CENTERS AND COXETER ELEMENTS

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Abstract. Suppose that G is a connected compact Lie group. We show that simple numerical information about the Weyl group of G can be used to obtain bounds, often sharp, on the size of the center of G. These bounds are obtained with the help of certain Coxeter elements in the Weyl group. Variants of the method use generalized Coxeter elements and apply to p-compact groups; in this case a splitting theorem emerges. The Lie group results are mostly known, but our arguments have a conceptual appeal.

1. Introduction

Suppose that G is a connected compact Lie group. The *center* $\mathcal{Z}_G$ of G is the subgroup consisting of elements which commute with everything else in G. In this paper we develop some simple general methods for obtaining information about $\mathcal{Z}_G$. The most striking technique applies if G is simply connected and gives a formula for the order of $\mathcal{Z}_G$ in terms of the exponents of the Weyl group W of G; the formula is derived by studying certain special elements in W called *Coxeter elements*. Many of our results apply in one way or another to p-compact groups. Much of what we do is well-known, but does not appear in the standard textbooks; we are particularly interested in minimizing the amount of case-by-case consideration and in eliminating analytical arguments.

Before stating our results we give some background on compact Lie groups. See [13] for a geometric approach to the structure theorems we use.

1.1. Product decompositions. A connected compact Lie group G has a finite covering group $\tilde{G}$ isomorphic to a product $T \times K$, where T is a torus and K is a simply connected compact Lie group. The group G is obtained from $\tilde{G}$ by forming a quotient $\tilde{G}/A$, where A is a finite subgroup of $\mathcal{Z}_{\tilde{G}}$, and the center of G is isomorphic to $\mathcal{Z}_{\tilde{G}}/A$. This isomorphism indicates that in order to understand $\mathcal{Z}_G$ it is enough to

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understand $\mathcal{Z}_{\tilde{G}}$ together with the subgroup A of $\mathcal{Z}_{\tilde{G}}$ which is used in constructing G from $\tilde{G}$.

Since the torus T is abelian, $\mathcal{Z}_{\tilde{G}}$ is just $T \times \mathcal{Z}_K$. We will mostly concentrate on studying $\mathcal{Z}_K$, which is known to be a finite abelian group. In principle this can be done once and for all. Say that K is *almost simple* if the quotient $K/\mathcal{Z}_K$ is simple (in the group-theoretic sense of having no nontrivial normal subgroups). Then any simply-connected compact Lie group K is isomorphic to a product $\prod_i K_i$ of simply connected almost simple factors, so that $\mathcal{Z}_K \cong \prod_i \mathcal{Z}_{K_i}$. The possible factors in this product decomposition are known and can be studied one-by-one; most of these possible factors have standard descriptions in terms of matrices. To be explicit, up to isomorphism a simply connected almost simple compact Lie group falls into one of four families or is one of five exceptional examples. The families are A_n (more concretely SU_{n+1}), B_n (Spin_{2n+1}), C_n (Sp_n), and D_n (Spin_{2n}); the exceptional examples are G_2, F_4, E_6, E_7, and E_8. The groups are often paired up with the Dynkin diagrams of the same names.

1.2. Maximal tori and Weyl groups. A compact Lie group G has a maximal torus T unique up to conjugacy; we will usually pick one out and denote it T_G (or just T if G is understood). The dimension of T is a numerical invariant of G called the *rank* ℓ_G (or ℓ) of G; the integer subscripts which appear above in the conventional designations for the simply connected almost simple groups correspond to the ranks of the groups, so that, for instance, E_7 has rank 7. The maximal torus T is of finite index in its normalizer $N_G(T)$, and the quotient $N_G(T)/T$ is called the *Weyl group* W_G (or W) of G. Let L_G denote $\pi_1(T_G)$ (this is the *Weyl lattice* of G) and let V_G denote $\mathbb{Q} \otimes L_G$. Then $L_G \cong \mathbb{Z}^\ell$, $V_G \cong \mathbb{Q}^\ell$. The conjugation action of W on T gives a homomorphism

$$W \to \mathrm{Aut}(T) \to \mathrm{Aut}(L_G) \to \mathrm{Aut}(V_G) \ .$$

If G is connected this homomorphism is faithful, and its image is a *reflection subgroup* (§2) of $\mathrm{Aut}(V_G) \cong \mathrm{GL}_\ell(\mathbb{Q})$, i.e., a subgroup generated by the reflection matrices it contains. The group G is almost simple if and only if the action of W on V_G affords an irreducible representation of W; the fundamental group $\pi_1(G)$ is finite, or equivalently, G has a finite center, if and only if $(V_G)^W = 0$.

1.3. Calculating centers. We can now describe our results about the center of a connected compact Lie group.

1.4. Theorem. *Suppose that G is a simply connected compact Lie group with maximal torus T and Weyl group W. Then the center $\mathcal{Z}_G$ is exactly the fixed point set T^W.*

1.5. *Remark.* There is a general formula for $\mathcal{Z}_G$ in [13, S8]; it turns out that for an arbitrary connected compact Lie group, $\mathcal{Z}_G$ differs from T^W by an elementary abelian 2-group.

Surprisingly, it turns out that in order to compute T^W in the simply-connected case it is enough to focus on a single special kind of element in W_G. Suppose that G is a compact Lie group of rank ℓ, and that $(L_G)^W = 0$; as mentioned above, this will always be the case if G is simply connected. Coxeter [7] [6] proves that it is possible to find ℓ reflections $s_1, \ldots, s_\ell$ which generate W; he goes on to show that, up to conjugacy in W, the product $s_1 s_2 \cdots s_\ell$ does not depend either on the choice of generating reflections or on the order in which they are written down. We will call such a product of generating reflections a *Coxeter element* of W, and denote it w_{Cox}. Let $\#(\mathcal{Z}_G)$ denote the order of the center of G.

