mathcal{D}} & \longrightarrow & Z_{\mathcal{C}\cup\mathcal{D}} \end{array} .$$

Proof. It is only necessary to check that for each subgroup $H \in \mathcal{C} \cup \mathcal{D}$, the indicated diagram becomes an ordinary homotopy pushout diagram upon taking H-fixed points, which is clear. (Note that the spaces on the left have empty H-fixed sets for $H \in \mathcal{C}$.) □

Suppose that the collection $\mathcal{D}$ consists of all conjugates of a single p-group D, and let γ and $\gamma^\natural$ be Bredon coefficient systems for G. Recall that $W = N/D$ acts on $\gamma(G/D)$ and $\gamma^\natural(G/D)$, as well as on L. As usual, $*$ represents the one-point G-space.

Lemma 5.5. *Assume that $\mathcal{C}$ is a collection of p-subgroups of G that does not contain D but contains all p-supergroups of D. Then $(*)_{\mathcal{C}} \to (*)_{\mathcal{C}\cup\mathcal{D}}$ gives isomorphisms on γ-homology and $\gamma^\natural$-cohomology if and only if the natural maps*

$$H_*\left(L_{hW};\gamma(G/D)\right) \to H_*\left(BW;\gamma(G/D)\right)$$
$$H^*\left(L_{hW};\gamma^\natural(G/D)\right) \leftarrow H^*\left(BW;\gamma^\natural(G/D)\right)$$

are isomorphisms.

Proof. The diagram of Lemma 5.4 with $Y = (*)_{\mathcal{C}}$ and $Z = *$ gives the homotopy pushout diagram

$$\begin{array}{ccc} (*_{\mathcal{C}})_{\mathcal{D}} & \longrightarrow & (*_{\mathcal{C}})_{\mathcal{C}\cup\mathcal{D}} \\ \downarrow & & \downarrow \\ (*)_{\mathcal{D}} & \longrightarrow & (*)_{\mathcal{C}\cup\mathcal{D}} \end{array} .$$

In the upper right corner, $(*_{\mathcal{C}})_{\mathcal{C}\cup\mathcal{D}}$ is G-equivalent to $(*)_{\mathcal{C}}$ (Section 2). Further, $\mathcal{D}$ consists of just one conjugacy class, allowing us to use Lemma 4.4 to obtain explicit formulas for $(*_{\mathcal{C}})_{\mathcal{D}}$ and $(*)_{\mathcal{D}}$. The result is the homotopy pushout diagram of G-spaces

$$\begin{array}{ccc} G\times_N \left(EW \times (*_{\mathcal{C}})^D\right) & \longrightarrow & (*)_{\mathcal{C}} \\ \downarrow & & \downarrow \\ G\times_N EW & \longrightarrow & (*)_{\mathcal{C}\cup\mathcal{D}} \end{array} .$$

In view of the Mayer-Vietoris property (Lemma 2.3), the only point to note is that the fixed point set $((*)_{\mathcal{C}})^D$ is homotopy equivalent to L via a W-equivariant map. This is proved in the third paragraph of [6, Pf. of 8.3]. □

Proof of Proposition 5.3. By Lemma 4.2, it is enough to check that for each $K \in \mathrm{Iso}(X)$ the map $(G/K)_{\mathcal{C}} \to (G/K)_{\mathcal{A}}$ gives a Γ^*_*-isomorphism. By Lemma 4.3, we need $G\times_K (*)_{\mathcal{C}\downarrow K} \to G\times_K (*)_{\mathcal{A}\downarrow K}$ to be a Γ^*_*-isomorphism. By definition of restriction of coefficients (2.5), this amounts to showing that $(*)_{\mathcal{C}\downarrow K} \to (*)_{\mathcal{A}\downarrow K}$ is a $\Gamma^*_*|_K$-isomorphism.

We proceed by downward induction on the number of conjugacy classes in $\mathcal{C}$, and the result is trivially true for $\mathcal{C} = \mathcal{A}$. We pass from $\mathcal{A}$ to $\mathcal{C}$ by removing conjugacy classes of subgroups one at a time, in such a way that smaller subgroups are removed before larger ones. The intermediate collections in this process are themselves closed under passage to p-supergroups. So assume that there exists a p-subgroup D of G, whose conjugates form a collection $\mathcal{D}$, such that $D \notin \mathcal{C}$ and $\mathcal{C} \cup \mathcal{D}$ is closed under passage to p-supergroups, and such that $(*)_{\mathcal{C}\cup\mathcal{D}} \to (*)_{\mathcal{A}}$ gives a $\Gamma^*_*|_K$-isomorphism. It is necessary to prove that $(*)_{\mathcal{C}} \to (*)_{\mathcal{C}\cup\mathcal{D}}$ gives a $\Gamma^*_*|_K$-isomorphism. Lemma 5.5 (with K playing the role of G) guarantees that this is the case. □

There remains the question of how to establish the isomorphisms (5.2). The following is an adaptation and strengthening of [6, 6.3]. The notation is the same as throughout the section.

Proposition 5.6. *Assume that M is a $\mathbb{Z}_{(p)}[W]$-module and that there exists an element of order p in W that acts trivially on M. Then D is M-prunable in K.*

Proof. Let J be the kernel of the action map $W \to \mathrm{Aut}(M)$. By [6, 6.3], which is essentially the homology version of the present proposition in the special case in which M is an $\mathbb{F}_p$-module, the natural twisted coefficient homology map

$$H_*\left(L_{hW}; \mathbb{F}_p[W/J]\right) \to H_*\left(BW; \mathbb{F}_p[W/J]\right)$$

is an isomorphism. By Shapiro's lemma, this is the same as the natural map

$$H_*(L_{hJ}; \mathbb{F}_p) \to H_*(BJ; \mathbb{F}_p)\,.$$

Since L_{hJ} and BJ are of finite type, we conclude that there is an isomorphism

$$H_*(L_{hJ}; \mathbb{Z}_{(p)}) \xrightarrow{\cong} H_*(BJ; \mathbb{Z}_{(p)})\,. \tag{5.7}$$

The proof is finished by comparing the Serre spectral sequences for the following diagram of fibrations:

$$\begin{array}{ccccc} L_{hJ} & \longrightarrow & L_{hW} & \longrightarrow & B(W/J) \\ \downarrow & & \downarrow & & \downarrow \\ BJ & \longrightarrow & BW & \longrightarrow & B(W/J) \end{array}\,.$$

The spectral sequences abut to homology or cohomology with twisted coefficients in M, and by (5.7), the vertical maps induce isomorphisms on the E^2-pages. The proposition follows. □

6. Fixed point sets

In this section, we study the fixed point sets of p-subgroups of Σ_n acting on $\mathcal{P}_n$. If $H \subseteq \Sigma_n$, we write $(\mathcal{P}_n)^H$ for the full subcategory of H-fixed partitions, that is, partitions λ with the property that $x \sim y \Rightarrow hx \sim hy$. Our main result in this section is the following proposition, which tells us that most p-subgroups of Σ_n have contractible fixed point sets on the partition complex.

