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Conjectural calculations of general linear group homology

W.G. Dwyer* and E.M. Friedlander*

ABSTRACT. We use etale homotopy theory together with the work of Artin and Verdier to translate some of the K-theory conjectures of Quillen and Lichtenbaum into explicit conjectures about general linear group homology.

§1. INTRODUCTION. Let F be an algebraic number field (a finite extension of the rational numbers $\mathbb{Q}$), let $\mathcal{O}$ be the ring of algebraic integers in F, and let ℓ be an ordinary prime number. Quillen [12] and Lichtenbaum [8] have offered some remarkable conjectures relating the (ℓ-adic) algebraic K-theory of $\mathcal{O}$ to more classical invariants of the field F. The most dramatic statement posits a connection between orders of various K-groups which are known to be finite and values of the zeta-function of F:

CONJECTURE 1.1. [8, 2.4] If F is totally real and m is an odd positive integer, then, up to powers of two,

$$\#K_{2m}(\mathcal{O})/\#K_{2m+1}(\mathcal{O}) = |\zeta(F, -m)|.$$

Another formulation, which is tied to 1.1 by known and suspected connections between values of the zeta-function and orders of etale cohomology groups, gives an explicit description of ℓ-adic K-theory in terms of etale cohomology:

CONJECTURE 1.2. [12, §9] If ℓ is odd or F is totally imaginary, there are isomorphisms

$$K_n(\mathcal{O}[1/\ell]) \otimes \mathbb{Z}_\ell \approx H^j_{cont}(\mathrm{Spec}\, \mathcal{O}[1/\ell]_{et}, \mathbb{Z}_\ell(i))$$

where $n = 2i - j$; $j = 1, 2$; and $n \geq 1$.

As in [6] (generalized according to §3 below if $\ell = 2$ and F has a real embedding) let $\hat{K}^{et}_*(\mathcal{O}[1/\ell])$ denote the ℓ-adic etale K-theory of $\mathcal{O}[1/\ell]$, which

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is defined in terms of the etale homotopy theory of Spec $\mathcal{O}[1/\ell]$. There is a natural map $\phi_* : K_*(\mathcal{O}[1/\ell]) \otimes \mathbb{Z}_\ell \to \hat{K}_*^{et}(\mathcal{O}[1/\ell])$. The following conjecture includes 1.2 [6, 8.8]:

CONJECTURE 1.3. ([6, §8] and §3 below) *The map*

$$\phi_* : K_n(\mathcal{O}[1/\ell]) \otimes \mathbb{Z}_\ell \to \hat{K}_n^{et}(\mathcal{O}[1/\ell])$$

is an isomorphism if $n \geq 1$.

The main technical attraction of 1.3 is that the map ϕ_* is *geometric*, that is, it can be identified with the homotopy map $\pi_*U \to \pi_*V$ induced by a certain map $\phi : U \to V$ between connected pointed spaces (see §3). In [6] we exploited this geometric quality to show that the map ϕ_* is often surjective. In this paper we observe that ϕ_* is an isomorphism iff ϕ induces a homology isomorphism $H_*(U,\mathbb{Z}/\ell) \to H_*(V,\mathbb{Z}/\ell)$. By construction $H_*(U,\mathbb{Z}/\ell)$ is exactly the Eilenberg-MacLane group homology of the infinite general linear group $GL(\mathcal{O}[1/\ell])$, and in some situations we can explicitly compute $H_*(V,\mathbb{Z}/\ell)$. This leads to some specific conjectures about general linear group homology (4.3, 4.6). These conjectures would be implied by the truth of 1.3.

Our main technique in computing $H_*(V,\mathbb{Z}/\ell)$ is to combine the etale homotopy theoretic methods of [6] with the etale cohomology calculations of Artin and Verdier [3] (cf. [9], [14]). The expert will appreciate that we can work only in those number-theoretic contexts in which class group considerations (e.g. numerators of Bernoulli numbers) do not come into play.

NOTATION. Throughout the paper, ℓ denotes a fixed prime, R the ring $\mathbb{Z}[1/\ell]$, F an algebraic number field, and $\mathcal{O}$ the ring of algebraic integers in F.

§2. ETALE HOMOLOGY OF NUMBER RINGS.