1.6. Theorem. *Suppose that G is a simply connected compact Lie group with maximal torus T, Weyl group W, and Coxeter element $w_{Cox} \in W$. Then the center $\mathcal{Z}_G$ is exactly the fixed point set $T^{\langle w_{Cox}\rangle}$, and $\#(\mathcal{Z}_G) = |\det(I - w_{Cox})|$. If the order of w_{Cox} is prime to p, then $\#(\mathcal{Z}_G)$ is also prime to p.*

Here I denotes the identity matrix of rank ℓ_G, we identify w_{Cox} with its image in $\mathrm{Aut}(L_G) \cong \mathrm{GL}_\ell(\mathbb{Z})$, and $\det(I - w_{\mathrm{Cox}})$ is the determinant of the matrix $(I - w_{\mathrm{Cox}})$. Theorem 1.6 was observed in the 50's, apparently on the basis of case-by-case inspection. Our proof (which uses ideas communicated to us by R. Steinberg) is more conceptual. The real impact of 1.6 comes from the fact that the eigenvalues $\{\alpha_i\}$ of a Coxeter element w_{Cox}, and hence the number $\det(I - w_{\mathrm{Cox}}) = \prod_i (1 - \alpha_i)$, can be determined (3.3) from the list of exponents of the reflection group W_G. This reduces the calculation of the order of $\mathcal{Z}_G$ to routine arithmetic (see 5.5).

Say that an element $w \in W_G$ is a *generalized Coxeter element* if $\det(I - w) \neq 0$, i.e., if w has no nonzero fixed vectors in its action on V_G. Any Coxeter element of W_G is a generalized Coxeter element. It turns out that whereas Coxeter elements determine the order of the center of a simply connected group G, generalized Coxeter elements give bounds for the order of the center of an arbitrary connected group.

1.7. Theorem. *Suppose that G is a connected compact Lie group with maximal torus T, Weyl group W, and generalized Coxeter element $w \in W$. Then the center $\mathcal{Z}_G$ is finite, and $\#(\mathcal{Z}_G)$ divides $|\det(I - w)|$. If the order of w is prime to p, then $\#(\mathcal{Z}_G)$ is also prime to p.*

One source of generalized Coxeter elements is the center of the Weyl group itself. It is possible to do a little bit better than 1.7 in this case.

1.8. Theorem. *Suppose that G is a simply connected almost simple compact Lie group with Weyl group W, and that $\mathcal{Z}_W$ is nontrivial. Then $\mathcal{Z}_W$ is $\mathbb{Z}/2$, and the nonidentity element $w \in \mathcal{Z}_W$ is a generalized Coxeter element which acts on T_G by inversion. The determinant $\det(I - w)$ is 2^ℓ, and $\mathcal{Z}_G$ is isomorphic to $(\mathbb{Z}/2)^s$ for some $s \leq \ell$.*

1.9. *Remark.* Suppose that G is a simply connected almost simple compact Lie group. Then the center of W_G is nontrivial as long as G is not E_6, A_n for $n > 1$, or D_{n+1}. This fact can be read off from the list of degrees of W_G (see 2.7).

Finally, there is one additional class of examples (which only partially overlaps the class of simply connected groups) for which the inclusion 5.2 is an equality.

1.10. Proposition. *Suppose that G is a connected compact Lie group of rank ℓ, with Weyl group $W \subset \operatorname{Aut}(L_G) \cong \operatorname{GL}_\ell(\mathbb{Z})$. If reduction mod 2 gives a monomorphism $W \to \operatorname{GL}_\ell(\mathbb{Z}/2)$, then $\mathcal{Z}_G = (T_G)^W$.*

1.11. *Remark.* Every finite subgroup of the kernel of the reduction map $\operatorname{GL}_\ell(\mathbb{Z}) \to \operatorname{GL}_\ell(\mathbb{Z}/2)$ is an elementary abelian 2-group. Thus the conclusion of 1.10 applies to G whenever the Weyl group of G has no normal elementary abelian 2-subgroups. The proof of 1.10 shows that the theorem remains true under the weaker hypothesis that the kernel of the mod 2 reduction map contains none of the reflections in W. Thus among the almost simple types, except for some forms of A_1, B_n, and C_n, $\mathcal{Z}_G = (T_G)^W$.

1.12. *Remark.* Combining the above theorems in a straightforward way with tables of Weyl group degrees gives the following information about the the centers of the simply connected almost simple compact Lie groups (§6): the center of A_n has order $(n + 1)$, the centers of B_n and C_n are cyclic of order 2, the center of D_{2n} is $(\mathbb{Z}/2)^2$, the center of D_{2n+1} is of order 4, the centers of G_2, F_4 and E_8 are trivial, the center of E_6 is $\mathbb{Z}/3$, and the center of E_7 is $\mathbb{Z}/2$. Of course in some cases it is possible to obtain these results and more by inspection (for instance it is elementary to check that the center of $A_n \cong \operatorname{SU}_{n+1}$ is $\mathbb{Z}/(n + 1)$), but in other cases the inspection is not so easy.

1.13. *Fundamental groups.* Suppose that G is a connected compact Lie group with finite fundamental group. In this case the universal cover of G is a simply connected almost simple compact Lie group K, and G is isomorphic to K/A for some finite subgroup A of $\mathcal{Z}_K$. In particular,

$\pi_1(G) \cong A$ is isomorphic to a subgroup of $\mathcal{Z}_K$. Since $T_G \cong T_K/A$, it is easy to see that $\ell_G = \ell_K$, and that W_G is conjugate to W_K as a subgroup of $\mathrm{GL}_\ell(\mathbb{Q})$. It follows that Theorem 1.7 gives bounds for $\#(\pi_1 G)$ in terms of determinants involving generalized Coxeter elements of W_G; we leave it to the reader to write this out. Similarly, Theorem 1.8 guarantees that if K is almost simple and the center of W_G is nontrivial, then $\pi_1 G$ is isomorphic to $(\mathbb{Z}/2)^s$ for $s \le \ell_G$.