Proposition 6.1. *Let $H \subseteq \Sigma_n$ be a p-group. If $(\mathcal{P}_n)^H$ is not contractible, then H is elementary abelian and acts freely on* $\mathbf{n}$.

The proof appears at the end of the section. For the analysis in this section, we need a stronger notion of "fixed," which will be applied for a subgroup $V \subseteq H$.

Definition 6.2. Let $V \subseteq \Sigma_n$ and let λ be a partition of $\mathbf{n}$. We say that λ is "strongly V-fixed" if $x \sim vx$ for all $v \in V$. We write $(\mathcal{P}_n)^V_{\text{st}}$ for the full subcategory of strongly V-fixed partitions.

For any partition λ of $\mathbf{n}$, we can define a minimal strongly V-fixed coarsening λ_V by $x \sim_{\lambda_V} y$ if there exists $v \in V$ such that $x \sim_\lambda vy$. The main tool for proving Proposition 6.1 is the following lemma.

Lemma 6.3. *Let H be a p-subgroup of Σ_n. Suppose there exists a nontrivial subgroup V of the center of H such that V never acts transitively on the equivalence classes of an object of $(\mathcal{P}_n)^H$. Then the nerve of $(\mathcal{P}_n)^H$ is contractible.*

Proof. We assert that the inclusion

$$(\mathcal{P}_n)^V_{\text{st}} \cap (\mathcal{P}_n)^H \longrightarrow (\mathcal{P}_n)^H \tag{6.4}$$

induces an equivalence on nerves and that the left side has an initial object. We construct a functorial retraction. Given an object λ of $(\mathcal{P}_n)^H$, it is easy to check that the coarsening λ_V is H-fixed because V is in the center of H. To know that λ_V is an object of $(\mathcal{P}_n)^V_{\text{st}} \cap (\mathcal{P}_n)^H$, we only need to know that λ_V is not the indiscrete partition, and this follows from the assumption that V does not act transitively on the set of equivalence classes of λ. Since λ_V is natural in λ, the assignment $\lambda \mapsto \lambda_V$ gives a functorial retraction of (6.4), and the morphism $\lambda \to \lambda_V$ gives the required natural transformation on $(\mathcal{P}_n)^H$. Finally, the partition of $\mathbf{n}$ by the orbits of V is nontrivial because V is nontrivial; it is also H-fixed because V is in the center of H, and it is therefore an initial object in $(\mathcal{P}_n)^V_{\text{st}} \cap (\mathcal{P}_n)^H$. □

To prove that H is elementary abelian in Proposition 6.1, we need a little group theory. Given a finite group H, let H/p be the maximal elementary abelian p-quotient of H, i.e., the quotient of H by the normal subgroup generated by commutators and p-th powers. Let $Z_p(H)$ be the maximal elementary abelian p-subgroup of the center of H.

Lemma 6.5. *If H is a p-group and is not elementary abelian, then the composite map $Z_p(H) \hookrightarrow H \twoheadrightarrow H/p$ has a nontrivial kernel.*

Proof. Let K be the kernel of $H \twoheadrightarrow H/p$. Because H is not elementary abelian, we know K is a nontrivial p-group, so $Z_p(K)$ is nontrivial. By the usual counting argument, the action of H on $Z_p(K)$ by conjugation has a fixed point $k \neq e$. Then $k \in Z_p(H)$, and k is in the kernel of $Z_p(H) \hookrightarrow H \twoheadrightarrow H/p$ by construction. □

The following proposition now gives the group-theoretic structure in Proposition 6.1.

Proposition 6.6. *If H is a p-subgroup of Σ_n and $(\mathcal{P}_n)^H$ is not contractible, then H is an elementary abelian p-group.*

Proof. Suppose that H is not elementary abelian; then by Lemma 6.5 the kernel of $Z_p(H) \to H \to H/p$ contains a subgroup $V \subseteq Z_p(H)$ of order p. We want to apply Lemma 6.3, so let λ be an object of $(\mathcal{P}_n)^H$, and suppose that V acts transitively on the equivalence classes of λ. In this case, since λ has more than one equivalence class, it must have exactly p equivalence classes. The action of H permutes the equivalence classes of λ, giving a map $H \to \Sigma_p$. Because the action of V on the equivalence classes of λ is transitive, the composite $V \hookrightarrow H \to \Sigma_p$ has nontrivial image $\mathbb{Z}/p \subseteq \Sigma_p$. But this contradicts our assumption that the composite $V \to H \to H/p$ is zero. We conclude that V cannot act transitively on the equivalence classes of any object of $(\mathcal{P}_n)^H$, and so by Lemma 6.3, $(\mathcal{P}_n)^H$ is contractible. □

Proof of Proposition 6.1. If $(\mathcal{P}_n)^H$ is not contractible, then we already know from Proposition 6.6 that H is an elementary abelian p-group. Further, by Lemma 6.3, we know that for any $\mathbb{Z}/p \subseteq H$, there exists a partition λ in $(\mathcal{P}_n)^H$ such that $\mathbb{Z}/p$ acts transitively on the equivalence classes of λ. Since λ is not indiscrete, λ must have exactly p equivalence classes, freely permuted by $\mathbb{Z}/p$, which therefore acts freely on $\mathbf{n}$.

□

7. Centralizers and involutions

In Section 4, we explained that if a Bredon coefficient functor is a p-constrained Mackey functor, then the Bredon homology and cohomology of a G-space X can be calculated using the approximation $X_{\mathcal{A}}$ of X constructed from the collection $\mathcal{A}$ of all p-subgroups of G. Our strategy now, in the special case $G = \Sigma_n$ and $X = \mathcal{P}_n$, is to further reduce the size of the approximating collection to the extent that the only p-subgroups $H \subseteq \Sigma_n$ that remain are those such that one or both of the following conditions hold:

- $(\mathcal{P}_n)^H$ is contractible, or
- H acts transitively on $\mathbf{n}$.

According to Proposition 6.1, the subgroups to be discarded ("pruned") are the elementary abelian p-subgroups D of Σ_n that act freely but nontransitively on $\mathbf{n}$. We do the pruning with Proposition 5.6, and in preparation for this we need to study how such D sit inside the isotropy subgroups of the action of Σ_n on $\mathcal{P}_n$. The goal is the following proposition, whose proof consitutes the remainder of this section. Here an "odd involution" means a permutation of order 2 that can be written as a product of an odd number of transpositions. The whole group Σ_n makes an appearance in the statement because the proposition will be applied to $*$, the trivial Σ_n-space, as well as to $\mathcal{P}_n$ itself.