The Artin-Verdier duality theorem for the etale cohomology of rings of integers in number fields [3] enables us to compute the etale homology of $\mathcal{O}$. This computation is just a re-interpretation of the duality theorem and related results as presented by Mazur [9] and Zink [14].

If A is a noetherian ring, the i'th etale homology group of A with trivial $\mathbb{Z}/k$ coefficients is the pro-abelian-group $H_i(\text{Spec } A_{et},\mathbb{Z}/k)$ described in [7, p. 68].

DEFINITION. *The* i'th *etale homology group of* A *with trivial (pro-finite) integral coefficients*, denoted $H_i(\text{Spec } A_{et})$, is the composite pro-abelian group $\{H_i(\text{Spec } A_{et}, \mathbb{Z}/k)\}$ in which the additional structure maps are induced by the natural surjections $\mathbb{Z}/k \to \mathbb{Z}/k'$ for $k' \mid k$.

The pro-groups $H_i(\mathrm{Spec}\ A_{et})$ are the analogues, for the etale topology on Spec A, of the ordinary homology groups of a space with trivial (i.e. untwisted) integral coefficients. Describing these pro-groups for a ring of algebraic integers $\mathcal{O}$ requires some notation. Let r_1 denote the number of real embeddings of $\mathcal{O}$. A non-zero element of $\mathcal{O}$ is said to be totally positive if its image under any real embedding of $\mathcal{O}$ is a positive real number. Let $\mathcal{O}^*_{pos}$ denote the multiplicative group of totally positive units of $\mathcal{O}$, and $Cl_{pos}(\mathcal{O})$ the ray class group of $\mathcal{O}$ (i.e. the group of fractional ideals modulo principal fractional ideals with totally positive generators). Let "^" denote profinite completion.

THEOREM 2.1. ([3], [9], [14]) <u>As above, let $\mathcal{O}$ be the ring of algebraic integers in a number field. Then there are natural isomorphisms</u>

$$H_i(\mathrm{Spec}\ \mathcal{O}_{et}) \simeq \begin{cases} \mathbb{Z}^\wedge & i = 0 \\ Cl_{pos}(\mathcal{O}) & i = 1 \\ (\mathcal{O}^*_{pos})^\wedge & i = 2 \\ (\mathbb{Z}/2)^{r_1} & i \text{ odd},\ i \geq 3 \\ 0 & i \text{ even},\ i \geq 3 \end{cases}$$

SKETCH OF PROOF. If $\mathcal{O}$ is totally imaginary $(r_1 = 0)$, Theorem 3.1 of [3] gives isomorphisms

$$H_q(\mathrm{Spec}\ \mathcal{O}_{et}, \mathbb{Z}/k) \to \mathrm{Ext}^{3-q}(\mathrm{Spec}\ \mathcal{O}_{et}; \mathbb{Z}/k, G_m)$$

(see also [9, 2.4] and [14, 3.2.1]). The Ext-groups on the right-hand-side can be calculated easily using [3, Cor. 1.5] (see also [9, p. 539]) and the theorem follows by passing to the limit in k. If $\mathcal{O}$ has a real embedding $(r_1 > 0)$ the theorem is proved by combining the more delicate duality theorem of [14, 3.2.1] with several long-exact sequence arguments.

From the calculation it is easy to derive information more directly relevant to K-theory about $\mathrm{Spec}\ \mathcal{O}[1/\ell]_{et}$. For each prime α of $\mathcal{O}$ above ℓ, let $\mathcal{O}_\alpha$ denote the completion of $\mathcal{O}$ at α and F_α the completion of the quotient field F of $\mathcal{O}$ at α. According to the decomposition lemma [9] [1] there is a homotopy pushout diagram

$$\begin{array}{ccc} \coprod_{\alpha \mid \ell} (\mathrm{Spec}\ F_\alpha)_{et} & \to & (\mathrm{Spec}\ \mathcal{O}[1/\ell])_{et} \\ \downarrow & & \downarrow \\ \coprod_{\alpha \mid \ell} (\mathrm{Spec}\ \mathcal{O}_\alpha)_{et} & \to & (\mathrm{Spec}\ \mathcal{O})_{et} \end{array}$$

