

Cartan Involutions and Normalizers of Maximal Tori

William G. Dwyer and Clarence W. Wilkerson, Jr.

*Dedicated to Brooke Shipley and Kevin Corlette
on the occasion of their wedding, Bastille Day, 2001.*

ABSTRACT. One consequence of Tits' well known work [11] on the structure of the normalizer of the maximal torus in a connected compact Lie group is that twice the k -invariant classifying the extension

$$\{e\} \rightarrow T_G \rightarrow N_G(T_G) \rightarrow W(G) \rightarrow \{e\}$$

is zero. In this note we observe that this conclusion follows directly from the existence of an unstable Adams map of type Ψ^{-1} on the classifying space BG . Work from the 1970's using etale methods or more recent diagrammatic methods produce a Ψ^α self-map of BG whenever α is relatively prime to the order of $W(G)$, so the k -invariant bound follows. However, the Lie algebra version of Ψ^{-1} (the Cartan involution) is classical. This note discusses the Cartan involution, and shows how for a connected compact Lie group it gives rise to a self map of type Ψ^{-1} .

Analogues of $\{\Psi^{-1}\}$ are not known for the general 2-compact group context of Dwyer-Wilkerson [4]. While this could be a possible divergence point for 2-compact group theory from classical Lie theory, the authors speculate that it is not.

1. Introduction

This paper springs from two decades of fascination with Tits' paper [11] and related work of Curtis-Wiederholf-Williams,[3]. Last year we

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were led to ask Kevin Corlette a rather ill-formed question :

If the compact Lie group G is a real form of the complex Lie group $G_{\mathbb{C}}$, then what sort of automorphisms of a maximal torus of G are induced from the complex conjugation automorphisms of $G_{\mathbb{C}}$?

Kevin pointed us to Helgason's book [7]. Chapter IX of this monograph carries out Elie Cartan's program for the classification of irreducible symmetric spaces. One step of this program establishes a correspondence between involutions on the complex Lie group or algebra and various real forms of the complex Lie group or algebra. In these terms, the "right answer" to our naive question is that for a given complex semisimple Lie algebra \mathfrak{g} the complex conjugation $\tau_{\mathfrak{g}_0}$ relative to a normal real Lie subalgebra \mathfrak{g}_0 maps some compact real Lie subalgebra \mathfrak{k} into itself. Promotion of this to the connected compact Lie group level justifies our main observation:

THEOREM 1.1. *Let G be a connected compact Lie group of rank l . There exists a maximal torus T_G and an Lie group automorphism Φ of G such that*

- (a) $\Phi(T_G) \subseteq T_G$ and
- (b) $\Phi|_{T_G}$ is the inverse homomorphism of T_G .
- (c) Φ induces the identity map on $W(G) = N_G(T_G)/T_G$.

Moreover, any two automorphisms of this type differ by an inner automorphism of G .

To illustrate, consider $GL_n(\mathbb{C})$. This has $GL_n(\mathbb{R})$ as a real form. Coordinate-wise complex conjugation pointwise fixes $GL_n(\mathbb{R})$. It also takes $U(n)$ into itself and induces the involution Φ of 1.1 . On the other hand, the transpose-inverse map on $GL_n(\mathbb{C})$ restricts to the same self-map of $U(n)$. Since these approaches depend on coordinate-wise descriptions, their analogues for general compact connected G are not immediate.

COROLLARY 1.2. *Let G be a connected compact Lie group of rank l with maximal torus T_G and Φ as in 1.1 . Then*

- (a) *There is a commutative diagram*

$$\begin{array}{ccccccccc}
 \{e\} & \longrightarrow & T_G & \longrightarrow & N_G(T_G) & \longrightarrow & W(G) & \longrightarrow & \{e\} \\
 \uparrow & & \uparrow i & & \uparrow j & & \uparrow id & & \uparrow \\
 \{e\} & \longrightarrow & (Z/2Z)^l & \longrightarrow & W^{ext}(G) & \longrightarrow & W(G) & \longrightarrow & \{e\} \\
 \uparrow & & \uparrow = & & \uparrow = & & \uparrow = & & \uparrow \\
 \{e\} & \longrightarrow & (T_G)^\Phi & \longrightarrow & (N_G(T_G))^\Phi & \longrightarrow & W(G)^\Phi & \longrightarrow & \{e\}
 \end{array}$$

Here $W^{ext}(G)$ (the “extended Weyl group”) is a finite subgroup of $N_G(T_G)$ of order $2^l \times |W(G)|$.