1.14. Generalizing to p-compact groups. A *p-compact group* X is a homotopy theoretic object which has the essential homotopy theoretic structure of the p-completion of a compact Lie group; see [10], [17], [18], or i[8]. Technically, a p-compact group is a pair $(X, \mathrm{B}X)$, where X is a p-complete space with finite mod p homology and $\mathrm{B}X$ is a classifying space for X (that is, the loop space $\Omega\,\mathrm{B}X$ is equivalent to X). For technical reasons it is also convenient to require that $\pi_0(X) \cong \pi_1(\mathrm{B}X)$ be a finite p-group. Like a compact Lie group, a p-compact group X has a center $\mathcal{Z}_X$ (which is another p-compact group), a fundamental group $\pi_1(X)$, a rank $\ell = \ell_X$, a maximal (p-complete) torus T_X, and a Weyl group $W = W_X$. The group W acts on T_X by a kind of conjugation. Let $L_X = \pi_1 T_X$ and $V_X = \mathbb{Q} \otimes L_X$. Then $L_X \cong (\mathbb{Z}_p)^\ell$ and $V_X \cong (\mathbb{Q}_p)^\ell$, where $\mathbb{Z}_p$ is the ring of p-adic integers and $\mathbb{Q}_p$ is its quotient field. The action of W on T_X induces a homomorphism

$$W \to \mathrm{Aut}(L_X) \to \mathrm{Aut}(V_X) \ .$$

If X is connected this homomorphism is injective and its image is a (generalized) reflection subgroup of $\mathrm{Aut}(V_X) \cong \mathrm{GL}_\ell(\mathbb{Q}_p)$, i.e., a finite subgroup generated by the (generalized) reflections it contains (§2). Say that X is *almost simple* if V_X affords an irreducible representation of W. Then any connected p-compact group X has a finite cover $\tilde{X}$ which can be written as a product $T \times \prod_i Y_i$, where T is a p-complete torus and each Y_i is a simply connected almost simple p-compact group. Moreover, X is obtained in a certain sense as a quotient $\tilde{X}/A$, where A is a finite subgroup of the center of $\tilde{X}$. We give versions of the above results for p-compact groups; these versions are somewhat stronger if p is odd and weaker if $p = 2$. In any case they lead to a proof of the following theorem.

Say that X is of *Lie type* G, where G is a connected compact Lie group, if $W_X \subset \mathrm{GL}_\ell(\mathbb{Q}_p)$ is obtained up to conjugacy by taking the image of $W_G \subset \mathrm{GL}_\ell(\mathbb{Q})$ under the usual inclusion $\mathrm{GL}_\ell(\mathbb{Q}) \subset \mathrm{GL}_\ell(\mathbb{Q}_p)$.

1.15. Theorem. *If X is a simply connected almost simple p-compact group which is not of Lie type, then $\mathcal{Z}_X$ is trivial.*

In view of the product decomposition theorem [12] for p-compact groups, this gives the following splitting result.

1.16. Theorem. *Any connected p-compact group can be written as a product $X \times \prod_i Y_i$, where X is a p-compact group of Lie type, and each Y_i is a simply connected almost simple p-compact group which is not of Lie type.*

1.17. Organization of the paper. Section 2 discusses (generalized) reflection groups and their centers, and in particular indicates how to determine the center of an irreducible reflection group from its list of degrees. Section 3 describes the key properties of Coxeter elements, and §4 moves on to generalized Coxeter elements. Section 5 has proofs of 1.4–1.10, and the next section draws conclusions about centers of particular groups. Section 7 extends our results to p-compact groups, while §8 presents the information about p-adic reflection groups which is combined with these extended results in §9 to prove 1.15 and 1.16.

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2. Reflection groups and their centers

In this section we recall some basic properties of finite reflection groups over fields of characteristic zero, and describe how the center of an irreducible reflection group can be determined by inspecting a finite collection of integers called the set of *degrees* of the reflection group.

2.1. Reflection groups. If k is a field of characteristic zero, an element w in $\mathrm{GL}_\ell(k)$ of finite order is said to be a (generalized) *reflection* if w is conjugate to a diagonal matrix which differs from the identity matrix in only one position. A finite subgroup W of $\mathrm{GL}_\ell(k)$ is said to be a *reflection subgroup* if it is generated by the reflections it contains. Let $V = k^\ell$. The reflection sugroup is *essential* if $V^W = 0$, and *irreducible* if W is nontrivial and V affords an irreducible representation of W. The number ℓ is called the *rank* of W.

2.2. *Remark.* If R is an integral domain with quotient field k, an element of $\mathrm{GL}_\ell(R)$ is said to be a reflection if it is a reflection when it is considered as an element of $\mathrm{GL}_\ell(k)$; similarly, a finite subgroup of $\mathrm{GL}_\ell(R)$ is said to be an (essential, irreducible) reflection subgroup if it has the corresponding properties as a subgroup of $\mathrm{GL}_\ell(k)$.

If G is a connected compact Lie group of rank ℓ, the Weyl group W_G can be viewed as a finite reflection subgroup of $\mathrm{GL}_\ell(\mathbb{Z})$; W_G is essential if and only if the fundamental group (equivalently, the center) of G is finite, and irreducible if and only if in addition the universal cover of G is compact and almost simple (equivalently, if and only if the universal cover of G cannot be split in a nontrivial way as a product of compact Lie groups).