of etale homotopy types. According to local class field theory [9], [13] the relative homology group $H_i((\mathrm{Spec}\ \mathcal{O}_\alpha)_{et}, (\mathrm{Spec}\ F_\alpha)_{et})$ is $(\mathcal{O}^*_\alpha)\hat{}$ if $i = 2$ and and zero otherwise. Calculating with the long exact homology sequence of the right-hand vertical map above then gives the following theorem. To simplify notation, let $A^\ell_{pos}(\mathcal{O})$ and $B^\ell_{pos}(\mathcal{O})$ denote the kernel and cokernel respectively of the natural map

$$(\mathcal{O}^*_{pos})\hat{} \to (\prod_{\alpha \mid \ell} \mathcal{O}^*_\alpha)\hat{}$$

THEOREM 2.2. <u>There are natural isomorphisms</u>

$$H_i(\mathrm{Spec}\ \mathcal{O}[1/\ell]_{et}) \simeq \begin{cases} \mathbb{Z}\hat{} & i = 0 \\ A^\ell_{pos}(\mathcal{O}) & i = 2 \\ (\mathbb{Z}/2)^{r_1} & i \text{ odd},\ i \geq 3 \\ 0 & i \text{ even},\ i \geq 3 \end{cases}$$

<u>as well as a short exact sequence</u>

$$0 \to B^\ell_{pos}(\mathcal{O}) \to H_1(\mathrm{Spec}\ \mathcal{O}[1/\ell]_{et}) \to Cl_{pos}(\mathcal{O}) \to 0$$

REMARK. By global class field theory it is possible to identify $H_1(\mathrm{Spec}\ \mathcal{O}[1/\ell]_{et})$ as a more or less evident quotient of the group of idèles of F.

§3. REFORMULATION OF THE QUILLEN-LICHTENBAUM CONJECTURE.

The purpose of this section is to reformulate the Quillen-Lichtenbaum conjecture into an assertion that the mod ℓ cohomology of the infinite general linear group over $\mathcal{O}$ can be calculated in terms of the mod ℓ cohomology of the reduced etale K-theory space. This observation, implicit in [6], be- ... the invariance property of Corollary 3.3.

As usual, R is the ring $\mathbb{Z}[1/\ell]$. For each $n \geq 0$ let GL_n denote the rank n general linear group scheme over R. According to [6, 2.5], for any noetherian R-algebra A there exists a natural map

$$\phi_n : BGL_n(A) = \mathrm{Hom}(A,BGL_n)_R \to \mathrm{Hom}_\ell(A,BGL_n)_R .$$

The maps ϕ_n pass to a limit map

$$\phi : BGL(A) \to \lim_{\substack{\to \\ n}} \mathrm{Hom}\ (A,BGL_n)_R .$$

If A has finite mod-ℓ etale cohomological dimension, we will let $\tilde{K}^{et}(A)$ denote the component of $\lim_{\substack{\to \\ n}} \mathrm{Hom}_\ell(A,BGL_n)_R$ which contains the image of ϕ.

PROPOSITION 3.1. <u>Suppose that $\mathcal{O}$ is the ring of algebraic integers in a number field, and that ℓ is odd or that $\ell = 2$ and $\sqrt{-1} \in \mathcal{O}$. Then the</u> ℓ-adic <u>Quillen Lichtenbaum conjecture for $\mathcal{O}$ is true iff the above map</u>

$$\phi : BGL(\mathcal{O}[1/\ell]) \to \tilde{K}^{et}(\mathcal{O}[1/\ell])$$

<u>induces an isomorphism on</u> mod ℓ <u>cohomology</u>.

PROOF. This follows immediately from [6; 4.5, 8.8] and the appropriate Whitehead theorem.

REMARK. If ℓ is odd or $\mathcal{O}$ has no real embeddings (i.e., if $\mathcal{O}$ has finite mod ℓ etale cohomological dimension) we will refer to the conjecture that the map ϕ of 3.1 induces an isomorphism on mod ℓ cohomology as the ℓ-adic <u>Quillen-Lichtenbaum conjecture</u> for $\mathcal{O}$, even in cases to which 3.1 does not apply.