Proposition 7.1. *Let $K \in \mathrm{Iso}(\mathcal{P}_n) \cup \{\Sigma_n\}$, and let $D \subseteq K$ be an elementary abelian p-subgroup of Σ_n that acts freely and nontransitively on $\mathbf{n}$. Let C be the centralizer of D in K. Then either*

1. *$p \mid [C : D]$, or*
2. *p is odd, and there is an odd involution in C that acts trivially on p-subgroups of the normalizer of D in K.*

We begin by describing the strategy of the proof of Proposition 7.1. Suppose $K \in \mathrm{Iso}(\mathcal{P}_n)$ is the subgroup of Σ_n that fixes a chain of partitions

$$\Lambda = (\lambda_0 < \lambda_1 < \ldots < \lambda_j),$$

i.e., K is the isotropy subgroup of a nondegenerate j-simplex in $\mathcal{P}_n$. Let C and D be as in Proposition 7.1. Let $A_1, \ldots, A_k$ be the orbits of the action of D on $\mathbf{n}$, and consider the extension of this action to an action of D^k on $\mathbf{n}$, where the i-th factor acts by the action of D on A_i and trivially on A_j for $j \neq i$. We look for elements in this copy of D^k that have order p in C/D. Sometimes the search fails for odd primes, and then we seek a suitable involution in C. This involution is obtained by picking two distinct integers $x, y \in \mathbf{n}$ with $x \sim_{\lambda_0} y$, letting $\sigma \in K$ be the involution that interchanges x and y while fixing the other elements of $\mathbf{n}$, and taking the product of all of the conjugates of σ by elements of D. Needless to say, there are details to be worked out.

We begin by assuming $K \neq \Sigma_n$ and setting up notation that we will use throughout the section for this case. Suppose that Λ and the groups K, D, and C are as above. Let $\hat{0}$ and $\hat{1}$ denote, respectively, the discrete and the indiscrete partition of $\mathbf{n}$, and for convenience, let

$\lambda_{-1} = \hat{0}$ and $\lambda_{j+1} = \hat{1}$. Let i be the smallest integer such that D acts transitively on $\mathrm{cl}(\lambda_i)$, the set of equivalence classes of λ_i. Consider the homomorphism $D \to \Sigma_{\mathrm{cl}(\lambda_i)}$ and write S for the kernel and $\overline{D}$ for the image, giving a short exact sequence

$$1 \to S \to D \to \overline{D} \to 1.$$

Lemma 7.2. *If $S \neq \{e\}$, then $p \mid [C : D]$.*

Proof. We write $\mathbf{n}$ as the union of two sets:

$$A = \{x \in \mathbf{n} : \exists\, d \in D \ s.t.\ x \sim_{\lambda_{i-1}} d \cdot 1\}$$
$$B = \mathbf{n} \backslash A$$

Because D does not act transitively on $\mathrm{cl}(\lambda_{i-1})$, we know $B \neq \emptyset$. Both A and B are stabilized by D, because they are unions of D-orbits, so $D \subseteq \Sigma_A \times \Sigma_B \subset \Sigma_n$. Let D_A and D_B denote the images of D in Σ_A and Σ_B, respectively, and likewise let S_A and S_B denote the images of S in Σ_A and Σ_B. Because D acts freely on $\mathbf{n}$, we know $D \cong D_A \cong D_B$ and $S \cong S_A \cong S_B$. A routine calculation shows that elements of $S_A \times S_B \subseteq \Sigma_n$ stabilize the chain Λ and centralize D. The off-diagonal elements are of order p and are not in D itself. □

If $S \neq \{e\}$, then Lemma 7.2 establishes Proposition 7.1. So suppose that $S = \{e\}$, i.e., suppose that D acts freely and transitively on the set $\mathrm{cl}(\lambda_i)$. Let J be the subgroup of K that pointwise fixes all $x \in \mathbf{n}$ such that x is not λ_i-equivalent to 1. Define an embedding $f : J \to K$ by

$$f(\sigma) = \prod_{d \in D} d\sigma d^{-1} \qquad \sigma \in J.$$

(The image of f is the diagonal copy of J in $D \ltimes J^D$.) A routine calculation establishes the following lemma.

Lemma 7.3. *Suppose $S = \{e\}$. Then $f(J)$ is a subgroup of K that centralizes D, and $f(J) \cap D = \{e\}$.*

Remark 7.4. Let $\lambda_i(1)$ denote the λ_i-equivalence class of 1. There is a commutative diagram of group monomorphisms

$$\begin{array}{ccc} J & \longrightarrow & \Sigma_{\lambda_i(1)} \\ {\scriptstyle f}\downarrow & & \downarrow \\ K & \longrightarrow & \Sigma_n \end{array}$$

An element of $\Sigma_{\lambda_i(1)}$ belongs to J if and only if it stabilizes the chain of partitions of $\lambda_i(1)$ obtained by restricting to $\lambda_i(1)$ the partitions λ_0, ..., λ_{i-1} of $\mathbf{n}$. In particular, if $g \in K$ and g carries $\lambda_i(1)$ to itself, then the restriction of g to $\lambda_i(1)$ belongs to J.

Corollary 7.5. *If $S = \{e\}$, then $C \backslash D$ contains an involution that is the product of $|D|$ transpositions.*

Proof. We know that D acts transitively on the set $\mathrm{cl}(\lambda_i)$, so $i \geq 0$, because $\lambda_{-1} = \hat{0}$ and by assumption D does not act transitively on $\mathbf{n} \cong \mathrm{cl}(\hat{0})$. We assert that $\lambda_i(1)$ contains distinct elements $x, x' \in \mathbf{n}$ such that $x \sim_{\lambda_0} x'$. For since λ_0 is not discrete, there exist distinct $y, y' \in \mathbf{n}$ such that $y \sim_{\lambda_0} y'$, and since λ_i is a coarsening of λ_0, this necessarily means $y \sim_{\lambda_i} y'$. Because D acts transitively on $\mathrm{cl}(\lambda_i)$, there exists $d \in D$ such that $dy \sim_{\lambda_i} 1$. Then taking $x = dy$ and $x' = dy'$ gives $x, x' \in \lambda_i(1)$ with $x \sim_{\lambda_0} x'$. We define σ to be the transposition that interchanges x and x', and $f(\sigma)$ is the desired involution. □

Proof of Proposition 7.1.

First, suppose that $K \neq \Sigma_n$.

If $p = 2$, then by combining Lemma 7.2 and Corollary 7.5 we know that $p \mid [C : D]$.

If p is odd and $p \nmid [C : D]$, then by Lemma 7.2 we know that $S = \{e\}$ and D acts freely and transitively on $\mathrm{cl}(\lambda_i)$, and by Lemma 7.3 we know that $p \nmid |f(J)|$. Corollary 7.5 gives us an involution $f(\sigma) \in f(J) \subseteq C$, and we will show that $f(\sigma)$ is actually in the centralizer of any p-subgroup of $N = N_K(D)$. There is an action of N on $\mathrm{cl}(\lambda_i)$, because $N \subseteq K$, and there is also an action of N on the set of D-orbits in $\mathbf{n}$. Thus we can define a map

$$N \to \Sigma_{\mathbf{n}/D} \times \Sigma_{\mathrm{cl}(\lambda_i)} \tag{7.6}$$

Each class of λ_i must contain exactly one element of each orbit of D, else D would not act both freely and transitively on $\mathrm{cl}(\lambda_i)$. It follows that the obvious projection maps give an isomorphism of N-sets

$$\mathbf{n} \cong (\mathbf{n}/D) \times \mathrm{cl}(\lambda_i).$$

This implies that (7.6) is a monomorphism.