If X is a p-compact group of rank ℓ, the Weyl group W_X can be viewed as a finite reflection subgroup of $\mathrm{GL}_\ell(\mathbb{Z}_p)$. Again, W_X is essential if and only if the fundamental group (equivalently, the center) of X is a finite p-group, and irreducible if and only if the universal cover of X cannot be split in a nontrivial way as a product of p-compact groups.

2.3. *Remark.* If W is an essential finite reflection subgroup of $\mathrm{Aut}(V)$, then there are product decompositions $V \cong \prod_j V_j$ and $W \cong \prod_j W_j$, such that W_j acts as an irreducible reflection group on V_j and W_j acts trivially on V_i for $i \neq j$. We are particularly interested in finite reflection groups over $\mathbb{Q}$ (Weyl groups of connected compact Lie groups), over $\mathbb{R}$ (Coxeter groups), over $\mathbb{C}$ (complex reflection groups), and over $\mathbb{Q}_p$ (Weyl groups of p-compact groups). A convenient source of information on irreducible reflection groups is [5].

2.4. Invariants and degrees. Suppose that $V = k^\ell$ and that W is a finite reflection subgroup of $\mathrm{Aut}(V)$. Let $V^\#$ be the k-dual of V. The action of W on V induces an action of W on $V^\#$ and hence an action on the symmetric algebra $\mathrm{Sym}(V^\#)$. We grade this symmetric algebra so that $V^\#$ itself is in degree one; the action of W then preserves the grading. It is a classical fact that the ring of invariants $(\mathrm{Sym}(V^\#))^W$ is itself a graded polynomial algebra over k with ℓ generators [4] [19] [15]. The collection $\{d_1, \ldots, d_\ell\}$ of degrees of a set of homogeneous polynomial generators is well-defined, and is called the set of *degrees* of W. The numbers $m_j = d_j - 1$ are referred to as the *exponents* of W.

2.5. *Remark.* The degrees of the irreducible finite reflection groups of various sorts are listed in [5]. We take these degrees as the basic numerical invariants of a reflection group. If R is an integral domain of characteristic zero, the degrees of a reflection group over R are computed by first extending scalars to the quotient field of R and then determining the degrees as above.

2.6. Centers. If W is a finite reflection group with degrees $\{d_j\}$, we let $\delta(W)$ denote $\gcd\{d_j\}$. The following theorem determines the center of an irreducible finite irreducible reflection group W in terms of $\delta(W)$.

2.7. Theorem. *Suppose that k is a field of characteristic zero, and that W is an irreducible finite reflection group of rank ℓ over k. Let $\delta = \delta(W)$. Then the center $\mathcal{Z}_W$ is a cyclic group of order δ, and acts on k^ℓ by scalar diagonal matrices. In particular, k contains the primitive roots of unity of order δ, and any generator of $\mathcal{Z}_W$ acts on k^ℓ by multiplication by such a primitive root.*

2.8. *Remark.* If k is the field of complex numbers, there is a convenient geometric interpretation of the center of the reflection group as the kernel of the homomorphism from W to the associated collineation group W' acting on the corresponding complex projective space. This connection is exploited in the Shephard-Todd classification scheme, [19].

The proof of 2.7 proceeds with a series of lemmas. Recall that an irreducible representation of a finite group W over a field k is said to be *absolutely irreducible* if the representation remains irreducible after extending scalars to the algebraic closure of k.

2.9. Lemma. *Suppose that k is a field of characteristic 0, and that V is a finite dimensional vector space over k which affords an absolutely irreducible representation of the finite group W. Then the center of W is cyclic and and acts on V by scalar diagonal matrices.*

Proof. Extend scalars to obtain $\bar{V} = \bar{k} \otimes_k V$, where $\bar{k}$ is the algebraic closure of k. Action by an element of the center of W on $\bar{V}$ is a W-map and hence by Schur's lemma a scalar unit. So $\mathcal{Z}_W$ is embedded in the units of $\bar{k}$, and hence, being finite, is cyclic. Let w be a generator. In $Aut(\bar{V})$, $w = \lambda \cdot I$, for some λ in $\bar{k}$. But then $\mathrm{trace}(c) = \dim(V)\lambda \in k$, not just in $\bar{k}$. Since $\mathrm{char}(k) = 0$, $\mathrm{trace}(c)/\dim(V) = \lambda \in k$ also. □

2.10. Lemma. *Let k be a field of characteristic zero and $W \subset \mathrm{GL}_\ell(k) = \mathrm{Aut}(V)$ an irreducible finite reflection group. Then the action of W on V affords an absolutely irreducible representation of W over k.*

Proof. Product decompositions of an essential reflection representation correspond to decompositions of the reflection group itself into products of subgroups generated by reflections (2.3, [12, 7.1]). Since the action of W on V is irreducible, W cannot be written as a product of subgroups generated by reflections. Conversely, since W cannot be written as such a product group, the induced reflection representation of W on $\bar{k} \otimes_k V$ is irreducible. □

2.11. Lemma. *Let k be a field of characteristic zero and V a finite dimensional k-vector space. Denote by $\mathrm{Sym}(V^\#)$ the algebra of polynomial functions on V. Suppose that W is a finite subgroup of $Aut(V)$ which acts irreducibly on V. If $\mathrm{Sym}(V^\#)^W$ is concentrated in degrees* *divisible by δ, then W contains a scalar diagonal matrix $\theta \cdot I$, where θ is a primitive root of unity in k of order δ.*

Proof. This is similar to an argument of Humphreys, [15, p. 82]. Temporarily replacing k by a larger field, we assume that k contains the δ'th roots of unity. Let S be the fraction field of $\mathrm{Sym}(V^{\#})$ and K that of $\mathrm{Sym}(V^{\#})^W$. Then $K \to S$ is a Galois field extension with Galois group W.