Let $f : X \to \mathrm{Spec}\ R_{et}$ be a map of pro-spaces. If X has finite mod ℓ cohomological dimension, let $\tilde{K}^{top}(X)$ denote the identity component of the direct limit

$$\lim_{\substack{\to \\ n}} \mathrm{Hom}_\ell(X,(BGL_n)_{et})_{R_{et}}$$

(See [6, 2.3]. The identity component of $\mathrm{Hom}_\ell(X,(BGL_n)_{et})_{R_{et}}$ is the component which contains the composite map $X \to \mathrm{Spec}\ R_{et} \to (BGL_n)_{et}$ induced by the section $\mathrm{Spec}\ R \to BGL_n$ corresponding to the identity matrix in $GL_n(R)$).

Observe that if A is a noetherian R-algebra of finite mod ℓ etale cohomological dimension, X a pro-space of finite mod ℓ cohomological dimension, and $X \to \mathrm{Spec}\, A_{et}$ a map, the commutative diagram

$$\begin{array}{ccc} X & \to & (\mathrm{Spec}\, A)_{et} \\ & \searrow \quad \swarrow & \\ & (\mathrm{Spec}\, R)_{et} & \end{array}$$

induces a map $\tilde{K}^{et}(A) \to \tilde{K}^{top}(X)$.

Let ζ_ℓ denote a primitive ℓ'th root of unity. If $X \to (\mathrm{Spec}\, R)_{et}$ is a map of pro-spaces, let $\tilde{X}$ denote the pullback to X of the $(\ell - 1)$-fold Galois covering $\mathrm{Spec}\, R[\zeta_\ell]_{et} \to \mathrm{Spec}\, R_{et}$.

PROPOSITION 3.2. Suppose that A is a noetherian R-algebra of finite mod ℓ etale cohomological dimension, X a pro-space of finite mod ℓ cohomological dimension, and $f : X \to (\mathrm{Spec}\, A)_{et}$ a map. Then if $\tilde{f} : \tilde{X} \to (\mathrm{Spec}\, A)^{\sim}_{et}$ induces an isomorphism on mod ℓ cohomology, the induced map $\tilde{K}^{et}(A) \to \tilde{K}^{top}(X)$ is a homotopy equivalence.

PROOF. Let $\mathbb{Z}_\ell(i)$ $(i \geq 0)$ denote the coefficient system of [6, §5]. It follows from the hypotheses, together with a limit argument [6, 2.9], that f induces isomorphisms

$$H^i_{cont}(\mathrm{Spec}\, A_{et}, \mathbb{Z}_\ell(i)) \to H^i_{cont}(X, \mathbb{Z}_\ell(i))$$

for any $i \geq 0$. The result now follows from [6, Proof of 5.1] and the obstruction theory spectral sequences [6, Proof of 2.11].

Assume for the moment that $\ell = 2$. In this case none of the above discussion applies to $\mathcal{O}[1/\ell]$ if the number ring $\mathcal{O}$ has a real embedding, since then $\mathcal{O}[1/\ell]$ has infinite mod ℓ etale cohomological dimension. However, there does exist a finite etale cover of $\mathrm{Spec}\, \mathcal{O}[1/\ell]$, namely $\mathrm{Spec}\, \mathcal{O}[1/\ell, \sqrt{-1}]$, which does have finite mod ℓ etale cohomological dimension. With this in mind we will extend the above machinery as follows.

A noetherian R-algebra A is said to have virtually finite mod ℓ etale cohomological dimension if there is a finite etale Galois covering $p : \mathrm{Spec}\, A' \to \mathrm{Spec}\, A$ with the property that $\mathrm{Spec}\, A'$ has finite mod ℓ etale cohomological dimension. For such an A, define $\tilde{K}^{et}(A)$ to be the "identity component" of the function space

