THE A-COMPLICATION OF A SPACE

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Abstract. Suppose that A is a pointed CW-complex. We look at how difficult it is to construct an A-cellular space B from copies of A by repeatedly taking homotopy colimits; this is determined by an ordinal number called the *complication* of B. Studying the complication leads to an iterative technique, based on resolutions, for constructing the A-cellular approximation $\mathrm{CW}_A(X)$ of an arbitrary space X.

1. Introduction

Suppose that A is a pointed CW complex. A space B is said to be A-*cellular* if it can be constructed from copies of A by repeatedly taking homotopy colimits. In this paper we introduce the notion of the A-*complication* of an A-cellular space B; this is an ordinal number, denoted $\kappa_A(B)$, which gives the minimal number of times it is necessary to repeat the homotopy colimit construction in order to get from A to B. We then describe an explicit way to construct B in terms of functorial resolutions which depend upon information about maps $A \to B$. The resolution process has to be iterated before it settles on the right result, and it turns out that the number of iterations required is controlled by $\kappa_A(B)$. From this point of view the complication determines how difficult it is to build B, starting with the space A and data about the maps $A \to B$.

Suppose now that X is an arbitrary pointed space. Let $\mathrm{Map}(A, X)$ denote the space of pointed maps from A to X, and say that a map $X \to Y$ is an A-*equivalence* if it induces a weak equivalence $\mathrm{Map}(A, X) \simeq \mathrm{Map}(A, Y)$. Dror-Farjoun [5] shows that up to homotopy there is a canonical A-cellular space $\mathrm{CW}_A(X)$ together with an A-equivalence $\mathrm{CW}_A(X) \to X$; this is the A-*cellular approximation to* X. (If X is A-cellular then $\mathrm{CW}_A(X) \simeq X$.) The resolution process mentioned above can be applied to X as well as to $\mathrm{CW}_A(X)$. It turns out that after the

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very first stage it gives the same result for these two spaces, and so, after a number of iterations controlled by $\kappa_A(\mathrm{CW}_A X)$, converges to $\mathrm{CW}_A(X)$. For this reason we extend the definition of A-complication to arbitrary spaces by setting $\kappa_A(X) = \kappa_A(\mathrm{CW}_A X)$, and we interpret this ordinal as measuring the difficulty of starting with data on maps $A \to X$ and building $\mathrm{CW}_A(X)$.

Our approach is to construct an increasing chain of categories

$$\mathbf{C}_0 \subset \mathbf{C}_1 \subset \cdots \subset \mathbf{C}_\alpha \subset \mathbf{C}_{\alpha+1} \cdots \tag{1.1}$$

depending on A and indexed by the ordinal numbers α. (These are full subcategories of the category of pointed spaces.) The category $\mathbf{C}_0$ contains all cofibrant spaces which are homotopy equivalent to retracts of wedges of copies of A. The category $\mathbf{C}_\alpha$, $\alpha > 0$ is defined inductively. Let $\mathbf{C}_{<\alpha}$ denote the union of the categories $\mathbf{C}_\beta$ for $\beta < \alpha$. Then $\mathbf{C}_\alpha$ contains all cofibrant spaces which up to homotopy equivalence and retracts can be written as $\operatorname{hocolim} G$, for some small category $\mathbf{D}$ and functor $G : \mathbf{D} \to \mathbf{C}_{<\alpha}$. It is clear that any A-cellular space B belongs to $\mathbf{C}_\alpha$ for some ordinal α; the least such α is the A-*complication* $\kappa_A(B)$ of B. As mentioned above, for a general X we set $\kappa_A(X) = \kappa_A(\mathrm{CW}_A X)$.

We also construct for any X a functorial sequence of spaces

$$F_0(X) \to F_1(X) \to \cdots \to F_\alpha(X) \to F_{\alpha+1}(X) \to \cdots \ .$$

The space $F_0(X)$ is equivalent to a wedge of copies of A, one for each map $A \to X$. If $\alpha = \beta + 1$ is a successor ordinal, then $F_\alpha(X)$ is equivalent to the realization or homotopy colimit of a resolution of X derived from iteratively applying the functor F_β to X. If α is a limit ordinal, then $F_\alpha(X) \simeq \operatorname{hocolim}_{\beta<\alpha} F_\beta(X)$. It is clear that $F_\alpha(X) \in \mathbf{C}_\alpha$. All of the spaces $F_\alpha(X)$ map in a coherent way to X.

Our first result gives a way to interpret the functors F_α conceptually. Recall the definition of $\mathbf{C}_{<\alpha}$ from above; if $\alpha = \beta + 1$ is a successor ordinal, then $\mathbf{C}_{<\alpha}$ is just $\mathbf{C}_\beta$.

1.2. Theorem. *Suppose that $\alpha > 0$ is an ordinal. Then the space $F_\alpha(X)$ is the canonical approximation to X by a homotopy colimit of spaces from $\mathbf{C}_{<\alpha}$ (4.4). More concretely, $F_\alpha(X)$ is a homotopy colimit, over the category of maps $B \to X$ with $B \in \mathbf{C}_{<\alpha}$, of the functor which assigns to the object $B \to X$ the space B itself.*

Since the category $\mathbf{C}_{<\alpha}$ in 1.2 is not small, part of the work in proving this theorem is to make sense of the indicated homotopy colimit. (The way in which this is done (3.11) explains why in the above statement $F_\alpha(X)$ is called "a" homotopy colimit instead of "the" homotopy colimit.) Roughly speaking, the theorem says that in spite of its awkward construction in terms of an explicit resolution, $F_\alpha(X)$ is the most natural functorial approximation to X by a homotopy colimit of A-cellular spaces of complication $< \alpha$.

We next show that the functors F_α are invariant under A-equivalences, and that they are ultimately successful at reconstructing A-cellular spaces.

1.3. Theorem. *Suppose that $\alpha > 0$ is an ordinal. Then a map $X \to Y$ is an A-equivalence if and only if the induced map $F_\alpha(X) \to F_\alpha(Y)$ is a homotopy equivalence.*

1.4. Theorem. *Suppose that α and β are ordinals with $\alpha > \beta$. Then if B belongs to $\mathbf{C}_\beta$, the natural map $F_\alpha(B) \to B$ is a homotopy equivalence. In particular $F_\alpha \circ F_\beta \simeq F_\beta$.*

1.5. *Remark.* Now let X be an arbitrary space and let $B = \mathrm{CW}_A(X)$. Consider the commutative ladder

$$\begin{array}{ccccccccc} F_0(B) & \to & \cdots & \to & F_\alpha(B) & \to & \cdots & B \\ \downarrow & & & & \downarrow & & & \downarrow \\ F_0(X) & \to & \cdots & \to & F_\alpha(X) & \to & \cdots & X \end{array}$$

By 1.3 the vertical maps indexed by α are equivalences for $\alpha > 0$, and in view of 1.4 the maps $F_\alpha(B) \to B$ are equivalences for $\alpha > \kappa_A(B) = \kappa_A(X)$. Thus the spaces $F_\alpha(X)$ give a sequence of ever more complicated approximations to $\mathrm{CW}_A(X)$, a sequence which hits the mark when $\alpha = \kappa_A(X) + 1$.

The following proposition is clear from Dror-Farjoun's construction of $\mathrm{CW}_A(X)$.

1.6. Proposition. *There exists an ordinal κ, depending on A, such that $\kappa_A(B) \leq \kappa$ for any A-cellular space B.*

Let $\kappa_A^{\max}$ denote the minimal such ordinal and let $\alpha = \kappa_A^{\max} + 1$. It follows from 1.3 and 1.4 that for any space X the space $F_\alpha(X)$ is homotopy equivalent to $\mathrm{CW}_A(X)$. In this way we have constructed $\mathrm{CW}_A(X)$ by an iterative functorial resolution process, although carrying the process to its conclusion potentially requires a transfinite reserve of patience.