(b) The k -invariant for the extension

$$\{e\} \rightarrow T_G \rightarrow N_G(T_G) \rightarrow W(G) \rightarrow \{e\}$$

has order dividing two in $H^2(W(G), T_G) = H^3(W(G), \pi_1(T_G))$.

(c) Let $W_p(G)$ be the Sylow- p subgroup of $W(G)$ and $N_{p,G}(T_G)$ be its preimage in $N_G(T_G)$. Then if p is an odd prime,

$$\{e\} \rightarrow T_G \rightarrow N_{p,G}(T_G) \rightarrow W_p(G) \rightarrow \{e\}$$

is a split exact sequence.

The article of Tits provides a group presentation for the finite group $W^{ext}(G)$ as well as an explicit formula for the k -invariant. Moreover, that work is carried out for linear algebraic groups over fields of arbitrary characteristic. The methods of the present note are particular to the compact connected Lie case and do not lead to explicit formulas or results for linear algebraic groups over other fields.

One can restate parts of the corollary in terms of classifying spaces. A self-map Ψ^α of the classifying space BG is said to be an Adams map of type α if there is a homotopy commutative diagram

$$\begin{array}{ccc} BG & \xrightarrow{\Psi^\alpha} & BG \\ \uparrow B_i & & \uparrow B_i \\ BT_G & \xrightarrow{B\alpha} & BT_G \end{array}$$

Here $B\alpha$ is the map induced by the α -power-map $\{t \rightarrow t^\alpha\}$ on T_G . Similar definitions apply in the p -compact group setting. In this case, α may be a p -adic unit.

THEOREM 1.3. (a) Let G be a connected compact Lie group. If an Adams map Ψ^{-1} exists as a self homotopy equivalence of BG , then the fibration

$$BT_G \rightarrow BN_G(T_G) \rightarrow BW(G)$$

has k -invariant of order in $H^3(W(G), \pi_1(T_G))$ dividing 2.

(b) Let (BX, X) be a connected p -compact group. If an Adams map Ψ^α exists as a self homotopy equivalence of BX , then the fibration

$$BT_X \rightarrow BN_X(T_X) \rightarrow BW(X)$$

has k -invariant annihilated by $(1 - \alpha)$. In particular, if $p > 2$ and α is a non-trivial root of unity in \mathbb{Z}_p , then the k -invariant is zero in $H^3(W(X), \pi_1(T_X))$.

There is as yet no analogue of 1.1 for 2-compact groups, and hence the analogue of 1.2 is not known to be true in this case. However, in many cases the ambient cohomology group for the k -invariant is directly computable. For example, Dwyer-Wilkerson, [6], show that if the degrees of the polynomial invariants of $W(X)$ are congruent to zero modulo four, then the group $H^3(BW(X), \pi_1(T_X))$ is an elementary abelian 2-group. Likewise, K. Andersen, [1] has shown that the p -primary part of this group is zero for any p -adic reflection group, if p is odd. Such computations can NOT account for all cases of 1.2 – for example, $H^3(BW(E_6), \pi_1(T_{E_6}))$ is known to contain an element of order 4.

We thank Kevin Corlette for answering our awkward questions. We also thank Freydoon Shahidi for pointing out the use of the term “Cartan involution” in [12] to describe the Lie algebra version of Ψ^{-1} .

2. Reduction of 1.1 to the complex semisimple Lie algebra case

PROPOSITION 2.1. *Let G be a connected compact centerfree Lie group. If there exists an automorphism Φ of G such that $\Phi(T_G) \subseteq T_G$ and $\Phi|_{T_G}$ is the inverse map, then Φ lifts to the universal cover G_{sc} of G . There is a maximal torus $T_{G_{sc}}$ of G_{sc} such that the lift restricted to $T_{G_{sc}}$ is the inverse map.*

THEOREM 2.2. *If the above described involution exists for all simply connected compact Lie groups H , then it exists for all connected compact Lie groups.*

LEMMA 2.3. *If G is a connected compact Lie group with maximal torus T_G , then any automorphism Φ of G that induces the inverse map on T_G must induce the identity map on $W(G)$.*

THEOREM 2.4. *Let G be a connected compact centerfree Lie group with Lie algebra \mathfrak{L}_G and maximal torus T_G . Then G is isomorphic to the identity component of $\text{Aut}(\mathfrak{L}_G)$ and the induced map $\varphi : \text{Aut}(\mathfrak{L}_G) \rightarrow \text{Aut}(G)$ is an isomorphism.*