Given an δ'th root of unity $\theta \in k$, define a field automorphism ϕ of S by setting $\phi(v) = \theta v$ for $v \in V^{\#}$ and then extending multiplicatively to $\mathrm{Sym}(V^{\#})$ and the fraction field. Since $\mathrm{Sym}(V^{\#})^W$ is concentrated in degrees divisible by δ, and $\phi(x_N) = \theta^N x = (\theta^\delta)^{N/\delta} x = x$ if $\delta | N$, we have that $\mathrm{Sym}(V^{\#})^W$ is fixed by ϕ. Since $K \to S$ is a Galois field extension, and hence normal, ϕ must be in the Galois group of S over K, namely W. Hence W contains the scalar diagonal matrix $\theta \cdot I$ and the element θ is in the original field. Notice that $\theta \cdot I$ is a central element in W. □

Proof of 2.7. This is a combination of the previous three lemmas. □

2.12. Cohomology bounds. Theorem 2.7 has an interesting implication for Weyl group cohomology.

2.13. Theorem. *Suppose that $R = \mathbb{Z}$ or $R = \mathbb{Z}_p$, and that W is an irreducible reflection subgroup of $\mathrm{GL}_\ell(R) = \mathrm{Aut}(L)$. Let $\delta = \delta(W)$. Then R contains a primitive root of unity θ of order δ, and for $M = L$ or $M = (\mathbb{Q} \otimes L)/L$,*

$$(1-\theta) \cdot \mathrm{H}^*(W; M) = 0 \ .$$

2.14. *Example.* We apply this theorem in the following way. If $R = \mathbb{Z}$ or $\mathbb{R} = \mathbb{Z}_2$, then the group of roots of unity in R is cyclic of order 2 and $\delta(W)$ is either 1 or 2. In the first case 2.13 gives no information; in the second case $\theta = -1$, $(1-\theta) = 2$, and the conclusion is that $\mathrm{H}^*(W; M)$ has exponent 2. If $R = \mathbb{Z}_p$ for p odd, then the group of roots of unity in R is cyclic of order $(p-1)$. If $\delta(W) = 1$ then 2.13 gives no information, otherwise θ is a nontrivial primitive root of unity, $(1-\theta)$ is a unit in R, and the conclusion is that $\mathrm{H}^*(W; M) = 0$.

2.15. *Remark.* If W is the Weyl group of a connected compact Lie group G, then the cohomology group $\mathrm{H}^3(W; L_G)$ is the home of the k-invariant or extension class that determines the group extension

$$1 \to T_G \to N_G(T_G) \to W \to 1 \, .$$

Thus 2.13 and 2.14 enable us to recover the result of Tits [24] that this k-invariant has order 2, although only for the cases (1.9) in which

G has type B_n, C_n, D_{2n}, G_2, F_4, E_7 or E_8. Related results for p-compact groups can be obtained by the same method. See [1] for a comprehensive statement.

Proof of 2.13. By 2.7, the quotient field of R contains the required primitive root of unity θ; since R is integrally closed in its field of fractions, $\theta \in R$. Again by 2.7, there is an element $w \in \mathcal{Z}_W$ which acts on L by multiplication by θ. For $M = L$ or $M = (\mathbb{Q} \otimes L)/L$, write

$$\mathrm{H}^*(W; M) = \mathrm{Ext}^*_{\mathbb{Z}[W]}(\mathbb{Z}, M) \ .$$

The element w acts on $\mathbb{Z}$ by the identity, and on M by multiplication by θ. These actions commute with the action of W, and so give two actions of w on $\mathrm{H}^*(W; M)$. By a naturality argument, these actions must be the same. See for example [14]. The result follows. □

3. Coxeter elements

In this section we study Coxeter elements in real reflection groups; these groups are also known as *Coxeter groups.* Any integral or rational reflection group (e.g., the Weyl group of a connnected compact Lie group) can be treated as a Coxeter group via the inclusion $\mathbb{Q} \subset \mathbb{R}$.

A *finite Coxeter group* W is a reflection subgroup of $\mathrm{GL}_\ell(\mathbb{R})$. One can assume that the matrices in W belong to the orthogonal group, but this is not always convenient or necessary. Let V denote $\mathbb{R}^\ell$. If the action of W is essential, then it is possible to find ℓ distinct reflections $\{s_1, \ldots, s_\ell\}$ in W which generate W. The product $w_{\mathrm{Cox}} = s_1 s_2 \ldots s_\ell$ is then called a *Coxeter element.* Note that w_{Cox} depends on the choice of the reflections $\{s_1, \ldots, s_\ell\}$ and on the order in which these reflections are written down in the product.

These Coxeter elements have amazing properties, some of which will come up in this section. We are mainly interested in proving the following theorem (which relates Coxeter elements to subreflection groups) and in recalling the two subsequent statements (which in particular give information about the eigenvalues of a Coxeter element).