$$\mathrm{Hom}^{\Gamma}(E\Gamma, \tilde{K}^{et}(A'))$$

where Γ is the Galois group of A' over A, $E\Gamma$ is a contractible space on which Γ acts freely, $\tilde{K}^{et}(A')$ is as defined above, and the action of Γ on $\tilde{K}^{et}(A')$ is through maps induced by the action of Γ on A'. It is easy to construct a map $\phi : BGL(A) \to \tilde{K}^{et}(A)$ and to check, using [6, 7.1], that neither $\tilde{K}^{et}(A)$ nor the map ϕ depend up to homotopy on the choice of A'; in particular, if A already has finite mod ℓ etale cohomological dimension, this more general definition of $\tilde{K}^{et}(A)$ agrees with the earlier one. For *any* number ring $\mathcal{O}$, we will now call the conjecture that the map $\phi : BGL(\mathcal{O}[1/\ell]) \to \tilde{K}^{et}(\mathcal{O}[1/\ell])$ induces an isomorphism on mod ℓ cohomology the ℓ-adic Quillen-Lichtenbaum conjecture for $\mathcal{O}$. It is easy to define what it means for a pro-space X to have virtually finite mod ℓ cohomological dimension, and to formulate and prove a generalization of Proposition 3.2. We will content ourselves with stating the following corollary of this generalization.

COROLLARY 3.3. *Let A be a noetherian R-algebra of virtually finite mod ℓ etale cohomological dimension, X a pro-space of virtually finite mod ℓ cohomological dimension, and $f : X \to \mathrm{Spec}\ A_{et}$ a map which induces an isomorphism on mod ℓ cohomology. Then if A contains a primitive ℓ'th root of unity the natural map $\tilde{K}^{et}(A) \to \tilde{K}^{top}(X)$ induced by f is a homotopy equivalence.*

PROOF. This follows easily from the fact that the structure map Spec A $\to$ Spec R factors through Spec $R[\zeta_\ell]$, where ζ_ℓ is a primitive ℓ'th root of unity.

§4. SPECIFIC CALCULATIONS

In this section we will use the results of §2 - §3 to calculate the space $\tilde{K}^{et}(A)$ explicitly for particular R-algebras A.

We will use BU to denote the topological classifying space of the infinite unitary group and BO the topological classifying space of the infinite orthogonal group. For any number k which is relatively prime to ℓ, $F\psi^k$ will stand for the space of [10], i.e., the homotopy fibre of the map $\psi^k - 1 : BU \to BU$. The symbol "$\underset{\ell}{\sim}$" will stand for ℓ-adic homotopy equivalence.

PROPOSITION 4.1. *There is an equivalence*

$$\tilde{K}^{et}(\mathbb{R}) \underset{\ell}{\sim} BO.$$

If $\mathbb{F}_q$ is a field with q elements, $(q,\ell) = 1$, then there is an equivalence

$$\tilde{K}^{et}(\mathbb{F}_q) \underset{\ell}{\simeq} F\psi^q$$

PROOF. By definition (§3) $\tilde{K}^{et}(\mathbb{R})$ is $\mathrm{Hom}^{\mathbb{Z}/2}(E\mathbb{Z}/2, \tilde{K}^{et}(\mathbb{C}))$. Since $\mathrm{Spec}\ \mathbb{C}_{et} \simeq *$, it follows from the remark in [6, proof of 5.1] and from the comparison theorem [2] that $\tilde{K}^{et}(\mathbb{C})$ is ℓ-equivalent, as a $\mathbb{Z}/2$-space, to BU with the standard action of complex conjugation. The first part of the proposition is therefore a consequence of [4]. The second part is a consequence of [6, 8.6] together with Quillen's calculations [10].

Now we are ready to work out a more substantial example. Set $\ell = 2$ and observe that the image of the prime 3, say, in the quotient group $\mathbb{Z}_2^*/\{\pm 1\}$ is a topological generator. It follows easily from Theorem 2.2 and a cohomology ring argument that the natural map

$$(\mathrm{Spec}\ \mathbb{R}_{et}) \vee (\mathrm{Spec}\ \mathbb{F}_3)_{et} \to \mathrm{Spec}\ \mathbb{Z}[1/2]_{et}$$

is a mod 2 (co-)homology equivalence. By Corollary 3.3 and the fact that a fibre square of mapping spaces arises from a wedge decomposition of the domain, there is a homotopy fibre square

$$\begin{array}{ccc} \tilde{K}^{et}(\mathbb{Z}[1/2]) & \to & \tilde{K}^{et}(\mathbb{R}) \\ \downarrow & & \downarrow \\ \tilde{K}^{et}(\mathbb{F}_3) & \to & \tilde{K}^{top}(*) \end{array}$$

In light of Proposition 4.1, this gives the following result.