We assert that the image of N in $\Sigma_{\mathbf{n}/D}$ is isomorphic to J. That the image contains a subgroup isomorphic to J follows from examining the composite

$$J \xhookrightarrow{f} C \hookrightarrow N \hookrightarrow \Sigma_{\mathbf{n}/D} \times \Sigma_{\mathrm{cl}(\lambda_i)},$$

since the image of $f(J)$ in $\Sigma_{\mathrm{cl}(\lambda_i)}$ is trivial. Conversely, let $\nu \in N$; we need to produce an element of J that has the same image as ν in $\Sigma_{\mathbf{n}/D}$. Since D acts freely and transitively on the equivalence classes of λ_i, there is a (unique) $d \in D$ such that $d\nu$ stabilizes $\lambda_i(1)$. By Remark 7.4, $d\nu \in J$, and since $d\nu$ and ν have the same image in $\Sigma_{\mathbf{n}/D}$, this proves that $\mathrm{im}\left(N \to \Sigma_{\mathbf{n}/D}\right) \cong J$.

Because $p \nmid |J|$, all p-subgroups of N map trivially to $\Sigma_{\mathbf{n}/D}$ and monomorphically to $\Sigma_{\mathrm{cl}(\lambda_i)}$. Meanwhile, $f(J)$ stabilizes each element (i.e., equivalence class) in $\mathrm{cl}(\lambda_i)$, and therefore maps trivially to $\Sigma_{\mathrm{cl}(\lambda_i)}$ and monomorphically to $\Sigma_{\mathbf{n}/D}$. Since the images of $f(J)$ and of a p-subgroup of N commute in $\Sigma_{\mathrm{cl}(\lambda_i)} \times \Sigma_{\mathbf{n}/D}$, we know that $f(J)$ actually commutes with p-subgroups of N. Therefore $f(\sigma)$ centralizes all p-subgroups of N.

Lastly, we consider the case $K = \Sigma_n$ itself, when C is the entire centralizer of D in Σ_n, and we must show that either $p \mid [C : D]$ or there exists an involution meeting the requirements of the proposition.

- If $D \neq \{e\}$, then we assert that $p \mid [C : D]$. For let λ_D denote the partition of $\mathbf{n}$ by orbits of D, and let Λ be the one-element chain λ_D. In the notation of Lemma 7.2 we have $S = D \neq \{e\}$ and so $p \mid [C : D]$.
- If $D = \{e\}$, then $C = \Sigma_n$. If $n \geq p$, then $p \mid |C| = [C : D]$ and we are done. If $n < p$, then p is odd. Any transposition in Σ_n satisfies the requirements of the proposition, since $N = \Sigma_n$ has no nontrivial p-subgroups.

□

8. Establishing the Pruning Criterion

We saw in Section 6 that if D is a subgroup of Σ_n such that $(\mathcal{P}_n)^D$ is not contractible, then D is an elementary abelian p-group that acts freely on $\mathbf{n}$. In this section, we introduce two conditions on a Mackey functor Γ for Σ_n which, taken together, will permit the use of Proposition 5.3 to discard these subgroups when they do not act transitively on $\mathbf{n}$.

We use the following notation throughout the section. Given $D \subseteq \Sigma_n$, we write $N_{\Sigma_n}(D)$ for the normalizer of D in Σ_n. It turns out to be useful to look at the embedding $\Sigma_n \hookrightarrow \mathrm{GL}_n \mathbb{R}$ given by permutations of the standard basis. We write $\mathrm{Cen}_{\Sigma_n}(D)$ and $\mathrm{Cen}_{\mathrm{GL}_n \mathbb{R}}(D)$ for the centralizers of D in Σ_n and $\mathrm{GL}_n \mathbb{R}$, respectively. We are specifically interested in subgroups D of isotropy groups of the partition complex. Let $K \in \mathrm{Iso}\,(\mathcal{P}_n) \cup \{\Sigma_n\}$, and suppose $D \subseteq K$. Let N and C denote, respectively, the normalizer and centralizer of D in K. Let $W = N/D$, and let L denote the poset of nontrivial p-subgroups of of W.

Definition 8.1. We say that Γ *satisfies the centralizer condition for D* if the kernel of $\mathrm{Cen}_{\Sigma_n}(D) \to \pi_0 \mathrm{Cen}_{\mathrm{GL}_n \mathbb{R}}(D)$ acts trivially on $\Gamma\,(\Sigma_n/D)$.

In the following definition, "odd involution" means an odd permutation of order 2.

Definition 8.2. Let p be an odd prime. We say that Γ *satisfies the involution condition for D* if any odd involution in $\mathrm{Cen}_{\Sigma_n}(D)$ acts on $\Gamma(\Sigma_n/D)$ by multiplication by -1.

Definition 8.3. We say that the Mackey functor Γ *satisfies the centralizer condition* (resp. *satisfies the involution condition*) if it satisfies the corresponding condition for all elementary abelian p-subgroups of Σ_n that act freely and nontransitively on $\mathbf{n}$.

Our main result in this section is the following proposition.

Proposition 8.4. *Let $D \subseteq \Sigma_n$ be an elementary abelian p-subgroup that acts freely and nontransitively on $\mathbf{n}$, and let Γ be a Mackey functor for Σ_n taking values in $\mathbb{Z}_{(p)}$-modules. Assume that*

- *Γ satisfies the centralizer condition for D, and*
- *if p is odd, Γ satisfies the involution condition for D.*

Then D is $\Gamma(\Sigma_n/D)$-prunable in K for any $K \in \mathrm{Iso}(\mathcal{P}_n) \cup \{\Sigma_n\}$ such that $D \subseteq K$.

Before the proof, we need a lemma from representation theory, which follows immediately from [5, Thm 1.3.4].