1.7. *Relationship to previous work.* In a sense, this paper is a result of combining Bousfield and Kan [1, XI.10] with long tower arguments from [3], turning the arrows around, and finding a more theoretical way to describe the results. To see the connection, let $R = \mathbb{Z}/p$, and construct a sequence of categories

$$\mathbf{D}_0 \subset \mathbf{D}_1 \subset \cdots \subset \mathbf{D}_\alpha \subset \cdots$$

depending on R and indexed by the ordinal numbers α. The category $\mathbf{D}_0$ contains all spaces which up to weak equivalence are products of Eilenberg-MacLane spaces $K(R,n)$, $n \geq 0$, and $\mathbf{D}_\alpha$ for $\alpha > 0$ is inductively defined to contain all spaces which up to weak equivalence and retracts are of the form can be written as $\operatorname{holim} G$, for some small category $\mathbf{D}$ and functor $G : \mathbf{D} \to \mathbf{D}_{<\alpha}$. (Here $\mathbf{D}_{<\alpha} = \cup_{\beta<\alpha}\mathbf{D}_\beta$.) Then for any space X it is possible to construct a functorial long tower of spaces

$$D_0(X) \leftarrow D_1(X) \leftarrow \cdots \leftarrow D_\alpha(X) \leftarrow D_{\alpha+1}(X) \leftarrow \cdots ,$$

where $D_0(X)$ is a topological version of RX, and $D_\alpha(X)$ for $\alpha > 0$ is a right homotopy Kan extension of the inclusion $J : \mathbf{D}_{<\alpha} \to \mathbf{S}_*$ along the functor J itself. (In language parallel to that of 1.2, $D_\alpha(X)$ is the canonical approximation to X by a homotopy limit of spaces from $\mathbf{D}_{<\alpha}$.) Bousfield and Kan show in a simplicial setting tha $D_1(X) = R_\infty(X)$, while the results of [3] implicitly show how to construct $D_\alpha(X)$ for $\alpha > 1$ in term of restricted cosimplicial resolutions, and prove that for sufficiently large α the space $D_\alpha(X)$ is the R-homology localization of X. (It is not known, however, that "sufficiently large" can be interpreted in a way which is independent of X, and it is not known whether an analogous construction of the homology localization succeeds for arbitrary homology theories.) The fact that the homology localization is not always reached at the first stage with $D_1(X) \simeq R_\infty X$ is a reflection of the fact that not all spaces are R-good. This is parallel in our setting to the fact that $\mathrm{CW}_A(X)$ is not always reached at the first stage with $F_1(X)$.

1.8. *Organization of the paper.* Section 2 has preliminary material on homotopy colimits, and §3 identifies certain functors, called *terminal functors*, which preserve homotopy colimits. A device using terminal functors allows the notion of homotopy colimit to be extended to some large diagrams (3.11). Section 4 discusses left homotopy Kan extensions, which are a kind of parametrized homotopy colimit, and in §5 there is a general homotopy invariance theorem for left homotopy Kan extensions. The kind of approximation referred to in 1.2 is a special kind of homotopy Kan extension (4.4), and section 6 describes a method for computing these approximations. This method depends on the availability of a certain kind of functor F associated to the approximating category $\mathbf{C}$. If such an F exists then $(\mathbf{C}, F)$ is called an *adapted pair*, and the approximation is computed by taking the homotopy colimit of a resolution constructed with the help of F. Section 7 focuses on constructing adapted pairs, and §8 uses the machinery from the previous sections to prove the main results. Finally, §9 briefly discusses a few examples: the cases in which A is a finite complex, a Moore space, or a sphere.

1.9. *Notation and terminology.* The symbol $\mathbf{S}_*$ stands for the category of pointed spaces, and throughout the paper spaces are usually assumed to have a chosen basepoint. (There are some obvious exceptions, such as simplices.) Along these lines, "hocolim" is the pointed homotopy colimit (§2). The notation Y_+ stands for Y with a disjoint basepoint added, and $X \rtimes Y$ for $X \wedge (Y_+)$. We sometimes say that $f\colon X \to Y$ is an *equivalence* (and write $X \simeq Y$) if f is a weak equivalence. A natural transformation between functors $\mathbf{D} \to \mathbf{S}_*$ is called a weak equivalence if it gives a weak equivalence for each object of $\mathbf{D}$. From a technical point of view we work with the model category structure on $\mathbf{S}_*$ due to Quillen ([14], see also [7, §8]); in particular, a space is *cofibrant* if it is a retract of a cell complex. A weak equivalence between cofibrant spaces is a homotopy equivalence.

2. Homotopy colimits

In this section we recall the notion of (pointed) homotopy colimit, and then give two examples involving simplicial spaces and restricted simplicial spaces.

Suppose that $\mathbf{D}$ is a small category, and that $F\colon \mathbf{D} \to \mathbf{S}_*$ is a functor. We use "$\operatorname{hocolim} F$" to denote the pointed homotopy colimit of F. This is defined as in [1, p. 328], with minor adjustments: the construction is made in the category $\mathbf{S}_*$, and the part of the construction due to the basepoints of the spaces involved is collapsed to the basepoint of $\operatorname{hocolim} F$. See also [10, §4]. The part of the unpointed homotopy colimit which is collapsed to form the basepoint of $\operatorname{hocolim} F$ is homeomorphic to the geometric realization of the nerve [1, XI 2.1] of $\mathbf{D}$; it follows easily that if the values of F are cofibrant and the nerve of $\mathbf{D}$ is contractible, then the pointed and unpointed homotopy colimits of F have the same homotopy type.

2.1. *Remark.* If $J : \mathbf{C} \to \mathbf{D}$ and $F : \mathbf{D} \to \mathbf{S}_*$ are functors, we let $J^*(F)$ denote the composite $F \circ J$. There are natural maps

$$\operatorname{hocolim} J^*(F) \to \operatorname{hocolim} F \to \operatorname{colim} F\ .$$

The homotopy colimit construction has the following basic invariance property.

2.2. Proposition. *(cf.* [1, XII 4.2]*) Suppose that* $\mathbf{D}$ *is a small category, and that* $t : F \to G$ *is a weak equivalence between functors* $\mathbf{D} \to \mathbf{S}_*$*. Suppose in addition that for each object* d *of* $\mathbf{D}$ *the spaces*

$F(d)$ and $G(d)$ are cofibrant. Then $\operatorname{hocolim} t\colon \operatorname{hocolim} F \to \operatorname{hocolim} G$ *is an equivalence.*

2.3. *Remark.* We will make use of $\operatorname{hocolim} F$ only in cases in which the values of F are cofibrant, and we write $\operatorname{hocolim}^{\mathbf{L}} F$ for $\operatorname{hocolim} F^{\mathrm{c}}$, where F^{c} is some naturally associated functor with cofibrant values which maps to F by an equivalence. For instance $F^{\mathrm{c}}(x)$ might be the geometric realization of the singular complex of $F(x)$. If F itself takes on cofibrant values, then $\operatorname{hocolim}^{\mathbf{L}} F$ is equivalent to $\operatorname{hocolim} F$.

2.4. *Remark.* The homotopy colimit can be viewed as the total derived functor in the sense of Quillen [9, II.7] associated to the colimit; roughly speaking, $\operatorname{hocolim}^{\mathbf{L}}$ is the construction closest to colim which takes equivalences between functors $\mathbf{C} \to \mathbf{S}_*$ to equivalences between spaces. See [7, §10] for a discussion of this in a special case. There is a slightly different construction of the homotopy colimit in [4, 2.16]. See also [2].

2.5. *Remark.* Given a functor $F : \mathbf{D} \to \mathbf{S}_*$, there exists a weak equivalence $F' \to F$, where F' is a functor with cofibrant values such that $\operatorname{colim} F'$ is cofibrant and the natural map $\operatorname{hocolim} F' \to \operatorname{colim} F'$ is a homotopy equivalence . See [4, §2] and [2, §16]. The functor F' can be taken to be a cofibrant replacement for F in an appropriate Quillen model structure on the functor category.

2.6. *Simplicial spaces.* Let Δ^{op} be the simplicial category, so that a *simplicial object* in some category $\mathbf{A}$ is a functor $X : \Delta^{\mathrm{op}} \to \mathbf{A}$, or more explicitly a collection $\{X_n\}_{n\geq 0}$ of objects of $\mathbf{A}$ together with face maps $d_i : X_n \to X_{n-1}$ and degeneracy maps $s_i : X_n \to X_{n+1}$, $0 \leq i \leq n$, satisfying the usual simplicial identities [12, §1]. The category Δ^{op} is the opposite of the category Δ whose objects are the finite ordered sets $[n] = \{0, \ldots, n\}$ and whose morphisms are the weakly monotone maps [12]. Let Δ_n be the topological n-simplex. The *geometric realization* of a simplicial space X is obtained from the space

$$\bigvee_n X_n \rtimes \Delta_n \tag{2.7}$$

by making the usual face and degeneracy identifications [12, §14]. If each X_n is cofibrant and the degeneracy maps $s_i : X_n \to X_{n+1}$ are cofibrations, then the geometric realization is equivalent to $\operatorname{hocolim} X$ (cf. [1, XII.3.4] or [15]).

2.8. *Restricted simplicial spaces.* We need a variation of the above. Let Λ^{op} be the *restricted simplicial category*; this is the subcategory of Δ^{op} whose morphisms are the composites of face maps d_i. This is the opposite of the category Λ whose objects are the finite ordered sets

$[n]$ and whose morphisms are the monotone injections. A *restricted simplicial object* in a category $\mathbf{A}$ [3, 3.13] is a functor $X : \Lambda^{\mathrm{op}} \to \mathbf{A}$, or, more explicitly, a collection $\{X_n\}_{n\geq 0}$ of objects of $\mathbf{A}$ together with maps $d_i : X_n \to X_{n-1}$, $0 \leq i \leq n$, such that $d_i d_j = d_{j-1} d_i$ for $i < j$. This is a "simplicial object without degeneracy maps." The *geometric realization* of a restricted simplicial space X is obtained from the one-point union 2.7 by making the identifications $(d_i x) \rtimes y \sim x \rtimes (d^i y)$, where $x \in X_n$, $y \in \Delta_{n-1}$, and $d^i : \Delta_{n-1} \to \Delta_n$ is the i'th face inclusion. In this case the geometric realization is homeomorphic to $\operatorname{hocolim} X$.