COROLLARY 2.5. *If G is a connected compact centerfree Lie group with maximal torus T_G and σ is any Lie algebra involution of \mathfrak{L}_G which restricts to \mathfrak{L}_{T_G} as multiplication by -1 then there is a Ψ^{-1} involution on G which induces σ . This in turn induces a Ψ^{-1} involution on BG .*

PROOF OF 2.1. The automorphism maps the fundamental group into itself, and hence the automorphism lifts to the universal cover. Let $t \in T_G$ be a topological generator. Let t' be in the inverse image

of t . Consider the topological closure X of the group generated by t' . It is the product of a torus T' and a finite cyclic group A of order N . Thus $(t')^N$ generates T' which maps onto T_G . \square

PROOF OF 2.2. A connected compact Lie group G is the quotient of the product of a torus T and a simply connected semisimple compact Lie group H by a central subgroup C of $T \times H$. Since the involution exists on each of T and H , it exists on the product $T \times H$. The central subgroup C is contained in each maximal torus of $T \times H$. Hence the involution maps C into itself. Therefore the involution is well defined on $G = (T \times H)/C$. \square

PROOF OF 2.3. The homomorphism $W(G) \rightarrow \text{Aut}(T_G)$ is an injection. Let $w \in W(G)$ with lift $n \in N_G(T_G)$. Assume that $\Phi(w) = w'$. For each $t \in T_G$,

$$\begin{aligned} \Phi(w(t)) &= (w(t))^{-1} = \Phi(n)\Phi(t)\Phi(n^{-1}) = \\ \Phi(n)(t^{-1})\Phi(n^{-1}) &= w'(t^{-1}) = (w'(t))^{-1} = (w(t))^{-1}. \end{aligned}$$

That is, $w(t) = w'(t)$ for $t \in T_G$. Hence $w = w'$ in $\text{Aut}(T_G)$ and thus in $W(G)$. \square

PROOF OF 2.4. G acts on G by conjugation. This sends $G_e = \mathfrak{L}_G$ into itself. If $g \in G$ induces the identity map on \mathfrak{L}_G , then conjugation by g is the identity on G , and $g \in \mathcal{Z}(G)$. But since G is center-free, $\mathbf{Adj} : G \rightarrow \text{Aut}(\mathfrak{L}_G)$ is a monomorphism. But it is known that $\text{Aut}(\mathfrak{L}_G)_{id}$ is a Lie group with its Lie algebra isomorphic to that of G . Since G is connected, $\mathbf{Adj}(G)$ falls within the identity component of $\text{Aut}(\mathfrak{L}_G)$. But since $\dim(\text{Aut}(\mathfrak{L}_G)) = \dim(\mathfrak{L}_G) = \dim G$, it must be that $G = \text{Aut}(\mathfrak{L}_G)_{id}$.

In particular there is a SES of Lie groups

$$\{e\} \rightarrow G \rightarrow \text{Aut}(\mathfrak{L}_G) \rightarrow \pi_0(\text{Aut}(\mathfrak{L}_G)) \rightarrow \{e\}.$$

That is, G is a normal subgroup of $\text{Aut}(\mathfrak{L}_G)$, so conjugation within $\text{Aut}(\mathfrak{L}_G)$ induces $\varphi : \text{Aut}(\mathfrak{L}_G) \rightarrow \text{Aut}(G)$. But the composition of φ with the natural map $\text{Aut}(G) \rightarrow \text{Aut}(\mathfrak{L}_G)$ is the identity, so φ is an isomorphism. \square

PROOF OF 2.5. From 2.4, $\varphi : \text{Aut}(\mathfrak{L}_G) \rightarrow \text{Aut}(G)$ is an isomorphism. If σ induces the -1 map on the Cartan subalgebra \mathfrak{L}_{T_G} , then $\exp(\sigma)$ induces the inverse map on T_G . \square

3. The Cartan involution - a Ψ^{-1} for semisimple Lie algebras

The existence of a Ψ^{-1} for complex semisimple Lie algebras is documented in Chapter VI, Corollary 7, of Serre, [9, 10] or Chapter IX of Helgason, [7]. Our expository strategy will be to examine the key example of \mathfrak{sl}_2 in detail before referencing the general results from [9, 10] and [7]. The exposition follows Serre, Chapter VI, [9, 10].