Lemma 8.5. *If $D \subseteq \mathrm{GL}_n \mathbb{R}$ is finite, then $\pi_0 \mathrm{Cen}_{\mathrm{GL}_n \mathbb{R}}(D)$ is an elementary abelian 2-group.*

Proof of Proposition 8.4 for p odd, p dividing $|C/D|$. Pick $x \in C/D$ of order p. If $\tilde{x} \in C$ is an inverse image of x, it is clear from Lemma 8.5 that $\tilde{x}$ belongs to the kernel of $C \to \pi_0 \mathrm{Cen}_{\mathrm{GL}_n \mathbb{R}}(D)$. In view of the centralizer condition, x acts trivially on $\Gamma(\Sigma_n/D)$. The desired conclusion follows from Proposition 5.6. □

Proof of Proposition 8.4 for p odd, p not dividing $|C/D|$. We will show that D is M-prunable in K with $M = \Gamma(\Sigma_n/D)$ by showing that all of the homology and cohomology groups in Definition 5.1 vanish. Let $\overline{C} = C/D$ and recall that $W = N/D$. The short exact sequence

$$1 \to \overline{C} \to W \to N/C \to 1 \tag{8.6}$$

shows that the maps of Definition 5.1 are induced by

$$(L_{h\overline{C}})_{h(N/C)} \to \left(B\overline{C}\right)_{h(N/C)}.$$

The Serre spectral sequence shows that it is enough to prove that the local coefficient groups $H_*(L_{h\overline{C}}; M)$, $H_*(B\overline{C}; M)$, $H^*(L_{h\overline{C}}; M)$, and $H^*(B\overline{C}; M)$ all vanish. We will handle the case $H_*(L_{h\overline{C}}; M)$; the others are similar. By Proposition 7.1, there exists an odd involution $\tau \in C$ that centralizes each of the p-subgroups of N, and τ projects to an involution $\overline{\tau} \in \overline{C}$. The element $\overline{\tau}$ acts trivially on the space L and, in view of the involution condition, acts by -1 on $\Gamma(\Sigma_n/D)$. Consider the Serre spectral sequence of

$$L \to L_{h\overline{C}} \to B\overline{C}\,.$$

Since M is a p-local and $\overline{C}$ has order prime to p, we know that $E^2_{i,j} = 0$ for $i > 0$, while the group $E^2_{0,j}$ is given by the coinvariants of the action of $\overline{C}$ on $H_j(L; M)$. However, $\overline{\tau} \in \overline{C}$ acts trivially on L and acts on M by -1, so $\overline{\tau}$ acts on $H_j(L; M)$ by -1. Since the coinvariants of this $\overline{\tau}$-action vanish, the groups $H_0\left(\overline{C}; H_j(L; M)\right)$ vanish for all j, and the Serre spectral sequence collapses to zero at E^2. □

The remainder of the proof of Proposition 8.4, for $p = 2$, requires two lemmas. In the statements, $L(G)$ denotes the nerve of the poset of nontrivial p-subgroups of a finite group G. As usual, the group G acts on $L(G)$ by conjugation.

Lemma 8.7. *Let G be a finite group with a normal subgroup H of order prime to p. Then $L(G)/H$ is isomorphic to $L(G/H)$.*

Proof. It is easy to see that the natural map $L(G) \to L(G/H)$ is surjective. We first check the desired isomorphism at the level of vertices of $L(G)/H$. Let Q be a p-subgroup of G/H and $\{Q_1, ..., Q_l\}$ the set of p-subgroups of G that project to Q; it is necessary to show that H acts transitively by conjugation on $\{Q_1, ..., Q_l\}$. Let $\tilde{Q}$ be the inverse image of Q in G. All of the subgroups $Q_1, ..., Q_l$ are contained in $\tilde{Q}$, and counting dictates that each of them is a Sylow p-subgroup of $\tilde{Q}$, so $\tilde{Q}$ itself acts transitively on $\{Q_1, ..., Q_l\}$. However, every element of $\tilde{Q}$ can be written (uniquely) as a product hx where $h \in H$ and $x \in Q_1$. This implies that the set of $\tilde{Q}$-conjugates of Q_1 is the same as the set of H-conjugates of Q_1, and hence that H acts transitively on $\{Q_1, ..., Q_l\}$.

To check 1-simplices, suppose that $P_1 \subseteq Q_1$ and $P_2 \subseteq Q_2$ are 1-simplices of $L(G)$ that project to the same simplex $P \subseteq Q$ in $L(G/H)$. As above, pick $h \in H$ that conjugates Q_1 to Q_2. Then h conjugates P_1 to a subgroup of Q_2 that projects in G/H to Q. Since the projection map $Q_2 \to Q$ is an isomorphism, it must be the case that h conjugates P_1 to P_2. This shows that H acts transitively on the 1-simplices of $L(G)$ that project to $P \subseteq Q$. The extension to higher simplices is similar. □

Lemma 8.8. *If G is a finite group with a nontrivial normal p-subgroup, then $L(G)$ is contractible.*

Proof. Let Z be a nontrivial normal p-subgroup of G. For any p-subgroup P of G there is a natural zig-zag of inclusions $P \to PZ \leftarrow Z$. This gives gives a chain of natural transformations between the identity functor on the category of nontrivial p-subgroups of G and the constant functor with value Z. □

Let C_0 denote the subgroup of C generated by D and the kernel of the map $C \to \pi_0 \operatorname{GL}_n \mathbb{R}$. It is clear from Lemma 8.5 that C/C_0 is an elementary abelian 2-group.

Proof of Proposition 8.4 for $p = 2$, p dividing $|C_0/D|$. This follows by the same reasoning as in the case p odd, p dividing $|C/D|$. □

Proof of Proposition 8.4 for $p = 2$, p not dividing $|C_0/D|$. By Proposition 7.1, p divides $|C/D|$, so under the assumption $p \nmid |C_0/D|$, it must be the case that $p \mid |C/C_0|$ and C/C_0 is nontrivial. Observe that C_0 is a normal subgroup of N, because it is generated by the normal subgroup D and the normal subgroup obtained by intersecting C with the kernel of $N \to \pi_0 \operatorname{GL}_n \mathbb{R}$. It follows that C/C_0 is a nontrivial normal 2-subgroup of N/C_0.

As usual, let L denote the nerve of the poset of nontrivial p-subgroups of W. We must show that $L_{hW} \to BW$ induces isomorphisms on twisted M-homology and M-cohomology, where $M = \Gamma(\Sigma_n/D)$. Let $\overline{C}_0$ denote C_0/D. As in (8.6), we have a short exact sequence

$$1 \to \overline{C}_0 \to W \to N/C_0 \to 1$$

and a Serre spectral sequence argument like that following (8.6) establishes that we need only show that the map $L_{h\overline{C}_0} \to B\overline{C}_0$ induces an isomorphism on M-homology and M-cohomology. Further, because of the definition of C_0 and the assumption that Γ satisfies the centralizer condition, the action of $\overline{C}_0$ on M is trivial, so we have untwisted coefficients. Consider the following commutative diagram comparing homotopy orbits to strict orbits:

$$\begin{array}{ccc} L_{h\overline{C}_0} & \longrightarrow & (*)_{h\overline{C}_0} \\ \downarrow & & \downarrow \\ L/\overline{C}_0 & \longrightarrow & * \end{array} .$$

Since M is p-local, it is enough to show that all of the maps in this diagram are p-local equivalences. By Lemma 8.7, the orbit space $L/\overline{C}_0$ is isomorphic to the nerve of the poset of nontrivial p-subgroups of N/C_0, and that poset is weakly contractible (Lemma 8.8), because N/C_0 has the nontrivial normal p-subgroup C/C_0. Thus the lower horizontal map is an equivalence. The right vertical map is a p-local equivalence because the order of $\overline{C}_0$ is prime to p. In the same way the isotropy groups of the action of $\overline{C}_0$ on L are all of order prime to p, and so the isotropy spectral sequence of this action [7, 2.4] shows that the left vertical map is a p-local equivalence as well. □

9. Results of approximating

In this section, we assemble the results from previous sections to establish the two main theorems announced in the introduction and restated below.