3. Terminal functors

In this section we point out that a functor captures homotopy colimits (3.1) if and only if it is terminal in an appropriate sense (3.4). We then check that the condition of being terminal is satisfied in some key examples involving (restricted) simplicial categories. We also use the notion of a terminal functor to extend the construction of homotopy colimits to certain diagrams indexed by large categories (3.11).

3.1. Capturing homotopy colimits. We will say that a functor $J : \mathbf{C} \to \mathbf{D}$ between small categories *captures (pointed) homotopy colimits* if for every $F : \mathbf{D} \to \mathbf{S}_*$ the natural map $\operatorname{hocolim}^{\mathbf{L}} J^*(F) \to \operatorname{hocolim}^{\mathbf{L}} F$ is an equivalence. There is a similar sense in which J can be said to capture unpointed homotopy colimits.

Given $J : \mathbf{C} \to \mathbf{D}$ and an object d of $\mathbf{D}$, let $d \downarrow J$ denote the category whose objects are pairs (c, f) where c is an object of $\mathbf{C}$ and $f : d \to J(c)$ is a map in $\mathbf{D}$; a morphism $(c, f) \to (c', f')$ is a map $g : c \to c'$ in $\mathbf{C}$ such that $J(g)f = f'$.

3.2. Definition. A functor $J : \mathbf{C} \to \mathbf{D}$ is said to be *terminal* if for each object d of $\mathbf{D}$ the nerve of the category $d \downarrow J$ is contractible.

3.3. *Remark.* An object d_0 of $\mathbf{D}$ is usually said to be *terminal* if for any object d of $\mathbf{D}$ the set $\operatorname{Hom}(d, d_0)$ has exactly one point. It is easy to see that d_0 is terminal in this sense if and only if the inclusion $J : \{d_0\} \to \mathbf{D}$ of the trivial subcategory with object d_0 is terminal in the sense of 3.2.

In general, in the situation of 3.2 let $\operatorname{Hom}(d, J)$ denote the functor $\mathbf{C} \to \mathbf{Sets}$ which sends $c \in \mathbf{C}$ to $\operatorname{Hom}(d, J(c))$. It is easy to check that the geometric realization of the nerve of $d \downarrow J$ is homeomorphic to the unpointed homotopy colimit of $\operatorname{Hom}(d, J)$, and so J is terminal if and only if for each object d of $\mathbf{D}$ the unpointed homotopy colimit of $\operatorname{Hom}(d, J)$ is equivalent to a single point.

3.4. Proposition. *Suppose that $J : \mathbf{C} \to \mathbf{D}$ is a functor between small categories. Then J captures pointed or unpointed homotopy colimits if and only if J is terminal.*

Proof. The "if" half of the statement appears in [6, 9.4], at least in a simplicial form in the unpointed case. The proof in the pointed case is identical. See also [10, 4.4] and [2]. The "only if" part is proved in the pointed case by looking at functors $F_d : \mathbf{D} \to \mathbf{S}_*$ given by $F_d(x) = \mathrm{Hom}_{\mathbf{D}}(d, x)_+$. The homotopy colimit of F_d is equivalent to S^0, while the homotopy colimit of $J^*(F_d)$ is obtained up to homeomorphism by adding a disjoint basepoint to the geometric realization of the nerve of $d \downarrow J$. □

3.5. *Example.* If d_0 is a terminal object of $\mathbf{D}$, then (3.2, 3.4) for any $F : \mathbf{D} \to \mathbf{S}_*$ the natural map $\mathrm{hocolim}^{\mathbf{L}} F \to F(d_0)$ is an equivalence.

3.6. Contractions. An *augmentation* for a (restricted) simplicial object X in the category $\mathbf{A}$ is an object X_{-1} of $\mathbf{A}$ together with a map $d_0 : X_0 \to X_{-1}$ such that $d_0 d_0 = d_0 d_1 : X_1 \to X_{-1}$. A *left contraction* for such an augmented X is a collection of maps $s_{-1} : X_n \to X_{n+1}$, $n \geq -1$, such that $d_0 s_{-1} = 1$ and $d_j s_{-1} = s_{-1} d_{j-1}$ for $j > 0$; if X is simplicial (as opposed to restricted simplicial) there are also identities $s_j s_{-1} = s_{-1} s_{j-1}$ for $j > -1$. A *right contraction* for X is a collection of maps $s_{n+1} : X_n \to X_{n+1}$ such that $d_{n+1} s_{n+1} = 1$ and $d_j s_{n+1} = s_n d_j$ for $j < n+1$; in the simplicial case there are also identities $s_j s_{n+1} = s_{n+2} s_j$ for $j \leq n+1$. See [8, 6.1], [3, 3.16], and [13] for similar formulations. Let ${}_l\Delta^{\mathrm{op}}$ and ${}_l\Lambda^{\mathrm{op}}$ be the extensions of Δ^{op} and Λ^{op} respectively which correspond to left contractions, and ${}_r\Delta^{\mathrm{op}}$ and ${}_r\Lambda^{\mathrm{op}}$ the extensions which corresponding to right contractions. All of these categories have objects $[n]$ for $n \geq -1$; the category ${}_l\Lambda^{\mathrm{op}}$, for instance, has morphisms $d_i : [n+1] \to [n]$, $s_{-1} : [n] \to [n+1]$ together with the identities required so that an augmented restricted simplicial object with a left contraction amounts to a functor $c\Lambda^{\mathrm{op}} \to \mathbf{S}_*$. There are commutative squares of categories and functors

$$\begin{array}{ccc} \Lambda^{\mathrm{op}} & \longrightarrow & \Delta^{\mathrm{op}} \\ \downarrow & & \downarrow \\ {}_l\Lambda^{\mathrm{op}} & \longrightarrow & {}_l\Delta^{\mathrm{op}} \end{array} \qquad \begin{array}{ccc} \Lambda^{\mathrm{op}} & \longrightarrow & \Delta^{\mathrm{op}} \\ \downarrow & & \downarrow \\ {}_r\Lambda^{\mathrm{op}} & \longrightarrow & {}_r\Delta^{\mathrm{op}} \end{array} . \tag{3.7}$$

3.8. Proposition. *All of the functors in display 3.7 are terminal.*

Proof. We first deal with the left-hand square. The upper functor is terminal by [3, 3.17], the one on the left by [3, p. 196], the lower one by

3.5 and the fact that both of the categories involved have $[-1]$ as a terminal object, and the one on the right by an easy argument depending on 3.4 and the fact that the other three functors are terminal.

The right-hand square differs from the left-hand square by the automorphism of Δ^{op} which in dimension n interchanges d_i and d_{n-i}. $\square$

3.9. *Example.* If X is a (restricted) simplicial space which admits both an augmentation by a space X_{-1} and an associated left or right contraction, then the map $\mathrm{hocolim}^{\mathbf{L}} X \to X_{-1}$ induced by the augmentation is an equivalence. This follows from observing that the categories ${}_l\Delta^{\mathrm{op}}$, ${}_l\Lambda^{\mathrm{op}}$, ${}_r\Delta^{\mathrm{op}}$ ${}_r\Lambda^{\mathrm{op}}$ all have $[-1]$ as a terminal object, and then combining 3.8 with 3.5.

3.10. *Example.* Consider a functor $J : \Delta \to \mathbf{D}$ or $\Lambda \to \mathbf{D}$, and let d be an object of $\mathbf{D}$. In these cases $\mathrm{Hom}(d, J)$ (see 3.3) is a (restricted) simplicial set; combining 3.3 with 2.6 and 2.8 shows that J is terminal if and only if for each d in $\mathbf{D}$ this restricted simplicial set has a contractible unpointed homotopy colimit. By 3.9, this will be the case if for each d the unique augmentation $\mathrm{Hom}(d, J) \to *$ admits either a left or a right contraction. (Note that along the lines of 2.8, the unpointed homotopy colimit of a restricted simplicial set can be easily described in terms of a geometrical realization.)