The simple real Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ of trace zero 2×2 real matrices has a familiar presentation. It has a real vector space basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The Lie bracket product is determined by

$$[h, h] = 0, [x, y] = h, [h, x] = 2x, [h, y] = -2y.$$

The complex simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is obtained by taking complex linear combinations of the generators $\{h, x, y\}$ from above. $\mathfrak{sl}_2(\mathbb{R})$ is an example of a *normal real form* of $\mathfrak{sl}_2(\mathbb{C})$, while $\mathfrak{sl}_2(\mathbb{C})$ is the complexification of $\mathfrak{sl}_2(\mathbb{R})$. In particular, one can speak of the complex conjugation automorphism τ on $\mathfrak{sl}_2(\mathbb{C})$ with respect to the real subalgebra $\mathfrak{sl}_2(\mathbb{R})$. Note that τ is linear over \mathbb{R} but not \mathbb{C} .

However, there is another involution σ on $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{R})$. σ is induced from the assignments $\{h \rightarrow -h, x \rightarrow -y, y \rightarrow -x\}$ and hence is linear over complex scalars.

σ corresponds to the transpose-inverse map on SL_2 . Explicitly, let t be a real parameter. Then

$$\exp(tx) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \exp(ty) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \text{ and } \exp(th) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

are one-parameter subgroups in SL_2 , with tangent vectors $\{x, y, h\}$. The action of σ induces

$$\begin{aligned} \exp(t\sigma(x)) &= \exp(-ty) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \\ \exp(t\sigma(y)) &= \exp(-tx) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \\ \exp(t\sigma(h)) &= \exp(-th) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \end{aligned}.$$

Notice that these latter matrices are indeed the transpose-inverse matrices of the first set and have tangent vectors $\{\sigma(x), \sigma(y), \sigma(h)\}$ respectively.

The definition of σ sets the stage for the discovery of a second real form within $\mathfrak{sl}_2(\mathbb{C})$. We use σ to define a second “complex conjugation” on \mathfrak{g} , which has as fixed points the new real form. Define κ to be the composition of σ and τ (these commute). Define \mathfrak{k} to be the fixed point set $\mathfrak{sl}_2(\mathbb{C})^\kappa$. It is apparent that \mathfrak{k} is a real sub-Lie algebra of $\mathfrak{sl}_2(\mathbb{C})$. One explicit basis for \mathfrak{k} is

$$t = ih, u = (x - y), v = i(x + y)$$

or

$$t = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, v = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The Lie brackets are $[t, u] = 2v$, $[t, v] = -2u$, $[u, v] = 2t$. These matrices are skew hermitian and describe the simple real Lie algebra $\mathfrak{su}_2(\mathbb{R})$. $\mathfrak{su}_2(\mathbb{R})$ is an example of a *compact real form* of $\mathfrak{sl}_2(\mathbb{C})$. $\mathfrak{sl}_2(\mathbb{C})$ is the complexification of $\mathfrak{su}_2(\mathbb{R})$ and κ is the complex conjugation on $\mathfrak{sl}_2(\mathbb{C})$ relative to $\mathfrak{su}_2(\mathbb{R})$.

The general semisimple case is modeled on the \mathfrak{sl}_2 example. The philosophy is that \mathfrak{g} is a free Lie product on copies of \mathfrak{sl}_2 's, modulo appropriate relations. The small dimension of \mathfrak{sl}_2 disguises two rather different possible presentations.

Let \mathfrak{g} be a complex semisimple Lie algebra, \mathfrak{h} a choice of Cartan subalgebra. Recall that $\alpha \in \mathfrak{h}^*$ is a **root** if the space $\mathfrak{g}^\alpha = [x \in \mathfrak{g} : [h, x] = \alpha(h)x, \text{ for any } h \in \mathfrak{h}]$ is not zero. The Killing bilinear form $B(\cdot, \cdot)$ defined on \mathfrak{g} as $(x, y) = \text{Trace}(\text{ad}(x)\text{ad}(y))$ restricts to a nondegenerate form on \mathfrak{h} . For each root α , there is a unique **coroot** $h_\alpha \in \mathfrak{h}$ with the property that $\alpha(h) = B(h, h_\alpha)$ for any $h \in \mathfrak{h}$. Let \mathbf{R}^+ be the positive roots and $\mathbf{S} \subset \mathbf{R}^+$ a base. Finally, $n(i, j) = B(h_j, h_i) = \alpha_j(h_i)$ are the Cartan integers. Notice that $n(i, i) = 2 = \alpha_i(h_i)$, but all other Cartan integers are non-positive.