PROPOSITION 4.2. *There is a 2-adic homotopy fibre square*

$$\begin{array}{ccc} \tilde{K}^{et}(\mathbb{Z}[1/2]) & \to & BO \\ \downarrow & & \downarrow \\ F\psi^3 & \to & BU \end{array}$$

COROLLARY 4.3. *Suppose that the 2-adic Quillen-Lichtenbaum conjecture is true for the ring $\mathbb{Z}$. Then*

(i) *there is a natural isomorphism*

$$H^*(GL^{top}(\mathbb{R})/GL(\mathbb{Z}), \mathbb{Z}/2) \simeq H^*(SU, \mathbb{Z}/2)$$

and

(ii) there is a natural filtration of $H^*(BGL(\mathbb{Z}), \mathbb{Z}/2)$ together with an isomorphism

$$GrH^*(BGL(\mathbb{Z}), \mathbb{Z}/2) \approx H^*(BO, \mathbb{Z}/2) \otimes H^*(SU, \mathbb{Z}/2)$$

REMARK. Here $GL^{top}(\mathbb{R})/GL(\mathbb{Z})$ denotes the direct limit over n of the homogeneous spaces $GL_n^{top}(\mathbb{R})/GL_n(\mathbb{Z})$, and SU denotes the infinite special unitary group.

REMARK. It appears from Proposition 4.2 that $\tilde{K}^{et}(\mathbb{Z}[1/2])$ is essentially identical to the space $JK(\mathbb{Z})$ which appears in the remarkable work of Bökstedt [5].

PROOF OF 4.3. Under the stated hypothesis, $BGL(\mathbb{Z}[1/2])^+$ is equivalent at 2 to $\tilde{K}^{et}(\mathbb{Z}[1/2])$. Since the composite of the map (§3) $BGL(\mathbb{Z}[1/2])^+ \to \tilde{K}^{et}(\mathbb{Z}[1/2])$ with the map (4.2) $\tilde{K}^{et}(\mathbb{Z}[1/2]) \to BO$ is induced by the natural map $GL(\mathbb{Z}[1/2]) \to GL^{top}(\mathbb{R})$, it follows from Proposition 4.2 that the homotopy fibre of $BGL(\mathbb{Z}[1/2])^+ \to BGL^{top}(\mathbb{R})$ is equivalent, at the prime 2, to the homotopy fibre of $F\psi^3 \to BU$, i.e., to U. By the localization theorem [11] and Quillen's calculations [10], the space $BGL(\mathbb{Z}[1/2])^+$ is equivalent, at 2, to the product $BGL(\mathbb{Z})^+ \times S^1$. One concludes that the homotopy fibre of $BGL(\mathbb{Z})^+ \to BGL^{top}(\mathbb{R})$, which can be identified with $[GL^{top}(\mathbb{R})/GL(\mathbb{Z})]^+$, is equivalent at 2 to SU. This gives (i). The map $BGL(\mathbb{Z})^+ \to BGL^{top}(\mathbb{R})$ is a map of infinite loop spaces; moreover, this map induces an injection on mod 2 cohomology, since $H^*(BGL^{top}(\mathbb{R}), \mathbb{Z}/2)$ can be detected on the group of diagonal matrices in $GL(\mathbb{Z})$. It follows that the Serre spectral sequence of this map collapses. This proves (ii).

Using the above ideas, it is possible to make a whole family of calculations at the prime 2 with little additional effort. Let $\mathbb{RP}^\infty (\sim B\mathbb{Z}/2)$ denote infinite real projective space and S^1 the circle. Observe that there is a map $\mathbb{RP}^\infty \vee S^1 \to (\mathrm{Spec}\ \mathbb{R})_{et} \vee (\mathrm{Spec}\ \mathbb{F}_3)_{et}$ which induces an isomorphism on [illegible] Let A be any ring finite, etale, and Galois over $\mathbb{Z}[1/2]$, and assume that the Galois group of A over $\mathbb{Z}[1/2]$ is a 2-group. Then if