3.11. Homotopy colimits over large categories. We will say that a category is *large* if the totality of its objects does not form a set, although the collection of morphisms between any two objects does form a set; for example, the category $\mathbf{S}_*$ is large. Suppose that $\mathbf{D}$ is a large category, and that $F : \mathbf{D} \to \mathbf{S}_*$ is a functor. There are clearly set-theoretic difficulties with forming $\mathrm{hocolim}^{\mathbf{L}} F$ (or even $\mathrm{colim}\, F$). However, if $\mathbf{D}$ is *terminally small* in the sense that there exists a small category $\mathbf{C}$ and a terminal functor $J : \mathbf{C} \to \mathbf{D}$, we will follow [1, XI.10.2] and say that $\mathrm{hocolim}^{\mathbf{L}} J^*(F)$ is *a homotopy colimit for* F. If the values of F are cofibrant, then $\mathrm{hocolim}\, J^*(F)$ is equivalent to $\mathrm{hocolim}^{\mathbf{L}} J^*(F)$, and so $\mathrm{hocolim}\, J^*(F)$ is also a homotopy colimit for F.

3.12. *Remark.* The argument of [1, XI.10.3] shows that the homotopy type of $\mathrm{hocolim}^{\mathbf{L}} J^*(F)$ does not depend upon the choice of terminal functor J.

4. Homotopy Kan extensions and approximations

Here we use objectwise homotopy colimits to define the *homotopy left Kan extension* of a functor $F : \mathbf{C} \to \mathbf{S}_*$ over a functor $J : \mathbf{C} \to \mathbf{D}$. Taking the homotopy left Kan extension of a suitable functor over itself (4.4) leads to the notion of *approximation*.

4.1. *Homotopy Kan extensions.* Suppose that $J : \mathbf{C} \to \mathbf{D}$ is a functor between small categories, and that F is a functor from $\mathbf{C}$ to some category $\mathbf{A}$ with colimits (e.g.. **Sets**). For each object d of $\mathbf{D}$, let $J \downarrow d$ denote the category whose objects are pairs (c, f), where c is an object of $\mathbf{C}$ and $f : J(c) \to d$ is a map in $\mathbf{D}$; a morphism $(c, f) \to (c', f')$ is a map $g : c \to c'$ in $\mathbf{C}$ such that $f' \circ (J(g)) = f$. If J is the inclusion of a subcategory, we sometimes write $\mathbf{C} \downarrow d$ and refer to this as the category of *objects of* $\mathbf{C}$ *over* d [11, II.6]. We let $F_d \colon J \downarrow d \to \mathbf{A}$ be given by $F_d(c, f) = F(c)$. The *left Kan extension* of F along J, denoted J_*F, is the functor $\mathbf{D} \to \mathbf{A}$ with

$$(J_*F)(d) = \operatorname{colim} F_d$$

See [11, X.4]. The construction J_*, considered as an operation on functors $\mathbf{C} \to \mathbf{A}$, is left adjoint to J^* [11, X.3]. If $\mathbf{A}$ is the category $\mathbf{S}_*$, the *homotopy left Kan extension* $J_{\mathrm{h}*}F$ is defined by the analogous formula

$$(J_{\mathrm{h}*}F)(d) = \operatorname{hocolim} F_d \ .$$

There is also a variant which is better-behaved if the values of F are not cofibrant:

$$(J_{\mathbf{L}*}F)(d) = \operatorname{hocolim}^{\mathbf{L}} F_d \ .$$

The construction $J_{\mathbf{L}*}$ can be viewed as the total derived functor in the sense of Quillen [9, II.7] associated to J_*; roughly speaking, $J_{\mathbf{L}*}$ is the construction closest to J_* which takes weak equivalences between functors $\mathbf{C} \to \mathbf{S}_*$ to weak equivalences between functors $\mathbf{D} \to \mathbf{S}_*$. See [10, §5] for a topological discussion, or [9, p. 439] for a simplicial treatment which includes a description [9, IX.1.4] of the model categories involved in applying Quillen's machinery. There is a more general discussion in [2, II.16].

4.2. *Transitivity.* Suppose that $J : \mathbf{C} \to \mathbf{D}$ and $K : \mathbf{D} \to \mathbf{E}$ are functors between small categories, and that F is a functor from $\mathbf{C}$ to say, **Sets**. The left Kan extension construction then satisfies the transitivity relation $(KJ)_*F = K_*(J_*F)$; this follows from the identity $(KJ)^* = J^*K^*$ and the fact that the left adjoint of a composite is the composite of the left adjoints. The homotopy left Kan extension construction has a similar property. If F takes values in $\mathbf{S}_*$, then

$$(KJ)_{\mathbf{L}*}F \simeq K_{\mathbf{L}*}(J_{\mathbf{L}*}F) \ .$$

See [6, 9.8] or [10, 5.5].

4.3. *Large categories.* If $J : \mathbf{C} \to \mathbf{D}$ is a functor between large categories and $F : \mathbf{C} \to \mathbf{S}_*$ is a functor, we can follow 3.11 and speak, under appropriate conditions, of *a homotopy left Kan extension of* F *along* J. In particular, we do this below when for each object d of $\mathbf{D}$ there is a terminal functor $J_d : \Lambda^{\mathrm{op}} \to J \downarrow d$ which is constructed in a way which is natural in d. In this case the functor which sends d to $\mathrm{hocolim}(F_d \circ J_d)$ is a homotopy left Kan extension of F.

4.4. *Approximation.* Suppose that $J : \mathbf{C} \to \mathbf{S}_*$ is the inclusion of a full subcategory; we can think of J in two ways, either as a map of index categories, or as a diagram of spaces indexed by $\mathbf{C}$. If there exists a left homotopy Kan extension of J (in the second sense) along J (in the first sense), we denote this extension $\alpha_{\mathbf{C}}(X)$; the natural map $\alpha_{\mathbf{C}}(X) \to X$ is the *canonical approximation to* X *by a homotopy colimit of spaces from* $\mathbf{C}$ (cf. [11, X.6]).

4.5. *Remark.* There is one case in which it is easy to compute $\alpha_{\mathbf{C}}(X)$, namely, if $J : \mathbf{C} \to \mathbf{S}_*$ is the inclusion of a full subcategory and $X \in \mathbf{C}$. In this case $\mathbf{C} \downarrow X$ has the identity map of X a terminal object, and so it follows from 3.5 and 3.12 that $\alpha_{\mathbf{C}}(X) \to X$ is a weak equivalence.

5. Invariance of homotopy Kan extensions

Suppose that $\mathbf{C}$ is a small full subcategory of $\mathbf{S}_*$, and recall that $\alpha_{\mathbf{C}}(X) \to X$ is the canonical approximation to X by a homotopy colimit of spaces from $\mathbf{C}$ (4.4). In this section we look at conditions which will guarantee that $\alpha_{\mathbf{C}}(X)$ is homotopy invariant as a functor of X. The main result is the following one.

5.1. Proposition. *Suppose that* $\mathbf{C}$ *is a small full subcategory of* $\mathbf{S}_*$ *which has the following two properties:*

1. *each object of* $\mathbf{C}$ *is cofibrant, and*
2. *if* A *is an object of* $\mathbf{C}$*, then each of the products* $A \rtimes \Delta_n$*,* $n \geq 0$ *also belongs to* $\mathbf{C}$*.*

Then $\alpha_{\mathbf{C}}$ *is homotopy invariant. More generally,* $\alpha_{\mathbf{C}}(X) \to \alpha_{\mathbf{C}}(Y)$ *is an equivalence whenever* $X \to Y$ *is an* A*-equivalence for each object* A *of* $\mathbf{C}$*.*

5.2. *Examples.* It may seem paradoxical that a construction like $\alpha_{\mathbf{C}}(X)$ described in terms of homotopy colimits might not be homotopy invariant. But consider the following example. Let $\mathbf{C}$ be any small full subcategory of $\mathbf{S}_*$ which contains only discrete spaces, but is large enough to contain both S^0 and the one-point space $*$. We will show that $\alpha_{\mathbf{C}}(X)$ is equivalent to X^δ (this is X with the discrete topology). In particular, $\alpha_{\mathbf{C}}(X)$ is *not* homotopy invariant as a functor of X. Let

$\mathbf{D} \subset \mathbf{C}$ be the subcategory which contains only S^0 with its identity map. We have functors

$$\mathbf{D} \xrightarrow{I} \mathbf{C} \xrightarrow{J} \mathbf{S}_* \quad \text{and also} \quad K = J \circ I \ .$$

Let Id be the identity functor on $\mathbf{S}_*$. Essentially by inspection, $K_{\mathbf{L}*}(K)$ is weakly equivalent to Id^δ_+; restricting this observation to $\mathbf{C}$ and using the fact that $X = X^\delta$ for $X \in \mathbf{C}$ gives $I_{\mathbf{L}*}(K) \simeq J_+$. By transitivity of homotopy Kan extensions (4.2),

$$J_{\mathbf{L}*}(J_+) \simeq J_{\mathbf{L}*}(I_{\mathbf{L}*}K) \simeq K_{\mathbf{L}*}(K) \simeq \mathrm{Id}^\delta_+ \ . \tag{5.3}$$

Let cS^0 be the constant functor with value S^0 (on whatever category is in question), and let $c*$ be the constant functor with value the one-point space. There is a map of homotopy pushout squares

$$\begin{array}{ccc} J_{\mathbf{L}*}(cS^0) & \longrightarrow & J_{\mathbf{L}*}(c*) \\ \downarrow & & \downarrow \\ J_{\mathbf{L}*}(J_+) & \longrightarrow & J_{\mathbf{L}*}(J) \end{array} \quad \longrightarrow \quad \begin{array}{ccc} cS^0 & \longrightarrow & c* \\ \downarrow & & \downarrow \\ \mathrm{Id}^\delta_+ & \longrightarrow & \mathrm{Id}^\delta \end{array} \ .$$

Since $J_{\mathbf{L}*}(c*)$ is trivially the constant functor with value a point, to show that $J_{\mathbf{L}*}(J)$ is equivalent to Id^δ it is enough to show that $J_{\mathbf{L}*}(cS^0)$ is weakly equivalent to the constant functor on $\mathbf{S}_*$ with value S^0. By inspection, the value of $J_{\mathbf{L}*}(cS^0)$ on a space X is the disjoint union of a basepoint with the nerve of the category $J \downarrow X$. This nerve is contractible, because $J \downarrow X$ has $* \to X$ as an initial object.