3.1. Theorem. *(R. Steinberg, E-mail) Suppose that W is a finite irreducible Coxeter group. Then no Coxeter element of W is contained in any proper reflection subgroup of W.*

3.2. Theorem. [15, 3.17–19] *Suppose that W is a finite essential Coxeter group.*

1. *The various Coxeter elements $w_{Cox} \in W$ are all conjugate in W.*
2. $V^{\langle w_{Cox} \rangle} = \{0\}$. *That is, 1 is not an eigenvalue of w_{Cox}.*

3. *If W is irreducible and* $\mathrm{order}(w_{Cox}) = h$*, then* w_{Cox} *has a one-dimensional eigenspace in* $\mathbb{C} \otimes V$ *for the eigenvalue* $\eta = e^{2\pi i/h}$.
4. *If W is irreducible, and* $v \in \mathbb{C} \otimes V$ *is a nonzero* w_{Cox}*-eigenvector for* η*, then no reflection in W carries v to itself.*

If W is a finite Coxeter group with degrees $\{d_j\}$ and exponents $\{m_j\}$, then the sum $\sum_j (2d_j - 1) = \sum_j (2m_j + 1)$ is called the *dimension* $\dim(W)$ of W. There is some fascinating numerology involving these quantities.

3.3. Theorem. [15, §3] *Let W be a finite Coxeter group with degrees* $\{d_1, \dots, d_\ell\}$ *and exponents* $\{m_1, \dots, m_\ell\}$*. Let h denote the order of a Coxeter element* $w_{Cox} \in W$*, and let* $\eta = e^{2\pi i/h}$.

1. $|W| = d_1 d_2 \dots d_n$.
2. *W contains exactly* $N = \sum_j m_j$ *reflections.*
3. *If W is the Weyl group of a Lie group G with finite fundamental group, then* $dim(G) = \dim(W)$.
4. *If W is irreducible, then* $h = (2\sum_j m_j)/\ell = \max\{d_j\}$.
5. *If W is irreducible, then* w_{Cox} *has eigenvalues* $\{\eta^{m_j}\}$.
6. *If W is irreducible, then* $|\det(1 - w_{Cox})| = \prod_j (1 - \eta^{m_j})$.

Remark. Note that if W is a finite Coxeter group of rank ℓ, the number of reflections in W is $(\dim(W) - \ell)/2$. This implies that if $W' \subset W$ is a subreflection group with $\dim(W') = \dim(W)$, then one must have $W' = W$, since the number of reflections in each group is the same.

3.4. *Remark.* If W is not irreducible, but $V^W = 0$, then V is a direct sum of irreducible W-spaces $\{V_i\}$, and W splits as a product of subreflection groups $\{W_i\}$ such that W_i acts irreducibly on V_i (2.3). In this case, the Coxeter element for W is the product of the Coxeter elements of the W_i. We leave it to the reader to extend the last three statements in 3.3 to cover this more general case.

3.5. *Example.* If W is the Weyl group of type A_n, then $W = \Sigma_{n+1}$ and the reflection representation is the reduced standard representation. The set of exponents is $\{1, 2, \dots, n\}$, so that $h = 2(\sum_i i)/n = (n+1)$. The Coxeter element w_{Cox} can be taken to be the standard $(n+1)$-cycle in Σ_{n+1}.

3.6. *Example.* If W is the Weyl group of E_8, the set of exponents is $\{1, 7, 11, 13, 17, 19, 23, 29\}$, so $h = 30$.

In general, W can have many generalized Coxeter elements, i.e., elements w with the property that $det(I - w) \neq 0$. The following theorem of Carter suggests that these arise as the Coxeter elements for maximal rank subreflection groups. Carter [2] proves this if W is a rational reflection group. It is possible to extend this to arbitrary Coxeter groups by checking the additional irreducible real cases H_3, H_4, and $I_2(m)$ [15, §2], which of course do not correspond to compact Lie groups. We state this extended theorem.

3.7. Theorem. *Let W be a finite essential Coxeter group of rank ℓ.*

1. [2] *Any $w \in W$ can be written as a product of ℓ or fewer reflections in W. The minimal number required is $n - dim(V^{\langle w \rangle})$.*
2. *An element $w \in W$ has $det(1 - w) \neq 0$ if and only if there exist a set of ℓ reflections $\{s_1, \ldots, s_\ell\}$ in W such that $w = s_1 \ldots s_\ell$ and the root spaces $\{\mathrm{image}(I - s_j)\}$ span V.*
3. *if $det(1 - w) \neq 0$, then w is a Coxeter element for the subgroup W' of W generated by the $\{s_j\}$ from (2).*

Proof. For the "if" part of (2), note that because each s_j is a reflection, each subspace $\mathrm{im}(I - s_j)$ has dimension 1, generated by some vector p_j. Under the given assumptions on $\{s_j\}$, the set $\{p_j\}$ forms a basis for V and $s_j(p_j) = -p_j$. Now the argument is the same as that of Carter, [3, 10.5.6]. For the "only if" direction of (2), observe that by (1) the element w is the product of at most ℓ reflections. If the associated (one-dimensional) root spaces fail to span V, it is easy to argue that $V^{\langle w \rangle} \neq 0$, which is assumed untrue. Therefore there must be exactly ℓ reflections in the product decomposition of w, and their root spaces must span V. Part (4) is immediate from the definition of Coxeter element. □

Proof of 3.1. We thank R. Steinberg for his help with this argument. Let w be a Coxeter element of W of order h, and let $\eta = e^{2\pi i/h}$. By (3.2)(4), the η-eigenspace of w acting on $\mathbb{C} \otimes V$ is spanned by a vector v which is not fixed by any reflection in W. Suppose that $w \in W'$ for some proper reflection subgroup. Suppose further to begin with that W' is a non-trivial product of reflection groups, $W = W_1 \times W_2$, acting on $V \cong V_1 \times V_2$. Then in $\mathbb{C} \otimes V$, $v = (v_1, v_2)$ and $w(v) = (w(v_1), w(v_2)) = \eta v = (\eta v_1, \eta v_2)$. Thus both v_1 and v_2 are in the η-eigenspace for η. Hence one is zero, say v_2. But W_2 acts trivially on V_1, so the reflections in W_2 fix $v = (v_1, 0)$. Hence W_2 must be the trivial group. We have concluded that W' is irreducible, and we must now show that $W' = W$. By 3.7(2), w is the product of ℓ reflections $\{s_\alpha\}$ in W' with rootspaces spanning V. If W'' is the subreflection group generated by the $\{s_\alpha\}$, then by 3.7(3) the element w is a Coxeter element for W''. One reasons as before that W'' cannot be written as a nontrivial product. So W'' must be irreducible and w is a Coxeter element for it. Hence $\dim(W'') = (h+1)\ell = \dim(W)$, so $W = W' = W''$ (see 3). □

4. Generalized Coxeter elements

We move on to consider a weaker notion of Coxeter element which is useful in dealing with reflection groups which are not necessarily real.