DEFINITION 3.1. A Weyl presentation for \mathfrak{g} relative to \mathfrak{h} and \mathbf{S} is a choice for each root $\alpha_i \in \mathbf{S}$ of a triple of elements $\{h_i, x_i, y_i\}$ from \mathfrak{g} such that the following conditions are satisfied:

(a) *Generators:*

for each $\alpha_i \in \mathbf{S}$ elements such that

- h_i is the coroot for α_i
- x_i is in the root space \mathfrak{g}^{α_i}
- y_i is in the root space $\mathfrak{g}^{-\alpha_i}$

(b) *The Weyl Relations:*

$$\begin{aligned} [x_i, y_i] &= h_i \\ [x_i, y_j] &= 0 \text{ if } i \neq j \\ [h_i, h_j] &= 0 \quad \forall (i, j) \\ [h_i, x_j] &= n(i, j)x_j \\ [h_i, y_j] &= -n(i, j)y_j \end{aligned}$$

(c) *The Ad-Nilpotency relations:*

$$\begin{aligned} \text{ad}(x_i)^{-n(i, j)+1}(x_j) &= 0 \text{ if } i \neq j \\ \text{ad}(y_i)^{-n(i, j)+1}(y_j) &= 0 \text{ if } i \neq j \end{aligned}$$

THEOREM 3.2 (Serre, [9, 10], Theorem 7). *In the above situation, the quotient of the free complex Lie algebra on abstract generators $\{H_i, X_i, Y_i\}_{\alpha_i \in \mathbf{S}}$ by the two sided Lie ideal generated by the Weyl and nilpotency relations is isomorphic to \mathfrak{g} . The map is the extension of $\{H_i \rightarrow h_i, X_i \rightarrow x_i, Y_i \rightarrow y_i\}_{\alpha_i \in \mathbf{S}}$.*

Notice in the theorem above, that although the roots, Killing form, etc. are used to describe the relations, they are not part of the formal structure in the free Lie algebra or the ideal of relations.

COROLLARY 3.3 (Serre, [9, 10], Corollary 7). *Given a Weyl presentation of \mathfrak{g} , the assignment*

$$h_i \rightarrow -h_i, \quad x_i \rightarrow -y_i, \quad y_i \rightarrow -x_i, \quad \alpha_i \in \mathbf{S}$$

has a unique extension to an automorphism σ of \mathfrak{g} and $\sigma^2 = \text{Id}_{\mathfrak{g}}$.

Define $\mathfrak{g}(\mathbb{R})$ to be the real span of the Lie monomials in the variables $\{h_i, x_i, y_i\}_{\alpha_i \in \mathbf{S}}$. In Helgason's terminology, [7], this corresponds to a *normal real form* of \mathfrak{g} . In this case, $\mathfrak{g}(\mathbb{R}) = \mathfrak{g}_0 = \mathfrak{g}^\tau$, where τ is the apparent complex conjugation.

The *compact real form* of \mathfrak{g} is more easily described using the Chevalley presentation of \mathfrak{g} . This uses more generators, but the relations are simpler:

Let \mathfrak{g} be a complex semisimple Lie algebra and \mathfrak{h} a Cartan subalgebra, with associated roots $\mathbf{R} \subset \mathfrak{h}^*$.

DEFINITION 3.4. A Chevalley presentation for \mathfrak{g} relative to \mathfrak{h} and \mathbf{R}^+ is a choice for each positive root α of elements $\{h_\alpha, x_\alpha, x_{-\alpha}\}_{\alpha \in \mathbf{R}^+}$ such that

- (a) $h_\alpha \in \mathfrak{h}$ is the coroot for α , $x_\alpha \in \mathfrak{g}^\alpha$, and $x_{-\alpha} \in \mathfrak{g}^{-\alpha}$
- (b) \mathfrak{g} is spanned as a complex vector space by $\{h_\alpha\}_{\alpha \in \mathbf{R}^+}$ and $\{x_\alpha, x_{-\alpha}\}_{\alpha \in \mathbf{R}^+}$.

- (c) $[x_\alpha, x_{-\alpha}] = h_\alpha$ for each $\alpha \in \mathbf{R}^+$.
- (d) $[h_\alpha, x_\alpha] = 2x_\alpha$ and $[h_\alpha, x_{-\alpha}] = -2x_{-\alpha}$
- (e) the structure constants $N_{\alpha,\beta}$ defined by $[x_\alpha, x_\beta] = N_{\alpha,\beta}x_{\alpha+\beta}$ are real numbers which are zero if $\alpha + \beta$ is not a root. Moreover, they should satisfy the relations $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$.