Theorem 1.1. *Suppose that Γ is a p-constrained Mackey functor for Σ_n and that Γ satisfies the centralizer condition and (if p is odd) the involution condition. If n is not a power of p, the groups $\tilde{H}_*^{Br}(\mathcal{P}_n^\diamond;\Gamma)$ and $\tilde{H}^*_{Br}(\mathcal{P}_n^\diamond;\Gamma)$ vanish. If $n = p^k$, then the map*

$$\Sigma_{n+} \wedge_{\mathrm{Aff}_k} (E\,\mathrm{GL}_{k\,+} \wedge B_k^\diamond) \longrightarrow \mathcal{P}_n^\diamond$$

is a Γ_*^*-isomorphism.

Theorem 1.2. *In the setting of Theorem 1.1, suppose that $n = p^k$. Then $\tilde{H}_j^{\mathrm{Br}}(\mathcal{P}_n^\diamond;\Gamma)$ and $\tilde{H}^j_{\mathrm{Br}}(\mathcal{P}_n^\diamond;\Gamma)$ vanish unless $j = k$. Moreover, with $M = \Gamma(\Sigma_n/\Delta_k)$ there are isomorphisms*

$$H_k^{\mathrm{Br}}(\mathcal{P}_n^\diamond;\Gamma) \cong \mathrm{St}_k \otimes_R M$$
$$H^k_{\mathrm{Br}}(\mathcal{P}_n^\diamond;\Gamma) \cong \mathrm{Hom}_R(\mathrm{St}_k, M)\,.$$

First we need a small lemma. Let G be a finite group, and let $\mathbf{G}$ denote the underlying set of G. The group G acts on $\mathbf{G}$ by left translation, and hence on the poset $\mathcal{P}(\mathbf{G})$ of nontrivial, proper partitions of $\mathbf{G}$.

Lemma 9.1. *The fixed point set $\mathcal{P}(\mathbf{G})^G$ is canonically isomorphic to the nerve of the poset of proper, nontrivial subgroups of G.*

Proof. The isomorphism is obtained by taking each G-fixed equivalence relation on $\mathbf{G}$ and assigning to it the subgroup of G represented by the equivalence class of the identity element. □

Proof of Theorem 1.1. Recall that $\mathcal{A}$ denotes the collection of all p-subgroups of Σ_n. Let $\mathcal{C}$ consist of all p-subgroups of Σ_n except the elementary abelian p-subgroups that act freely on $\mathbf{n}$, and $\mathcal{D}$ the collection containing all elementary abelian p-subgroups of Σ_n that act freely and transitively on $\mathbf{n}$. The collection $\mathcal{D}$ is empty unless $n = p^k$, in which case it consists entirely of conjugates of the subgroup Δ_k. Certainly $\mathcal{C}$ is closed under passage to p-supergroups, and $\mathcal{D}$ is initial in $\mathcal{C} \cup \mathcal{D}$. Let $X = \mathcal{P}_n$, and consider the commutative diagram of Σ_n-spaces

$$\begin{array}{ccccc} X_{\mathcal{C}\cup\mathcal{D}} & \longrightarrow & X_{\mathcal{A}} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ (*)_{\mathcal{C}\cup\mathcal{D}} & \longrightarrow & (*)_{\mathcal{A}} & \longrightarrow & * \end{array} . \tag{9.2}$$

By Proposition 4.6, the horizontal arrows in the right-hand square are Γ_*^*-isomorphisms because Γ is p-constrained. By Lemma 3.8, Γ takes values in $\mathbb{Z}_{(p)}$-modules, so Propositions 5.3 and 8.4 imply that the horizontal arrows in the left-hand square are Γ_*^*-isomorphisms. If $n \neq p^k$, then $\mathcal{D}$ is empty, while Proposition 6.1 tells us that the left vertical map is an equivalence on fixed point sets of subgroups in $\mathcal{C}$, hence a Σ_n-equivalence and a therefore Γ_*^*-isomorphism. This gives Theorem 1.1 for $n \neq p^k$.

Suppose $n = p^k$, so that $\mathcal{D}$ consists of conjugates of Δ_k. We can use the explicit formula of Lemma 4.4 to give a formula for $X_{\mathcal{D}}$ once we know X^{Δ_k}. As a Δ_k-set, $\mathbf{n}$ is isomorphic to Δ_k acting on itself by left translation, so by Lemma 9.1, we find that $X^{\Delta_k} \cong B_k$. Lemma 5.4 with $Y = \mathcal{P}_n$ and $Z = *$ tells us that the left square of the following diagram is a homotopy pushout:

$$\begin{array}{ccccc} \Sigma_n \times_{\mathrm{Aff}_k} (E\,\mathrm{GL}_k \times B_k) & \longrightarrow & X_{\mathcal{C}\cup\mathcal{D}} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma_n \times_{\mathrm{Aff}_k} (E\,\mathrm{GL}_k) & \longrightarrow & (*)_{\mathcal{C}\cup\mathcal{D}} & \longrightarrow & * \end{array} .$$

Taking cofibers vertically gives us the result. □

For the next lemma, suppose that G is a finite group, and that R is the ring $\mathbb{Z}_{(p)}[G]$. Let $\overline{R}$ denote the quotient ring $\mathbb{F}_p[G]$.

Lemma 9.3. *Suppose that M is a finitely-generated R-module which is p-torsion free and has the property that $\overline{M} = M/pM$ is a projective $\overline{R}$-module. Then M is a projective R-module.*

Proof. Let $\overline{N}$ be any $\overline{R}$-module. Since M is p-torsion free, there are change of rings isomorphisms

$$\mathrm{Ext}^i_R(M, \overline{N}) \cong \mathrm{Ext}^i_{\overline{R}}(\overline{M}, \overline{N}) .$$

Since $\overline{M}$ is projective over $\overline{R}$, it follows that $\mathrm{Ext}^i_R(M, \overline{N})$ vanishes for $i \geq 1$. Now suppose that N is a finitely-generated R-module which is p-torsion free, and let $\overline{N} = N/pN$. Applying $\mathrm{Ext}^*_R(M, -)$ to the short exact sequence

$$0 \to N \xrightarrow{p} N \to \overline{N} \to 0$$

gives an exact sequence

$$\cdots \to \operatorname{Ext}^1_R(M,N) \xrightarrow{p} \operatorname{Ext}^1_R(M,N) \to \operatorname{Ext}^1_R(M,\overline{N}) \to \cdots .$$

But it is easy to see that $\operatorname{Ext}^1_R(M,N)$ is a finitely-generated module over $\mathbb{Z}_{(p)}$, and, since the group on the right of the above sequence vanishes, multiplication by p gives a surjective map on this module. The structure theorem for finitely-generated modules over a PID gives $\operatorname{Ext}^1_R(M,N) = 0$. Now choose a surjection $f\colon F \to M$, where F is a finitely-generated free module over R. The above considerations show that $\operatorname{Ext}^1_R(M,N) = 0$ for $N = \ker(f)$, which implies that f has a section and that M is projective. □

Remark 9.4. For example, in the notation of Theorem 1.2, the module St_k is a projective R-module, because its reduction modulo p is the usual Steinberg module over $\mathbb{F}_p[GL_k]$. For more details, see [1, §6].