4.1. Definition. Let R be $\mathbb{Z}$ or $\mathbb{Z}_p$. If W is a finite subgroup of $\mathrm{GL}_\ell(R)$ (in our applications a reflection subgroup), an element $w \in W$ is said to be a *generalized Coxeter element* if $\det(I - w) \neq 0$.

If R is as above and W is a subgroup of $\mathrm{GL}_\ell(R)$, we let L be the module R^ℓ on which W acts, V the vector space $\mathbb{Q} \otimes L$, and $\breve{T}$ the quotient module V/L.

4.2. *Remark.* In the case in which $R = \mathbb{Z}$ and W is the Weyl group of a connected compact Lie group of rank ℓ, then $\breve{T}$ is naturally isomorphic as a W-module to the group of elements of finite order in T_G. If $R = \mathbb{Z}_p$ and W is the Weyl group of a connected p-compact group of rank ℓ, then $\breve{T}$ is naturally isomorphic as a W-module to the *p-discrete torus* $\breve{T}_X$ [10].

4.3. Theorem. *Suppose that R is $\mathbb{Z}$ or $\mathbb{Z}_p$, that W is a finite subgroup of* $\mathrm{GL}_\ell(R)$*, and that $w \in W$ is a generalized Coxeter element of order k. Then the group $\breve{T}^W$ is finite, and its order divides* $\det(I - w)$ *in R. Moreover $k \cdot \breve{T}^W = 0$.*

Proof. Let $C = \langle w \rangle$, so that C has order k. The definition of a generalized Coxeter element implies that $V^C = 0$, so from the long exact group cohomology sequence for the coefficients $0 \to L \to V \to \breve{T} \to 0$, it follows that $\breve{T}^C = \mathrm{H}^0(C; \breve{T}) \cong \mathrm{H}^1(C; L)$. Hence these groups are finite and have exponent at most k. Since $\breve{T}^W \subset \breve{T}^C$, $\breve{T}^W$ is also finite, and $k \cdot \breve{T}^W = 0$. Let $\nu = 1 + w + w^2 + \cdots + w^{k-1} \in \mathbb{Z}[C]$. Then $\nu \cdot L \subset \nu \cdot V \subset V^C = 0$, so that

$$\mathrm{H}^1(C; L) = \ker(\nu : L \to L)/(I - w)L = L/(I - w)L\ .$$

By the theory of elementary divisors the quotient group on the right has order $|\det(I - w)|$ if $R = \mathbb{Z}$ case and has order given by the p-th part of this if $R = \mathbb{Z}_p$. □

4.4. *Example.* In the Weyl group of E_8, the Coxeter element w_{Cox} has order 30 (3.6). Since the exponents $\{m_j\}$ are each relatively prime to 30, the 15-th power and the 10-th power of the Coxeter element are generalized Coxeter elements of order 2 and 3, respectively. By 4.3,

$\mathrm{H}^0(W;\breve{T})$ is annihilated by 2 and 3. Hence this group is zero. One can also calculate from 3.3(5) that $|\det(1-w_{\mathrm{Cox}})| = 1$.

5. Centers of Lie groups

In this section we give various descriptions of the center of a connected simply connected compact Lie group, and then various order or exponent bounds on the centers of other compact Lie groups.

5.1. Simply connected compact groups. Suppose first that G is a connected compact Lie group. It is a classical fact that every element of G is contained inside some maximal torus. Since all maximal tori in G are conjugate, it follows immediately that the center of G is contained inside the center of the normalizer of T_G. In other words,

$$\mathcal{Z}_G \subset (T_G)^{W_G} \ . \tag{5.2}$$

This inclusion is not in general an equality. See [13, §8] for a detailed description of $\mathcal{Z}_G$ in terms of the group structure of $N_G(T_G)$; in general, the quotient $(T^W)/\mathcal{Z}_G$ is an elementary abelian 2-group. However, the inclusion 5.2 *is* an equality if G is simply connected (Theorem 1.4). The proof of this depends upon the following result of Borel.

5.3. Theorem. [22] [23] *If G is a simply connected compact Lie group and $g \in G$ then the centralizer of g in G is connected.*

Proof of 1.4. As above, $\mathcal{Z}_G \subset T^W$. Suppose that $t \in T^W$. Consider the centralizer $C_G(t)$. By 5.3, $C_G(t)$ is connected. It is clear that T is a maximal torus for $C_G(t)$ and that the Weyl group $W_{C_G(t)}$ is the subgroup of W that fixes t. By the choice of t, then, $W_{C_G(t)} = W$. The dimension of G is determined by W and the dimension of $C_G(t)$ by $W_{C_G(t)}$ (see 3.3(3)). So $\dim(C_G(t)) = \dim(G)$. Since G and $C_G(t)$ are compact connected manifolds, we must have $G = C_G(t)$. Hence $t \in \mathcal{Z}_G$. This is true for all $t \in T^W$, so $C_G = T^W$. □

The Weyl group of a connected compact Lie group G is a finite Coxeter group, and so has a unique conjugacy class of elements called *Coxeter elements* (see §3). From 5.2 it is immediately clear that for any Coxeter element $w_{\mathrm{Cox}} \in W_G$ there is an inclusion

$$\mathcal{Z}_G \subset (T_G)^{\langle w_{\mathrm{Cox}}\rangle} \ . \tag{5.4}$$

It is somewhat surprising that even this inclusion is an equality if G is simply connected (Theorem 1.6).