X is defined by the homotopy pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{Spec} A_{et} \\ \downarrow & & \downarrow \\ \mathbb{R}P^\infty \vee S^1 \to (\operatorname{Spec} \mathbb{R})_{et} \vee (\operatorname{Spec} \mathbb{F}_3)_{et} & \to & \operatorname{Spec} \mathbb{Z}[1/2]_{et} \end{array}$$

the map $X \to \operatorname{Spec} A_{et}$ induces an isomorphism on mod 2 (co-) homology, and the consequent equivalence $\tilde{K}^{et}(A) \to \tilde{K}^{top}(X)$ allows as above an explicit calculation of $\tilde{K}^{et}(A)$. Here are some examples of the spaces X that can arise (ζ_{2^n} denotes a primitive 2^n'th root of unity, and $\bar\zeta_{2^n}$ its complex conjugate):

Ring A	Space X
$\mathbb{Z}[1/2, \zeta_{2^n}]$, $n \geq 2$	$S^1 \vee \ldots \vee S^1$, $2^{n-2} + 1$ summands
$\mathbb{Z}[1/2, \zeta_{2^n} + \bar\zeta_{2^n}]$, $n \geq 2$	$S^1 \vee \mathbb{R}P^\infty \vee \ldots \vee \mathbb{R}P^\infty$, 2^{n-2} $\mathbb{R}P^\infty$ summands
$\mathbb{Z}[1/2, \zeta_{2^n} - \bar\zeta_{2^n}]$, $n \geq 3$	$S^1 \vee \ldots \vee S^1$, $2^{n-3} + 1$ summands

In fact, rings A of the specified type are in bijective correspondence, via X, with suitable finite covering spaces of $\mathbb{R}P^\infty \vee S^1$.

Suppose now that ℓ is an odd prime and let $\mathcal{O} = \mathbb{Z}[\zeta_\ell]$, where ζ_ℓ is a primitive ℓ'th root of unity. Assume in addition that $\mathcal{O}$ is <u>regular</u> in the sense of number theory, i.e., that the ideal class group of $\mathcal{O}$ contains no ℓ-torsion.

LEMMA 4.4. <u>There are isomorphisms</u>

$$H_i(\operatorname{Spec} \mathcal{O}[1/\ell]_{et}, \mathbb{Z}/\ell) \simeq \begin{cases} \mathbb{Z}/\ell & i = 0 \\ (\mathbb{Z}/\ell)^{(\ell+1)/2} & i = 1 \\ 0 & i \geq 2 \end{cases}$$

PROOF. Let α be the unique prime of $\mathcal{O}$ above ℓ, $\mathcal{O}_\alpha$ the completion of $\mathcal{O}$ at α, and $f : (\mathcal{O}^*)^\wedge \to \mathcal{O}_\alpha^*$ the natural map. By Theorem 2.2 and a counting argument, it is enough to show that

(i) ker f is uniquely ℓ-divisible, and

(ii) coker f has no ℓ-torsion.

By a diagram chase (i) and (ii) together are implied by

(i)' f induces an isomorphism on ℓ-torsion subgroups, and

(ii)' $f \otimes \mathbb{Z}/\ell$ is injective

Condition (i)' is obvious, since the ℓ-torsion subgroups of $(\mathcal{O}^*)^\wedge$ and $\mathcal{O}^*_\alpha$ are each generated by a primitive ℓ'th root of unity. For (ii)', let u be a unit of $\mathcal{O}$ which represents a non-zero element of $\ker(f \otimes \mathbb{Z}/\ell)$, so that u is not an ℓ'th power in $\mathcal{O}$ but becomes an ℓ'th power in $\mathcal{O}_\alpha$. Adjoining an ℓ'th root of u to the quotient field F of $\mathcal{O}$ then produces a degree ℓ abelian extension of F which is clearly unramified away from α and is split and therefore unramified at α itself. This contradicts the assumption that the class group of $\mathcal{O}$ contains no ℓ-torsion.