Proof of Theorem 1.2. Let $G = \mathrm{GL}_k$, and $R = \mathbb{Z}_{(p)}[G]$, and let $Y = \Sigma_{n+} \wedge_{\mathrm{Aff}_k} (EG_+ \wedge B_k^\diamond)$. All of the isotropy subgroups of Y are conjugate to Δ_k, so the calculation of Lemma 4.4 tells us that there are isomorphisms

$$\begin{aligned}\tilde{H}^{\mathrm{Br}}_*(Y;\Gamma) &\cong \tilde{H}_*((B_k^\diamond)_{\tilde{h}G}; M)\\ \tilde{H}^*_{\mathrm{Br}}(Y;\Gamma) &\cong \tilde{H}^*((B_k^\diamond)_{\tilde{h}G}; M)\,.\end{aligned}$$

Consider the local coefficient homology Serre spectral sequence

$$E^2_{i,j} = \operatorname{Tor}^R_i(\tilde{H}_j(B_k^\diamond;\mathbb{Z}_{(p)}), M) \Rightarrow H_*((B_k^\diamond)_{\tilde{h}G}; M)\,.$$

This collapses at E^2 to give the desired homology calculation, because $\tilde{H}_j(B_k^\diamond;\mathbb{Z}_{(p)})$ vanishes for $j \neq k$, and, by Remark 9.4, gives a projective R-module for $j = k$. Again, see [1, §6] for a more detailed account. A similar calculation with a local coefficient cohomology Serre spectral sequence completes the proof. □

10. Our coefficient functors

In this section we describe some particular Mackey functors for Σ_n, and show that they are p-constrained and satisfy the involution and centralizer conditions (Definitions 8.2 and 8.3). It follows that Theorems 1.1 and 1.2 apply to these functors.

Recall that Σ_n acts on the one-point compactification S^n of $\mathbb{R}^n$ by permuting coordinates, and hence on the j-fold smash product S^{nj}. For us, a functor from spectra to spectra is *additive* if it respects equivalences and preserves finite coproducts up to equivalence.

Definition 10.1. Suppose that j is a fixed integer, with j odd if p is odd, and that F is an additive functor from spectra to spectra. For each finite Σ_n-set T, define the graded abelian group $\Gamma_F(T)$ by

$$\Gamma_F(T) = \pi_* F\left((\Sigma^\infty T_+ \wedge S^{nj})_{\tilde{h}\Sigma_n}\right) .$$

Our main result is Proposition 10.2, below. For the statement, a functor Γ from $\omega(\Sigma_n)$ to graded abelian groups is said to be a Mackey functor if each graded constituent is a Mackey functor in the usual sense. Similarly, Γ satisfies the centralizer condition, etc., if each graded constituent does. Let $L_{(p)}$ denote the functor on spectra given by localization at p, which has the effect of tensoring homotopy groups with $\mathbb{Z}_{(p)}$.

Proposition 10.2. *The assignment $T \mapsto \Gamma_F(T)$ extends naturally to a Mackey functor for Σ_n that satisfies the centralizer condition. If $F \to F \circ L_{(p)}$ is an equivalence, then Γ_F is p-constrained and (if p is odd) Γ_F satisfies the involution condition.*

Example 10.3. If $F(X) = H\mathbb{F}_p \wedge X$, then $\Gamma_F(T)$ is the $\mathbb{F}_p$-homology of the relevant reduced Borel construction and in this case Proposition 10.2 and Theorem 1.1 together give a relatively conceptual approach to the the homological calculations of Arone-Mahowald in [2].

Example 10.4. One advantage of our approach is that it applies in situations in which explicit calculation is impossible, e.g., when $F = L_{(p)}$. Here $\Gamma_F(T)$ is the p-local *stable homotopy* of the relevant Borel construction. The calculation of Bredon homology in this case provides a key ingredient for a new proof of some theorems of Kuhn [10] and Kuhn-Priddy [11] on the Whitehead conjecture. The Bredon cohomology relates to work of Behrens [3] at the prime 2 on the collapse of the homotopy spectral sequence of the Goodwillie tower of the identity functor evaluated at S^1, though our Bredon cohomology results also apply at odd primes. We intend to pursue this in another paper.

Example 10.5. Another interesting example to which our results apply is the functor

$$F(X) = (E \wedge X)_K.$$

Here E is the Morava E-theory and the subscript $_K$ denotes localization with respect to Morava K theory. This example and others similar to it were considered recently by Rezk [15] and Behrens [4]. It seems that our methods can be used to recover some of their calculations. For example, Lemma 5.6 of [4] seems to be closely related to our main theorem, applied to the functor F above.

The proof of Proposition 10.2 will appear after some preparation. Suppose that G is a finite group. Given a finite G-set S, let $\Theta(S) = \Sigma^\infty S_+$.

Lemma 10.6. *The assignment $S \mapsto \Theta(S)$ extends naturally to an additive functor from $\omega(G)$ to the homotopy category of spectra with an action of G.*

Remark. The homotopy category in question involves taking the category of spectra with an action of G and localizing with respect to equivariant maps that are equivalences of the underlying nonequivariant spectra. As usual, a functor to the homotopy category of spectra is additive if it preserves coproducts.

Proof of Lemma 10.6. Let $f = (S \xleftarrow{\alpha} V \xrightarrow{u} T)$ be a morphism in $\omega(G)$. We first define the stable transfer map $\alpha_!\colon \Theta(S) \to \Theta(V)$ associated with the finite covering α. The stable transfer map depends on a well-known natural stable equivalence $\Theta(T) \xrightarrow{\simeq} \Theta(T)^\vee$, where the superscript $^\vee$ denotes Spanier-Whitehead dual. This map is defined when T is a finite set, and it is natural with respect to isomorphisms of T. In particular, if T is a set with an action of G, then it is a G-equivariant map. $\alpha_!$ is defined to be the composite map

$$\alpha_!\colon \Theta(S) \xrightarrow{\simeq} \Theta(S)^\vee \xrightarrow{\alpha^\vee} \Theta(V)^\vee \xleftarrow{\simeq} \Theta(V).$$

$\alpha_!$ is a well-defined map in the homotopy category of spectra. If α is a G-map, then $\alpha_!$ is a well-defined map in the homotopy category of spectra with an action of G. Next, the map $u : V \to T$ directly induces a map $u_*\colon \Theta(V) \to \Theta(T)$. The map $\Theta(f)\colon \Theta(S) \to \Theta(T)$ is defined to be the composite $u_*\alpha_!$. Additivity is clear. Functoriality follows from the standard fact that given a pullback diagram on the left of finite G-sets, the diagram on the right commutes in the homotopy category of spectra with an action G.