The situation changes drastically if $\mathbf{C}$ is enlarged to contain spaces which are homotopy equivalent to S^0 but not discrete. In fact, let $\mathbf{C}$ be any small full subcategory of $\mathbf{S}_*$ which contains the spaces $(\Delta_n)_+$ for $n \geq 0$ as well as the one-point space $*$. We will sketch a modification of the above argument which shows that $\alpha_{\mathbf{C}}(X) \to X$ is an equivalence for every space X and so, in particular, that $\alpha_{\mathbf{C}}$ is homotopy invariant. Consider the functor $I : \Delta \to \mathbf{C}$ with $I([n]) = (\Delta_n)_+$. We have functors

$$\Delta \xrightarrow{I} \mathbf{C} \xrightarrow{J} \mathbf{S}_* \quad \text{and also} \quad K = J \circ I \ .$$

There is a natural transformation $K \to cS^0$ which is a weak equivalence on each object of Δ, and so there are weak equivalences of functors

$$I_{\mathbf{L}*}(K) \simeq I_{\mathbf{L}*}(cS^0) \simeq I_{\mathrm{h}*}(cS^0)$$

where the last equivalence comes from the fact that S^0 is cofibrant. Looking at 2.6 shows that for any space X, $(I_{\mathrm{h}*}(cS^0))(X)$ is obtained by adding a disjoint basepoint to the geometric realization of the singular complex of X, and it follows that $I_{\mathbf{L}*}(K) \simeq J_+$. Similarly, $K_{\mathbf{L}*}(K) \simeq \mathrm{Id}_+$. Transitivity of homotopy Kan extension gives as in

5.3 that $J_{\mathbf{L}*}(J_+) \simeq \mathrm{Id}_+$, and the disjoint basepoints are disposed of as before.

Proof of 5.1. We will use some machinery from [10]. If $\mathbf{D}$ is a small category, F is a contravariant functor $\mathbf{D} \to \mathbf{S}_*$ (a right $\mathbf{D}$-module) and G is a covariant functor $\mathbf{D} \to \mathbf{S}_*$ (a left $\mathbf{D}$-module), we write $F \otimes^{\mathrm{h}}_{\mathbf{D}} G$ for what in the notation of [10, §3] would be described as the quotient $B(Y, \mathbf{D}, X)/B(*, \mathbf{D}, *)$; this reduced bar construction is the *pointed homotopy coend* of F and G. This has a homotopy invariance property analogous to that of the homotopy colimit; in fact, $\mathrm{hocolim}\, G$ is exactly $S^0 \otimes^{\mathrm{h}}_{\mathbf{D}} G$. There is also a variant $F \otimes^{\mathbf{L}}_{\mathbf{D}} G$, which is defined as $F^{\mathrm{c}} \otimes^{\mathrm{h}}_{\mathbf{D}} G^{\mathrm{c}}$ (2.3) and is better-behaved if the values of F and G are not cofibrant. If $J : \mathbf{D} \to \mathbf{E}$ and $F : \mathbf{D} \to \mathbf{S}_*$ are functors, the homotopy Kan extension $J_{\mathrm{h}*}(F)$ can be identified as $\mathbf{E}_+ \otimes^{\mathrm{h}}_{\mathbf{D}} F$, where for short $\mathbf{E}_+$ denotes the $(\mathbf{E}, \mathbf{D})$-bimodule which assigns to a pair of objects (e, d) the pointed set $\mathrm{Hom}_{\mathbf{E}}(J(d), e)_+$.

Consider the functors

$$\mathbf{C} \xrightarrow{I} \mathbf{C} \times \Delta \xrightarrow{J} \mathbf{C} \xrightarrow{K} \mathbf{S}_* \ ,$$

where $I(A) = (A, [0])$, $J(A, [n]) = A \rtimes \Delta_n$, and K is the inclusion functor. The composite KJI is naturally equivalent to K. Let L denote the composite KJ. The diagram gives maps

$$(KJI)_{\mathrm{h}*}(K) = K_{\mathrm{h}*}(K) \to L_{\mathrm{h}*}(L) \to K_{\mathrm{h}*}(K)$$

such that the composite is the identity map. This shows that the approximation functor $\alpha_{\mathbf{C}} = K_{\mathrm{h}*}(K)$ is a retract of $L_{\mathrm{h}*}(L)$, and so in order to complete the proof it is enough to show that $L_{\mathrm{h}*}(L)$ has the appropriate invariance property. The functor L is weakly equivalent to the functor L' given by $L'(A, [n]) = A$, so in fact it is enough to treat $L_{\mathrm{h}*}(L')$. Let $P : \mathbf{C} \times \Delta \to \mathbf{C}$ be the projection functor given by $P(A, [n]) = A$. Since $L' = P^*(K)$, we are faced with the functor which in hocoend notation is

$$(\mathbf{S}_*)_+ \otimes^{\mathrm{h}}_{\mathbf{C} \times \Delta} K$$

where K is implicitly considered a left $(\mathbf{C} \times \Delta)$-module via P. Any functor $F : \mathbf{C} \to \mathbf{S}_*$ is equivalent to $\mathbf{C} \otimes^{\mathrm{h}}_{\mathbf{C}} F$ (see [10, 3.1(5)]; homotopy Kan extending over the identity functor leaves functors unchanged up to homotopy). After substituting $\mathbf{C} \otimes^{\mathrm{h}}_{\mathbf{C}} K$ for K and reparenthesizing [10, 3.1(3)], we are left with

$$\big((\mathbf{S}_*)_+ \otimes^{\mathrm{h}}_{\mathbf{C} \times \Delta} \mathbf{C}\big) \otimes^{\mathrm{h}}_{\mathbf{C}} K \ .$$

Recall that this expression represents a functor which depends upon a space X. The way in which X enters the picture is through the functor on the left, which can be decoded as the homotopy left Kan extension, over the projection map $P : (\mathbf{C} \times \Delta)^{\mathrm{op}} \to \mathbf{C}^{\mathrm{op}}$ of the functor H_X which assigns to a pair $(A, [n])$ the pointed set $\mathrm{Hom}_{\mathbf{S}_*}(A \rtimes \Delta_n, X)_+$. Taking a homotopy left Kan extension over a projection map such as this is a particularly simple process; the value $P_{\mathrm{h}*}(H_X)$ on an object A is just the homotopy colimit of the original functor over the category $P^{-1}(A) \cong \Delta^{\mathrm{op}}$. (Compare [10, §6]. The simplest way to see this is to calculate that the inclusion $P^{-1}(A) \to A \downarrow P$ is terminal, by noticing that each one of *its* under categories has a terminal object and hence a contractible nerve.) Let $\mathrm{Map}(A, X)$ denote the space of pointed maps from A to X. The conclusion, via 2.6 and inspection, is that the functor $P_{\mathrm{h}*}(H_X)$ assigns to an object A of $\mathbf{C}$ the geometric realization of the singular complex of $\mathrm{Map}(A, X)_+$. The proposition now follows from the fact that an A-equivalence $X \to Y$ induces a weak equivalence $\mathrm{Map}(A, X)_+ \to \mathrm{Map}(A, Y)_+$. □

For later use we record the following result, which is proved by the above observations and is implicit in [10, §6]; see also [2, III.24].

5.4. Proposition. *Suppose that* $\mathbf{C}$ *and* $\mathbf{D}$ *are small categories, and that* $F\colon\ :\mathbf{C} \times \mathbf{D} \to \mathbf{S}_*$ *is a functor. For each* $d \in \mathbf{D}$ *let* $F_d : \mathbf{C} \to \mathbf{S}_*$ *be given by* $F_d(x) = F(x, d)$*, and let* $G : \mathbf{D} \to \mathbf{S}_*$ *be given by* $G(d) = \mathrm{hocolim}^{\mathbf{L}} F_d$*. Then there is a natural weak equivalence*

$$\mathrm{hocolim}^{\mathbf{L}} F \simeq \mathrm{hocolim}^{\mathbf{L}} G\ .$$

We will use this in the case in which α is a limit ordinal and $\mathbf{D}$ is the ordered set (category), also denoted α, of all ordinals below α. A functor $F : \alpha \to \mathbf{S}_*$ is just a directed system $(X_\beta)_{\beta<\alpha}$. In that case we will need the following observation.