By Lemma 4.4, it is possible to choose a map g from the wedge of $(\ell + 1)/2$ circles to $\operatorname{Spec} \mathcal{O}[1/\ell]_{et}$ which induces an isomorphism on mod ℓ (co-) homology. Let $\theta : \pi_1 \operatorname{Spec} \mathbb{Z}[1/\ell]_{et} \to (\mathbb{Z}^*)^\wedge$ be the homomorphism given by the action of the Galois group of the maximal unramified extension of $\mathbb{Z}[1/\ell]$ on the ℓ-primary roots of unity. It is clear that the map g can be chosen so that under the composite

$$\pi_1(\bigvee_{\frac{\ell+1}{2}} S^1) \xrightarrow{g_\#} \pi_1 \operatorname{Spec} \mathcal{O}[1/\ell]_{et}$$
$$\to \pi_1 \operatorname{Spec} \mathbb{Z}[1/\ell]_{et} \xrightarrow{\theta} (\mathbb{Z}^*_\ell)^\wedge$$

all of the circle generators except the first, say, map trivially and the generator of the fundamental group of the first circle maps to a topological generator of

$$\ker(\mathbb{Z}^*_\ell \to (\mathbb{Z}/(\ell - 1)^*)^\wedge \simeq (\mathbb{Z}_\ell)^\wedge$$

Find a rational prime q whose image in $\mathbb{Z}^*_\ell$ is a topological generator of $\ker(\mathbb{Z}^*_\ell \to (\mathbb{Z}/\ell - 1)^*)$, choose a map $\operatorname{Spec} \mathbb{F}_q \to \operatorname{Spec} \mathcal{O}[1/\ell]$, and replace the first circle summand in the domain of g above by $(\operatorname{Spec} \mathbb{F}_q)_{et}$; this gives a mod ℓ (co-) homology isomorphism

$$(\operatorname{Spec} \mathbb{F}_q)_{et} \vee \bigvee_{\frac{\ell-1}{2}} S^1 \xrightarrow{g'} \operatorname{Spec} \mathcal{O}[1/\ell]_{et}$$

One can calculate using the machinery of [6, §5] that if $h : S^1 \to \operatorname{Spec} \mathbb{Z}[1/\ell]_{et}$ is a map with the property that the composite $\theta h_\# : \pi_1 S^1 \to (\mathbb{Z}_\ell^*)^\wedge$ is trivial, then $\tilde{K}^{top}(S^1)$, computed with respect to the map h, is ℓ-equivalent to the unpointed function space BU^{S^1}. Along the lines of Proposition 4.2 this leads to the following conclusion.

PROPOSITION 4.5. *Let ℓ be an odd regular prime, $\mathcal{O} = \mathbb{Z}[\zeta_\ell]$, and q a rational prime whose image in $\mathbb{Z}_\ell^*$ is a topological generator of* $\ker(\mathbb{Z}_\ell^* \to (\mathbb{Z}/\ell - 1)^*)$. *Then there is an ℓ-adic homotopy fibre square*

$$\begin{array}{ccc} \tilde{K}^{et}(\mathcal{O}[1/\ell]) & \to & BU^W \\ \downarrow & & \downarrow \\ F\psi^q & \to & BU \end{array}$$

where W is the wedge of $(\ell - 1)/2$ circles, BU^W denotes the function space of (unpointed) maps from W to BU, and the right-hand vertical map is evaluation at the basepoint.

Since BU is an infinite loop space, the function space BU^W, W as in Proposition 4.5, is equivalent to a product $BU \times (\prod_{\frac{\ell-1}{2}} U)$

COROLLARY 4.6. *In the setting of* Proposition 4.5 *there is an ℓ-adic equivalence*

$$\tilde{K}^{et}(\mathcal{O}[1/\ell]) \underset{\ell}{\simeq} F\psi^q \times (\prod_{\frac{\ell-1}{2}} U)$$

Consequently, if the ℓ-adic Quillen-Lichtenbaum conjecture is true for $\mathcal{O}$ there is a cohomology isomorphism

$$H^*(BGL(\mathcal{O}),\mathbb{Z}/\ell) \simeq H^*(F\psi^q \times SU \times (\prod_{\frac{\ell-3}{2}} U),\mathbb{Z}/\ell)$$

REMARK. It would be interesting to obtain similar results without the assumption that ℓ is regular. There are strong reasons to believe that this would be difficult.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NOTRE DAME
Notre Dame, Indiana 46556

DEPARTMENT OF MATHEMATICS
NORTHWESTERN UNIVERSITY
Evanston, Illinois 60201