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ C & \xrightarrow{v} & D \end{array} \qquad\qquad \begin{array}{ccc} \Theta(A) & \xrightarrow{u_*} & \Theta(B) \\ {\scriptstyle\alpha_!}\uparrow & & \uparrow{\scriptstyle\beta_!} \\ \Theta(C) & \xrightarrow{v_*} & \Theta(D) \end{array}.$$

∎

Lemma 10.7. *Let $u : T \to S$ be a map of finite G-sets for which the fibers have cardinality prime to p. Let $f = \left(S \xleftarrow{u} T \xrightarrow{u} S\right)$ as a map in $\omega(G)$. The $\Theta(f)$ is a p-local equivalence.*

Proof. The assertion is that $\Theta(f)\colon \Theta(S) \to \Theta(S)$, which is a G-map, is a p-local equivalence of underlying spectra. The map $\Theta(f)$ can be written as a sum indexed by the G-orbits in S, so it is enough to consider the case in which S is a transitive G-set and all of the fibers of $T \to S$ have a single cardinality m with m prime to p. In this case, the standard transfer arguments show that $\Theta(f) = u_* u_!$ gives a self-map of $\Theta(S)$ which is multiplication by m on integral homology, and the result follows. □

In order to verify the centralizer and involution conditions for Γ_F, we work unstably to understand the G-action on an orbit space (in our case, a Borel construction). Suppose that Y is a left G-space, that D is a subgroup of G with normalizer N, and that we let G act diagonally on the left of $G/D \times Y$. There is an isomorphism

$$(G/D \times Y)/G \xrightarrow{\cong} Y/D \tag{10.8}$$

given by taking the equivalence class $\langle (gD, y) \rangle$ to the equivalence class $\langle g^{-1}y \rangle$. Recall that N acts on G/D by G-maps, so the left side of (10.8) has an action of N by $n \cdot \langle (gD, y)\rangle = \langle (gn^{-1}D, y)\rangle$. The right side has an action by $n \cdot \langle y \rangle = \langle ny \rangle$, and (10.8) is an isomorphism of N-spaces.

We actually want to apply this when X is a pointed G-space and $Y = EG \times X$ with the diagonal G-action. In this case, we find an isomorphism of N-spaces $(G/D \times X)_{hG} \xrightarrow{\cong} X_{hD}$, and reducing and taking suspension spectra gives

$$\left(\Sigma^\infty (G/D_+) \wedge X\right)_{\tilde{h}G} \xrightarrow{\cong} \Sigma^\infty X_{\tilde{h}D}, \tag{10.9}$$

where N acts on the domain through its action on G/D.

Proof of Proposition 10.2. First we verify that Γ_F is a p-constrained Mackey functor for Σ_n. We define an additive functor from spectra with a Σ_n-action to spectra by

$$\beta(Z) = (Z \wedge S^{nj})_{\tilde{h}\Sigma_n}.$$

It follows from Lemma 10.6 that $F \circ \beta \circ \Theta$ is an additive functor from $\omega(\Sigma_n)$ to the homotopy category of spectra. Applying π_* to obtain Γ_F yields a additive functor from $\omega(\Sigma_n)$ to graded abelian groups, i.e., a Mackey functor.

Next we consider the centralizer condition. Let D be a subgroup of Σ_n, and let C be the centralizer of D in Σ_n. The action of C on S^{nj} is by D-equivariant maps, and hence passes to an action of C on $(S^{nj})_{\tilde{h}D}$. If an element $c \in C$, regarded as an element of $\mathrm{GL}_n\,\mathbb{R}$, is in the identity component of the centralizer of D in $\mathrm{GL}_n\,\mathbb{R}$, then the action of c on S^{nj} is homotopic to the identity through D-equivariant maps. Hence c induces a self-map of $(S^{nj})_{\tilde{h}D}$ that is homotopic to the identity. However, according to the discussion following (10.8), the action of C on $\Gamma_F(G/D)$ referenced in Definition 8.1 is the same as that induced by the action of C on $\beta(\Theta(G/D)) \sim (S^{nj})_{\tilde{h}D}$ from the action of C on S^{nj}. It follows that Γ_F satisfies the centralizer condition.

Suppose that $F \circ L_{(p)} \sim F$, and let f be a morphism of $\omega(G)$ determined as in Definition 3.6 by a covering of finite Σ_n-sets with fibers of cardinality prime to p. According to Lemma 10.7, $\Theta(f)$ is a p-local equivalence, and it follows immediately that $\beta(\Theta(f))$ is a p-local equivalence. Hence $L_{(p)}(\beta(\Theta(f)))$ is an equivalence, whence $\Gamma_F(f)$ is an isomorphism and Γ_F is p-constrained.

Finally, suppose that $F \circ L_{(p)} \sim F$, that p is odd, that D and C are as above, and that τ is an odd involution in C. Say that a spectrum X is *localized away from* 2 if the composite $X \to X \vee X \to X$ of the pinch map followed by the fold map is a self-equivalence of X. This is equivalent to $\pi_* X$ being a module over $\mathbb{Z}[1/2]$. If $t : X \to X$ is an involution on such an X, say that t is *of odd type* if $t-1 : X \to X$ is an equivalence, a condition equivalent to t acting by -1 on $\pi_* X$. It is clear that an additive functor from spectra to spectra preserves both spectra localized away from 2 and involutions of odd type on such spectra. As above, the action of τ on $\Gamma_F(\Sigma_n/D)$ which figures in Definition 8.2 is induced by the action of τ on S^{nj}. Since τ is an odd involution, τ acts by a map of degree -1 on S^n and hence, since j is odd, by a map of degree -1 on S^{nj}. Consequently, τ is an involution of odd type on $L_{(p)}\Sigma^\infty S^{nj}$. By additivity, τ gives an involution of odd type on

$$(L_{(p)}\Sigma^\infty S^{nj})_{\tilde{h}D} \sim L_{(p)}(\Sigma^\infty S^{nj}_{\tilde{h}D}) \sim L_{(p)} \circ \beta \circ \Theta(G/D)\,.$$

Again by additivity, applying F preserves the odd type property of τ, and taking π_* then shows that Γ_F satisfies the involution condition. □

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