5.5. Proposition. *Let* α *be a limit ordinal and* $(X_\beta)_{\beta<\alpha}$ *a directed system of spaces. Suppose that there is some* $\beta_0 < \alpha$ *such that for all* $\gamma > \beta_0$ *the map* $X_{\beta_0} \to X_\gamma$ *is a weak equivalence. Then* $\mathrm{hocolim}^{\mathbf{L}}_{\beta<\alpha} X_\beta$ *is naturally weakly equivalent to* X_β*.*

Proof. Interpret $(X_\beta)_{\beta<\alpha}$ as a functor $F : \alpha \to \mathbf{S}_*$. Let $[\beta_0, \alpha)$ denote the category of all ordinals γ with $\beta_0 \le \gamma < \alpha$. It is easy to check that the inclusion $J : [\beta_0, \alpha) \to \alpha$ is terminal (3.2), and so $\mathrm{hocolim}^{\mathbf{L}} F \simeq \mathrm{hocolim}^{\mathbf{L}} J^*F$ (3.4). Let F' be the constant functor on $[\beta_0, \alpha)$ which assigns to each γ some cofibrant space weakly equivalent to $F(\beta_0)$; it is clear that there is a natural transformation $F' \to J^*(F)$ which is a weak equivalence on each object, so that $\mathrm{hocolim}\, J^*(F) \simeq \mathrm{hocolim}\, F'$ (2.2). Finally, $\mathrm{hocolim}\, F'$ is homeomorphic to $F'(\beta_0) \rtimes N$, where N is the geometric realization of the nerve of the category $[\beta_0, \alpha)$; we refer to [1, XII.3.5] (in the special case in which the functor Y in question is constant and has value the one-point space) for the fact that N is contractible. □

6. How to compute approximations

Suppose that $\mathbf{C}$ is a full subcategory of $\mathbf{S}_*$ (not necessarily small) such that all of the objects of $\mathbf{C}$ are cofibrant. In this section we describe a technique for computing the homotopy approximations (4.4) $\alpha_{\mathbf{C}}(X) \to X$. The technique depends on the existence of a functor F on $\mathbf{S}_*$ which has a special relationship to $\mathbf{C}$ (6.2), and $\alpha_{\mathbf{C}}(X)$ is then constructed as the homotopy colimit or realization of a resolution of X derived from this functor. At the end we give a criterion for the homotopy approximations we construct to be homotopy invariant as functors of X.

6.1. Adapted pairs. An *augmented functor on* $\mathbf{S}_*$ is a functor $F\colon \mathbf{S}_* \to \mathbf{S}_*$ together with a natural transformation from F to the identity functor. We write this natural transformation $\epsilon_F(X)\colon F(X) \to X$, or just $\epsilon(X)$ if F is understood.

6.2. Definition. Suppose that $\mathbf{C}$ is a full subcategory of $\mathbf{S}_*$ such that all of the objects of $\mathbf{C}$ are cofibrant. An augmented functor F on $\mathbf{S}_*$ is said to be *adapted to* $\mathbf{C}$ if

1. F takes values in $\mathbf{C}$, and
2. for each space B in $\mathbf{C}$, the map $\epsilon(B)\colon F(B) \to B$ has a section (i.e. a right inverse).

If F is adapted to $\mathbf{C}$, then $(\mathbf{C}, F)$ is called an *adapted pair*.

6.3. *Remark.* A morphism $(\mathbf{C}, F) \to (\mathbf{C}', F')$ of adapted pairs is an inclusion $\mathbf{C} \subset \mathbf{C}'$ together with a natural transformation $F \to F'$ which respects the augmentations of the two functors involved. The functor F in an adapted pair $(\mathbf{C}, F)$ is not in any sense determined by $\mathbf{C}$; for instance, if $(\mathbf{C}, F)$ is an adapted pair, so is $(\mathbf{C}, F^n)$ for any $n > 1$.

Now suppose that $(\mathbf{C}, F)$ is an adapted pair and that X is a space. Associated to F is a restricted simplicial space $\mathbf{F}X$ (see 2.8) with $(\mathbf{F}X)_n = F^{n+1}(X)$ and

$$((\mathbf{F}X)_n \xrightarrow{d_i} (\mathbf{F}X)_{n-1}) = (F^{n+1}X \xrightarrow{F^i\epsilon(F^{n-i}X)} F^nX) .$$

The restricted simplicial identities amount to diagrams which commute because ϵ is a natural transformation. This object is augmented by the map $\epsilon(X) : (\mathbf{F}X)_0 = FX \to X$, and so can be interpreted as a kind of resolution of X by values of F. We define a functor $\hat{F}$ by taking the homotopy colimit or realization of the resolution (2.8):

$$\hat{F}(X) = \operatorname{hocolim} \mathbf{F}X \ .$$

The augmentation of $\mathbf{F}X$ induces a natural map $\hat{F}(X) \to X$, so that $\hat{F}$ is itself an augmented functor.

Since F takes values in $\mathbf{C}$, the resolution together with its augmentation can be considered as a functor $\mathbf{F}X : \Lambda^{\text{op}} \to \mathbf{C}\downarrow X$ (4.1). Our constructions depend on the following property of this functor.

6.4. Proposition. *If* $(\mathbf{C}, F)$ *is an adapted pair, then for any* $X \in \mathbf{S}_*$ *the resolution functor*

$$\mathbf{F}X : \Lambda^{\text{op}} \to \mathbf{C}\downarrow X$$

is terminal (3.2).

The proof of this is below. In particular, 6.4 implies that the categories $\mathbf{C}\downarrow X$ are terminally small (3.11). The following result is immediate (4.4).

6.5. Proposition. *If* $(\mathbf{C}, F)$ *is an adapted pair, then the natural map* $\hat{F}(X) \to X$ *is an approximation map* $\alpha_{\mathbf{C}}(X) \to X$*, i.e.,* $\hat{F}$ *is a left homotopy Kan extension of the inclusion functor* $J : \mathbf{C} \to \mathbf{S}_*$ *over itself.*

6.6. *Remark.* In particular, if $(\mathbf{C}, F)$ is an adapted pair and $X \in \mathbf{C}$, then (4.5) the map $\hat{F}(X) \to X$ is an equivalence.

Proof of 6.4. Write Hom_X for the set of morphisms in the category $\mathbf{C}\downarrow X$. To verify that $\mathbf{F}X$ is terminal in $\mathbf{C}\downarrow X$, it is enough to check that for each object $B \to X$ of this category the restricted simplicial set $\operatorname{Hom}_X(B, \mathbf{F}X)$ admits a suitable left contraction (3.6). Since F is adapted to $\mathbf{C}$, we can choose a section $s : B \to F(B)$ of the augmentation $F(B) \to B$, and define the contraction by letting s_{-1} : $\operatorname{Hom}_X(B, (\mathbf{F}X)_n) \to \operatorname{Hom}_X(B, (\mathbf{F}X)_{n+1})$ send a map $f : B \to F^{n+1}(X)$ to the composite

$$B \xrightarrow{s} FB \xrightarrow{F(f)} F^{n+2}(X) \ .$$

It is routine to verify the necessary identities. □

6.7. Invariance properties of approximations. We now study when the construction $\alpha_{\mathbf{C}}(X) = \hat{F}(X)$ described above is homotopy invariant as a functor of X. In fact under fairly mild conditions the construction satisfies an even stronger invariance property.