THEOREM 3.5. *For \mathfrak{g} a complex Lie algebra and any choice of Cartan subalgebra \mathfrak{h} , there exists a Chevalley presentation.*

In fact, Chevalley, [2], produces a choice of $\{x_\alpha\}$ for which the associated structure constants are integers derivable from the chains of roots.

Given a Chevalley presentation and a base $\mathbf{S} \subset \mathbf{R}^+$, one can produce a Weyl presentation by using the choices $\{h_\alpha, x_\alpha, y_\alpha = x_{-\alpha}\}$ for $\alpha \in \mathbf{S}$. We call this the induced Weyl presentation.

COROLLARY 3.6. *Given a Chevalley presentation of $(\mathfrak{g}, \mathfrak{h})$, and a choice of base \mathbf{S} , define σ using the induced Weyl presentation. Then $\sigma(h) = -h$ for $h \in \mathfrak{h}$ and $\sigma(x_\alpha) = -x_{-\alpha}$ for each root α .*

COROLLARY 3.7. *Given a Chevalley presentation of \mathfrak{g} and a choice of base $\mathbf{S} \subset \mathbb{R}$, then $\text{Span}_{\mathbb{R}}(\{h_\alpha, x_\alpha, x_{-\alpha}\}_{\mathbf{R}^+})$ is the $\mathfrak{g}(\mathbb{R})$ defined relative to the induced Weyl presentation.*

COROLLARY 3.8. *Given a Chevalley presentation of \mathfrak{g} , then $\mathfrak{k} = \text{Span}_{\mathbb{R}}(\{ih_\alpha, x_\alpha - x_{-\alpha}, i(x_\alpha + x_{-\alpha})\}_{\mathbf{R}^+})$ is a real Lie algebra with complexification \mathfrak{g} . The Killing form on \mathfrak{k} is negative definite. If G is a connected compact Lie group with Lie algebra \mathfrak{L}_G which complexifies to \mathfrak{g} , then \mathfrak{L}_G is isomorphic to \mathfrak{k} .*

Note that if one defines κ as before, then the compact real form of \mathfrak{g} is \mathfrak{g}^κ . In particular, $\sigma(\mathfrak{g}^\kappa) \subseteq \mathfrak{g}^\kappa$ and the restriction induces $\{h \rightarrow -h\}$ on the Cartan subalgebra $H = \text{Span}_{\mathbb{R}}(\{ih_\alpha\}_{\mathbf{R}^+})$ of \mathfrak{k} .

4. Extensions and automorphisms - the proof of 1.2

LEMMA 4.1. *Let Φ be an automorphism of finite order of the group B which maps the normal abelian subgroup $A \subseteq B$ into itself and which induces the identity map on the quotient $C = B/A$. Let $[\Phi]$ denote the cyclic group generated by Φ in $\text{Aut}(B)$. If $H^1([\Phi], A) = 0$, then*

(a) *the extension $\{e\} \rightarrow A \rightarrow B \rightarrow C \rightarrow \{e\}$ is induced from the extension*

$$\{e\} \rightarrow A^\Phi \rightarrow B^\Phi \rightarrow C^\Phi = C \rightarrow \{e\}.$$

(b) *the order of the k -invariant of the first extension in $H^2(C, A)$ divides the order of the k -invariant of the second extension in $H^2(C, A^\Phi)$.*

(c) if the action of Φ on A is as $-\text{Id}_A$, then A^Φ and hence $H^2(C, A^\Phi)$ are annihilated by 2. Hence the k -invariant of (1) has order 2.

PROOF OF 4.1. Consider the induced sequence in cohomology:

$$\{e\} \rightarrow A^\Phi \rightarrow B^\Phi \rightarrow C^\Phi \rightarrow H^1([\Phi], A) \rightarrow H^1([\Phi], B).$$

From the hypothesis, $H^1([\Phi], A) = 0$, so one has the commuting diagram of exact sequences of groups

$$\begin{array}{ccccccccc} \{e\} & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \{e\} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow = \\ \{e\} & \longrightarrow & A^\Phi & \longrightarrow & B^\Phi & \longrightarrow & C^\Phi & \longrightarrow & \{e\} \end{array}$$

That is, the top extension is induced from the lower extension. \square

LEMMA 4.2. *If*

(a) $A = T = (S^1)^\ell$ and $\phi : A \rightarrow A$ by $\{t \rightarrow t^{-1}\}$, or

(b) $A = (Z/p^\infty Z)^\ell$ and $\phi : A \rightarrow A$ is $\{a \rightarrow \alpha(a)\}$ where α is a root of unity in \mathbb{Z}_p ,
then $H^1([\phi], A) = 0$.