6.8. Proposition. *Suppose that* $(\mathbf{C}, F)$ *is an adapted pair. Assume that that the category* $\mathbf{C}$ *is closed under the functors* $B \mapsto B \rtimes \Delta_n$*,* $n \geq 0$*. Then if* $X \to Y$ *is a map of spaces which is a* B*-equivalence for every object* B *of* $\mathbf{C}$*, the induced map* $\hat{F}(X) \to \hat{F}(Y)$ *is an equivalence.*

Proof. This almost follows from a combination of 6.5 and 5.1, but there is a smallness hypothesis in 5.1 which is not necessarily satisfied here. Suppose that $X \to Y$ is a map of the indicated type. Let $\mathbf{D}$ be the smallest full subcategory of $\mathbf{C}$ which contains the spaces F^nX, F^nY, $n \geq 1$, and is closed under the constructions $B \mapsto B \rtimes \Delta_n$, $n \geq 0$. It is clear that $\mathbf{D}$ is in fact a small category. The argument of 6.4 shows that the functors $\mathbf{F}X : \Lambda^{\mathrm{op}} \to \mathbf{D} \downarrow X$ and $\mathbf{F}Y : \Lambda^{\mathrm{op}} \to \mathbf{D} \downarrow Y$ are terminal, so that $\hat{F}(X)$ and $\hat{F}(Y)$ compute respectively $\alpha_{\mathbf{D}}(X)$ and $\alpha_{\mathbf{D}}(Y)$. By 5.1, these approximations are equivalent. $\square$

7. Building adapted pairs

In this section we start with an adapted pair $(\mathbf{C}, F)$ in which $\mathbf{C}$ is closed under equivalences (7.4) and construct a directed system (6.3)

$$(\mathbf{C}, F) = (\mathbf{C}_0, F_0) \to (\mathbf{C}_1, F_1) \to \cdots \to (\mathbf{C}_\alpha, F_\alpha) \to \cdots$$

of adapted pairs, indexed by the ordinal numbers. Although this is a general process, we use notation reminiscent of §1 because it is the techniques of this section that yield the objects described in §1. It is useful to start with some definitions.

7.1. Definition. Suppose that $\mathbf{C}$ is a full subcategory of $\mathbf{S}_*$ such that all of the objects of $\mathbf{C}$ are cofibrant. The category $\mathrm{Hocolim}(\mathbf{C})$ is then the category of all spaces Y for which there exists a small category $\mathbf{D}$ and functor $G : \mathbf{D} \to \mathbf{C}$ with $Y = \operatorname{hocolim} G$.

7.2. Definition. Suppose that G is an augmented functor. Then G^{fib} is the augmented functor constructed by factoring each natural map $G(X) \to X$ in some fixed functorial way into a composite $G(X) \to G^{\mathrm{fib}}(X) \to X$, where $G(X) \to G^{\mathrm{fib}}(X)$ is a cofibration and a weak equivalence, and $G^{\mathrm{fib}}(X) \to X$ is a fibration.

The categories $\mathbf{C}_\alpha$ are defined inductively. To start with, $\mathbf{C}_0$ is the category $\mathbf{C}$. Suppose that $\alpha > 0$, and that $\mathbf{C}_\beta$ has been defined for all $\beta < \alpha$, such that $\mathbf{C}_\gamma \subset \mathbf{C}_\beta$ if $\gamma < \beta$. Let $\mathbf{C}_{<\alpha}$ denote the increasing union $\cup_{\beta<\alpha}\mathbf{C}_\beta$. Then $\mathbf{C}_\alpha$ is defined to be the full subcategory of $\mathbf{S}_*$ whose objects consist of all cofibrant spaces which up to homotopy are retracts of objects of $\mathrm{Hocolim}(\mathbf{C}_{<\alpha})$.

Let $F_0 = F$. Suppose that $\alpha > 0$, and that we have already constructed a directed system $\{F_\beta\}_{\beta<\alpha}$ of augmented functors, such that

F_β takes values in $\mathbf{C}_\beta$. For any space X there is a functor

$$\rho^X_\alpha : \alpha \times \Lambda^{\mathrm{op}} \to \mathbf{C}_{<\alpha} \downarrow X \tag{7.3}$$

given by $\rho_X(\beta, [n]) = (\mathbf{F}_\beta X)_n = F^{n+1}_\beta(X)$. This functor is built in such a way that it is a combination of all of the F_β–resolutions of X for $\beta < \alpha$ (§6). Let $G_\alpha(X) = \operatorname{hocolim} \rho^X_\alpha$; it is clear that G_α is an augmented functor, and we define F_α inductively by letting $F_\alpha = G^{\mathrm{fib}}_\alpha$ (7.2). By construction the functor F_α takes values in $\mathbf{C}_\alpha$. Notice that if $\beta < \alpha$ then the restriction of ρ^X_α to $\beta \times \Lambda^{\mathrm{op}}$ is exactly ρ^X_β; we thus get compatible maps

$$G_\beta(X) = \operatorname{hocolim} \rho^X_\beta \to \operatorname{hocolim} \rho^X_\alpha = G_\alpha(X), \quad 0 < \beta < \alpha$$

which, by the functoriality of the factorization process used in 7.2, pass to compatible maps

$$F_\beta(X) = G^{\mathrm{fib}}_\beta(X) \to G^{\mathrm{fib}}_\alpha(X) = F_\alpha(X), \quad 0 < \beta < \alpha.$$

We leave it to the reader to construct $F_0 \to F_1$.

The main result of this section is the following one. Say that the category $\mathbf{C}$ is *closed under equivalences* if whenever A and B are cofibrant spaces which are homotopy equivalent, then both are in $\mathbf{C}$ if either one is.

7.4. Proposition. *Suppose that* $(\mathbf{C}, F)$ *is an adapted pair and that* $\mathbf{C}$ *is closed under equivalences. Then each of the pairs* $(\mathbf{C}_\alpha, F_\alpha)$ *constructed above is also an adapted pair. Moreover, for* $\alpha > 0$ *the natural map* $F_\alpha(X) \to X$ *is the canonical approximation to* X *by a homotopy colimit of spaces from* $\mathbf{C}_{<\alpha}$ *(4.4).*

7.5. *Remark.* The construction above may seem elaborate, but the functors F_α are much less complicated than they look; the objects ρ^X_α appear only as a device for obtaining the correct maps $F_\beta \to F_\alpha$. In fact, if $\alpha = \beta + 1$ is a successor ordinal then $F_\alpha(X) \simeq \hat{F}_\beta(X)$; if α is a limit ordinal, the $F_\alpha(X) \simeq \operatorname{hocolim}_{\beta<\alpha} F_\beta(X)$.

To see this, suppose first that $\alpha = \beta + 1$ is a successor ordinal. The category α then has the element β as a terminal object (maximal element), and it follows easily from 5.4 or direct calculation that the inclusion functor

$$\Lambda^{\mathrm{op}} \to \alpha \times \Lambda^{\mathrm{op}}, \quad [n] \mapsto (\beta, [n]) \tag{7.6}$$

is terminal (3.2). The restriction of ρ^X_α to Λ^{op} over this inclusion functor is the resolution functor $\mathbf{F}_\beta X$, and so by 3.4 the resulting maps

$$\hat{F}_\beta(X) = \operatorname{hocolim} \mathbf{F}_\beta X \to \operatorname{hocolim} \rho^X_\alpha = G_\alpha(X) \to F_\alpha(X)$$

are equivalences. If α is a limit ordinal then, by 5.4, $\operatorname{hocolim} \rho_\alpha^X$ is equivalent to $\operatorname{hocolim}_{\beta<\alpha} \hat{F}_\beta(X)$. Since as above $\hat{F}_\beta(X)$ is equivalent to $F_{\beta+1}(X)$, we obtain an equivalence

$$F_\alpha(X) \simeq \operatorname{hocolim}_{\beta<\alpha} F_{\beta+1}(X) \simeq \operatorname{hocolim}_{\beta<\alpha} F_\beta(X) \ .$$

Proof of 7.4. We prove both statements by induction. By assumption, $(\mathbf{C}, F) = (\mathbf{C}_0, F_0)$ is an adapted pair. Suppose that $\alpha > 0$ and that for each $\beta < \alpha$ $(\mathbf{C}_\beta, F_\beta)$ is an adapted pair; suppose as well that if $0 < \beta < \alpha$ then F_β is the indicated approximation functor. We prove first that F_α has the desired approximation property. By definition (4.4), this will be the case if for each X the functor ρ_α^X (7.3) is terminal. Suppose that $\alpha = \beta + 1$ is a successor ordinal. Then the composite of ρ_α^X with the inclusion functor 7.6 is terminal (see 6.4), and, as mentioned above, the inclusion functor itself is terminal. By 3.4, ρ_α^X is also terminal. Suppose on the other hand that α is a limit ordinal; by 5.4 and 5.5, in order to show that ρ_α^X is terminal it is enough to show that for each $B \in \mathbf{C}_{<\alpha} \downarrow X$ and all sufficiently large ordinals $\gamma < \alpha$, the restricted simplicial set $\operatorname{Hom}_X(B, \mathbf{F}_\gamma X)$ has a contractible realization (see the proof of 6.4 for the notation here). But this is clear: if $B \in \mathbf{C}_\beta$, it is sufficient to choose $\gamma \geq \beta$, and then the desired contraction is exhibited in the proof of 6.4.