PROOF OF 4.2. (a) Consider first the $(S^1)^\ell$ case. It suffices to check for $A = S^1$. The short exact sequence of $[\phi]$ modules

$$0 \rightarrow \mathbb{Z}^- \rightarrow \mathbb{R}^- \rightarrow S^1 \rightarrow \{e\}$$

induces

$$0 \rightarrow H^1([\phi], S^1) \rightarrow H^2([\phi], \mathbb{Z}^-) \rightarrow 0.$$

Here \mathbb{Z}^- and \mathbb{R}^- denote the sign actions of $[\phi]$. But $H^2([\phi], \mathbb{Z}^-) = 0$, so $H^1([\phi], S^1) = 0$.

(b) This is trivial if $p \neq 2$. If $p = 2$, use the sequence of $[\phi]$ modules

$$0 \rightarrow \mathbb{Z}_2^- \rightarrow \mathbb{Z}_2^- \otimes \mathbb{Q} \rightarrow \mathbb{Z}/2^\infty \mathbb{Z} \rightarrow 0$$

as above, noting that $H^2([\phi], \mathbb{Z}_2^-) = 0$. \square

PROOF OF 1.2, ASSUMING 1.1. Since $\Phi(T_G) \subseteq T_G$, it follows that $\Phi(N_G(T_G)) \subseteq N_G(T_G)$. The restriction of Φ to $N_G(T_G)$ satisfies the hypotheses of 4.1, so 1.2 follows. \square

5. The classifying space case

A nice exposition of the implications of Tits' work for p -completed classifying spaces is done in Neuman, [8]. One point to mention is that one must use fiberwise p -completions to obtain the proper p -complete analogue of the usual $BT_G \rightarrow BN_G(T_G) \rightarrow BW(G)$ fibration.

Even for the classical Lie case, it is convenient to use some methods from the authors' p -compact group work. Recall from [5] that if one defines $\mathfrak{W}(X)$ as the monoid of self-equivalences of the fibration $BT_X \rightarrow BX$ for (BX, X) a connected p -compact group, then $\pi_0(\mathfrak{W}(X)) = W(X)$ and each component is contractible. For p -compact groups, $BN_X(T_X)$ is defined as the Borel construction of the $\mathfrak{W}(X)$ action on BT_X . If (BX, X) is the Bousfield-Kan p -completion of (BG, G) , for G a connected Lie group, then the fibration $BT_X \rightarrow BN_X(T_X) \rightarrow BW(X)$ agrees up to fiber homotopy equivalence with the fiberwise p -completion of $BT_G \rightarrow BN_G(T_G) \rightarrow BW(G)$.

We can combine this particular characterization of $BN_X(T_X)$ with general mapping space facts to conclude that in the p -compact group setting, a self-equivalence of BX compatible with BT_X induces a self-equivalence of $BN_X(T_X)$:

LEMMA 5.1. *Let $f : X \rightarrow Y$ be a fibration and W_f be the space of self-equivalences of f . Let $\phi : X \rightarrow X$ and $\phi' : Y \rightarrow Y$ be homotopy equivalences such that*

$$\begin{array}{ccc} X & \xrightarrow{\quad} & X \\ & \downarrow f & \downarrow f \\ Y & \xrightarrow{\quad} & Y \end{array}$$

is a commutative diagram. Then there exists self-equivalences ψ and ψ' of $EW_f \times_{W_f} X$ and BW_f respectively such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad} & X \\ \downarrow i & & \downarrow i \\ EW_f \times_{W_f} X & \xrightarrow{\quad} & EW_f \times_{W_f} X \\ \downarrow p & & \downarrow p \\ BW_f & \xrightarrow{\quad} & BW_f \end{array}$$

commutes up to homotopy.

COROLLARY 5.2. *Let (BX, X) be a connected p -compact group with maximal torus T_X and normalizer of the maximal torus $N_X(T_X)$. Given self-homotopy equivalences ϕ_X and ϕ_T of BX and BT_X such that the diagram*

$$\begin{array}{ccc} BX & \xrightarrow{\quad} & BX \\ & \downarrow \phi_X & \\ \uparrow_{Bi} & & \uparrow_{Bi} \\ BT_X & \xrightarrow{\quad} & BT_X \\ & \downarrow \phi_T & \end{array}$$

commutes up to homotopy, there exists ϕ_N a self-homotopy equivalence of $N_X(T_X)$ such that

$$\begin{array}{ccc} BX & \xrightarrow{\quad} & BX \\ & \downarrow \phi_X & \\ \uparrow_{Bj} & & \uparrow_{Bj} \\ BN_X(T_X) & \xrightarrow{\quad} & BN_X(T_X) \\ & \downarrow \Phi_N & \\ \uparrow_{Bi} & & \uparrow_{Bi} \\ BT_X & \xrightarrow{\quad} & BT_X \\ & \downarrow \phi_T & \end{array}$$

commutes up to homotopy.