What remains is to prove that F_α is adapted to $\mathbf{C}_\alpha$, or in other words to verify 6.2(2). Note that for any space $B \in \mathbf{C}_{<\alpha}$ the natural map $F_\alpha(B) \to B$ is a weak equivalence; this follows (4.5) from the approximation property proved above. Suppose that $X \in \operatorname{Hocolim}(\mathbf{C}_{<\alpha})$. Given 2.5 and the fact that $\mathbf{C}_{<\alpha}$ is closed under equivalences, X is homotopy equivalent to an appropriate Y, where Y is cofibrant and $Y = \operatorname{colim} G$ for some $G : \mathbf{D} \to \mathbf{C}_{<\alpha}$ with $\operatorname{hocolim} G \simeq \operatorname{colim} G$. Consider the commutative diagram

$$\begin{array}{ccc} \operatorname{hocolim} F_\alpha \circ G & \longrightarrow & F_\alpha(Y) \\ \downarrow & & \downarrow{\scriptstyle \epsilon_{F_\alpha}(Y)} \\ \operatorname{hocolim} G & \xrightarrow{\simeq} & Y \end{array}$$

in which the vertical maps are induced by the augmentation of F_α. The left vertical map is an equivalence because of the invariance property of hocolim (2.2) and the observation above that $F_\alpha(B) \simeq B$ for each $B \in \mathbf{C}_{<\alpha}$ (6.6). It follows that the map $\epsilon_{F_\alpha}(Y)$ has a section up to homotopy. Since the map is a fibration and the base is cofibrant, the homotopy lifting property implies that this homotopy section can be deformed into a genuine section s_Y of $\epsilon_{F_\alpha}(Y)$. Any X in $\mathbf{C}_\alpha$ is up to homotopy a retract of such a Y. Given X, choose maps $u : X \to Y$

and $v : Y \to X$ such that vu is homotopic to the identity map of X. The composite

$$X \xrightarrow{u} Y \xrightarrow{s_Y} F_\alpha(Y) \xrightarrow{F_\alpha(v)} F_\alpha(X)$$

is then a section up to homotopy of $\epsilon_{F_\alpha}(X)$. As above, this can be deformed into a genuine section. □

8. The main results

Here we prove the results described in §1. The first step is to construct the chain 1.1 of categories. In fact we will construct a chain

$$(\mathbf{C}_0, F_0) \to (\mathbf{C}_1, F_1) \to \cdots \to (\mathbf{C}_\alpha, F_\alpha) \cdots$$

of adapted pairs (6.2). Recall that A is a fixed cofibrant object of $\mathbf{S}_*$. The category $\mathbf{C}_0$ is defined to be the full subcategory of $\mathbf{S}_*$ containing all cofibrant spaces B which are homotopy equivalent to wedges of copies of A. Let G be the augmented functor which assigns to X the wedge $\vee_f A$, where the wedge is taken over the set of all maps $A \to X$; the augmentation $G(X) \to X$ is the obvious one which on the wedge summand of $G(X)$ corresponding to the map f is the map f itself. The functor F_0 is defined to be G^{fib} (7.2). It is clear that $(\mathbf{C}_0, F_0)$ satisfies the conditions of 6.2.

The pairs $(\mathbf{C}_\alpha, F_\alpha)$ for $\alpha > 0$ are now obtained by the method of §7. Theorem 1.2 follows from 7.4, and Theorem 1.4 from 7.4 and 4.5.

Proof of 1.3. We first assume that $X \to Y$ is an A-equivalence, and show by induction on α that $F_\alpha(X) \simeq F_\alpha(Y)$ for $\alpha > 0$. If $\alpha = \beta + 1$ is a successor ordinal this is a consequence of 7.5 and 6.8; note that by construction each space B in $\mathbf{C}_\beta$ is A-cellular, so that $X \to Y$ is a fortiori a B-equivalence. If α is a limit ordinal the result follows from 2.2 and the description of F_α in 7.5.

Suppose on the other hand that $F_\alpha(X) \simeq F_\alpha(Y)$ for some $\alpha > 0$. An induction using the result of the paragraph above shows that $F^n_\alpha(X) \simeq F^n_\alpha(Y)$ for $n \geq 1$, and hence by 2.2 and 7.5 that $F_{\alpha+1}(X) \simeq F_{\alpha+1}(Y)$. Further induction along these lines leads to the conclusion that $F_\beta(X) \simeq F_\beta(Y)$ for all $\beta \geq \alpha$. Choosing β sufficiently large (see 1.5, which depends only on the "if" part of 2.2) gives $\mathrm{CW}_A(X) \simeq \mathrm{CW}_A(Y)$, and hence that $X \to Y$ is an A-equivalence. □

9. Examples

Here we look at some particular choices for the space A: finite complexes, Moore spaces, and spheres.

9.1. Finite complexes. It is probably not surprising that if A is a finite complex there cannot be spaces of large A-complication.

9.2. Proposition. *Suppose that A is a finite complex. Then any X has $\kappa_A(X) \le \omega$, where ω is the first infinite ordinal. Moreover, $F_\omega(X)$ is equivalent to* $\mathrm{CW}_A(X)$.

9.3. *Remark.* In general, $\kappa_A(X) \le \omega$ would only imply $F_{\omega+1}(X) \simeq \mathrm{CW}_A(X)$ (1.4).

Proof of 9.2. To prove the result, it is enough to show that the natural map $F_\omega(X) \to X$ is an A-equivalence. Say that a space K is a finite A-cellular complex if it is built from a point by attaching a finite number of copies of A and its suspensions. In order to complete the proof it is enough to show that in any commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & F_\omega(X) \\ \downarrow & & \downarrow {\scriptstyle \epsilon(X)} \\ L & \longrightarrow & X \end{array}$$

in which K and L are finite A-cellular complexes, there is a lift $L \to F_\omega(X)$ which makes the diagram commute up to homotopy. The finiteness assumption on A implies that K is a finite complex in the ordinary sense, and so the map $A \to F_\omega(X) = \mathrm{hocolim}_n\, F_n(X)$ actually factors through $F_{n_0}(X)$ for some n_0. Consider the map of squares

$$\begin{array}{ccc} F_\omega(K) & \longrightarrow & F_\omega F_{n_0}(X) \\ \downarrow & & \downarrow \\ F_\omega(L) & \longrightarrow & F_\omega(X) \end{array} \quad \longrightarrow \quad \begin{array}{ccc} K & \longrightarrow & F_{n_0}(X) \\ \downarrow & & \downarrow \\ L & \longrightarrow & X \end{array}$$

induced by the augmentation of the functor F_ω. Since K, L, and $F_{n_0}(X)$ all lie in $\mathbf{C}_N$ for sufficiently large N, the map of squares is an equivalence except at the lower right-hand corner (1.4). In particular, the map $L \simeq F_\omega(L) \to F_\omega(X)$ gives up to homotopy the required lift. □

9.4. The Moore space. We next show that in some cases the bound from 9.2 is sharp.

9.5. Proposition. *Suppose that A is a Moore space $M(\mathbb{Z}/p, n)$, $n \ge 1$. Then $\kappa_A^{max} = \omega$, i.e.,there exists spaces X with $\kappa_A(X) = \omega$.*

Proof. It is elementary to see that $M(\mathbb{Z}/p^\infty, n+1)$ is an A-cellular space. By 9.2, $\kappa_A(M(\mathbb{Z}/p^\infty, n+1)) \le \omega$. We will show that the complication of this space actually equals ω, by showing with an induction on n that if B is a space with $\kappa_A(B) = n$ for some integer n, then each reduced integral homology group of B is killed by some power of p which depends only on the integer n and the degree of the homology group. This is clear if $\kappa_A(B) = 0$. Inductively, any space B of complication n can be expressed as a homotopy colimit of spaces of complication $(n-1)$. There is an associated first quadrant homology spectral sequence [1, XII.5.6] which in the limit gives each homology group of B up to a finite filtration. Inspecting the E^2-page shows that the filtration quotients $\mathrm{Gr}_j \tilde{\mathrm{H}}_k(B)$, $0 \le j \le k$, are subquotients of groups of the form $\oplus_\alpha \tilde{\mathrm{H}}_{k-j}(Y_{\alpha,j})$, where $\{Y_{\alpha,j}\}$ is a collection of spaces of complication $(n-1)$. The result follows immediately. □

9.6. The sphere. Suppose that A is a sphere S^n; in this case a space is A-cellular if and only if it is $(n-1)$-connected. Stover [16] shows by an explicit construction that every $(n-1)$ connected space is weakly equivalent to the homotopy colimit of a functor whose values are equivalent to wedges of copies of S^k, $k \ge n$. It is immediate that such a wedge has A-complication ≤ 1; it follows that for any space X, $\kappa_A(X) \le 2$.

With more work it is probably possible to do better. We briefly indicate how to accomplish this for $A = S^1$. Any 1-connected space X is weakly equivalent to the geometric realization of the classifying space of a simplicial group GX with the property that each constituent $(GX)_n$ is a free group. For instance, take GX to be the Kan loop group [12, §26] of the singular complex of X. One can take the geometric realization of the classifying space of each of the constituent groups $(GX)_n$; by functoriality these form a simplicial space Y. Of course each space Y_n is equivalent to a wedge of circles. It is not hard to see that X is equivalent to the homotopy colimit or realization of Y, and so $\kappa_A(X) \le 1$.

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