THEOREM 5.3. *If (BX, X) is a connected p -compact group and Φ is a self homotopy equivalence of BX such that the diagram*

$$\begin{array}{ccc} BX & \xrightarrow{\quad} & BX \\ & \downarrow \Phi & \\ \uparrow_{Bi} & & \uparrow_{Bi} \\ BT_X & \xrightarrow{\quad} & BT_X \\ & \downarrow B(\alpha) & \end{array}$$

commutes up to homotopy for some α a p -adic unit, then

(a) the fibration

$$BT_X \rightarrow BN_X(T_X) \rightarrow BW(X)$$

is classified up to f.h.e by the homotopy class of $\eta : BW(X) \rightarrow B(\text{Aut}(BT_X))$.

(b) This element when interpreted as an element of $H^3(BW(X), \pi_1(T_X))$ is annihilated by $(1 - \alpha)$.

(c) if α is a non-trivial root of unity and $p > 2$, then $\eta = 0$.

COROLLARY 5.4. *If G is a compact connected Lie group and Φ is a self homotopy equivalence of BG such that*

$$\begin{array}{ccc} BG & \xrightarrow{\quad} & BG \\ & \underset{\Phi}{\phantom{\xrightarrow{\quad}}} & \\ \uparrow_{Bi} & & \uparrow_{Bi} \\ BT_G & \xrightarrow[B(-1)]{\quad} & BT_G \end{array}$$

commutes up to homotopy, then the fibration

$$BT_G \rightarrow BN_G(T_G) \rightarrow BW(G)$$

is classified up to f.h.e by $\eta \in H^3(BW(G), \pi_1(T_G))$ and $2(\eta) = 0$.

PROOF OF 5.3. Fibrations with BT_X as fiber and BW as base are classified up to f.h.e class by a map $\eta : BW(X) \rightarrow B \text{Aut}(BT_X)$. Since $BT_X = K((\mathbb{Z}_p^\ell, 2)$, $\pi_0(\text{Aut}(BT_X)) \approx \text{GL}(\ell, \mathbb{Z}_p)$, and $B \text{Aut}(BT_X)$ has a twisted 2-stage Postnikov resolution

$$\begin{array}{c} B^2T_X = K((\mathbb{Z}_p)^\ell, 3) \\ \downarrow_i \\ B \text{Aut}(BT_X) \quad . \\ \downarrow_p \\ B \text{GL}(\ell, \mathbb{Z}_p) \end{array}$$

Hence, given the action map $\rho : W(X) \rightarrow \text{GL}(\ell, \mathbb{Z}_p)$, η is determined by a section of the induced fibration over $BW(X)$ pulled back by $B\rho$. The homotopy class of this section corresponds to η , an element of $H^3(W(X), \pi_1(T_X))$.

By Lemma 5.1, Φ has a restriction to $BN_X(T_X)$. The induced action on the coefficients gives that $\alpha\eta = \eta$, so $(1 - \alpha)(\eta) = 0$ in $H^3(W(X), \pi_1(T_X))$. If $p > 2$ and α is a root of unity in \mathbb{Z}_p , then $(1 - \alpha)$ is a p -adic unit and hence η is zero. \square

PROOF OF 5.4. The proof of the characterization of the classifying map as an element $\eta \in H^3(W(G), \pi_1(T_G))$ is the same as in 5.3. But, since $W(G)$ is finite,

$$H^3(W(G), \pi_1(T_G)) = \prod_p H^3(W(G), \pi_1(T_G) \otimes \mathbb{Z}_p).$$

The characterization of the Weyl space $\mathbb{W}(G)$ as a discrete space is only true after appropriate p -completions, however. Hence we argue somewhat indirectly. The fiberwise p -completion of $BT_G \rightarrow BN_G(T_G) \rightarrow$

$BW(G)$ produces the p -compact data for 5.3, so for $p = 2$, the order of the 2-complete component of η divides 2, and for each odd prime, the order is 1. Hence the order of η divides 2. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556

E-mail address: william.g.dwyer@nd.edu

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907-1395

E-mail address: wilker@math.purdue.edu