Proof of 1.6. Let $t \in T_G$ be fixed by w_{Cox}. The centralizer $C_G(t)$ is connected and hence its Weyl group is a subreflection group of the

Weyl group W of G. Since t is fixed by w_{Cox}, certainly $w_{\text{Cox}} \in W_{C_G(t)}$. By 3.1, $W_{C_G(t)} = W$. As in the above proof of 1.4, this gives $C_G(t) = G$.

Let $L = L_G$, $T = T_G$, and $C = \langle w_{\text{Cox}} \rangle$. It is easy to see that $T \cong (\mathbb{R} \otimes L)/L$ as a module over W. Since w_{Cox} is a Coxeter element, $(\mathbb{R} \otimes L)^C = 0$, so a long exact cohomology sequence (cf. proof of 4.3) gives an isomorphism

$$T^C \cong \mathrm{H}^1(C; L) .$$

The order of the group on the right is calculated in the proof of 4.3 as given by $|\det(I - w_{\text{Cox}})|$. □

5.5. *Remark.* Theorem 1.6 gives a simple algorithm for calculating the order of the center of a simply connected almost simple compact Lie group G. It consists of the following steps.

1. Determine the exponents (2.4) of W_G by consulting a table.
2. Combine this list of exponents with 3.3(5) to obtain the eigenvalues of the action of a Coxeter element $w_{\text{Cox}} \in W_G$ on V_G.
3. Use 1.6 to calculate $\#(\mathcal{Z}_G) = |\det(1 - w_{\text{Cox}})|$.

A little Galois theory shows that since w_{Cox} is represented by a rational matrix in the reflection representation of W_G, the power p^ν of a prime p which divides $|\det(1 - w_{\text{Cox}})|$ can be determined as follows. Let $\{m_j\}$ be the set of exponents for W_G, let ℓ be the rank of G, and let $h = (2\sum m_j)/\ell$ be the order of w_{Cox}. Let c_i be the number of exponents m_j such that the denominator of m_j/h (in lowest terms) is equal to p^i. Then $p^\nu = (p/(p-1))c_1 + \sum_{i>1} c_i$.

5.6. Other connected groups. We now prove some general results about centers which are simpler to work with than the somewhat cumbersome formula from [13, S8].

Proof of 1.7. The inclusion 5.2 is valid for any connected compact Lie group G, as is the analogue of 5.4, with w_{Cox} replaced by *any* element of W_G. Thus the second half of the above proof of 1.6 also gives 1.7. □

Proof of 1.8. By 2.14, the center of W_G is of order 2, generated by an element w which acts on $L = L_G$ by (-1) and hence on $T_G = (\mathbb{R} \otimes L)/L$ by inversion. Then $\det(1 - w) = 2^\ell$, so that w is a generalized Coxeter element. The center of G is a subgroup of $(T_G)^{\langle w \rangle} \cong (\mathbb{Z}/2)^\ell$ and so is of the indicated form. □

Proof of 1.10. We will use the notation of [13, 8.1]. It follows from [13, 8.2] that the $\mathcal{Z}_G = (T_G)^W$ if for each reflection s in W_G there is an equality $\sigma(s) = F(s)$. As is clear from [13, 8.7], it is enough to show that $H(s) = F(s)$. In the notation of [13, 8.7], consider the exact sequence

$$\{0\} \to H(s) \to T \to C \to \{0\}$$

and the induced exact sequence of elements of exponent 2

$$\{0\} \to {}_2H(s) \to {}_2T \to {}_2C \to \{0\} ,$$

where ${}_2C \cong \mathbb{Z}/2$ because C is a circle group. (This induced sequence is exact because $H(s)$, as a torus, is divisible and hence injective as an abelian group.) By assumption the action of s on ${}_2T \cong \mathbb{Z}/2 \otimes \pi_1(T)$ is nontrivial, and by construction the action of s on ${}_2H(s)$ is trivial. It follows immediately that $({}_2T)^{\langle s\rangle} = {}_2H(s)$. Now because of the exact sequence displayed in [13, 8.7] and the fact that $H(s)$ is injective, $F(s) = T^{\langle s\rangle}$ is isomorphic as an abelian group either to $H(s)$ or to $H(s) \oplus \mathbb{Z}/2$. Since we have shown that ${}_2F(s) = {}_2H(s)$, it must be that $F(s) = H(s)$. □

6. Centers of specific groups

In this section we apply the results of the previous section to obtain information about the Weyl groups of the simply connected almost simple compact Lie groups. We assume that for these groups the list of exponents of the corresponding Weyl group is known.

We will work out the case A_n in more detail than the others in order to illustrate the kinds of calculations that come up. The Weyl group is the symmetric group Σ_{n+1}, its exponents are $\{1, 2, 3, \ldots, n\}$, the order of w_{Cox} is $h = n + 1$, and the eigenvalues of w_{Cox} are $e^{2\pi ij/(n+1)}$, $j = 1, \ldots, n$ (3.3). These eigenvalues are exactly the roots of the polynomial

$$\prod_{j=1}^{n} \left(x - e^{2\pi ij/(n+1)}\right) = \frac{1 - x^{n+1}}{1 - x} = 1 + x + x^2 + \cdots + x^n .$$

The determinant $\det(1 - w_{\text{Cox}})$ is then the value of this polynomial at $x = 1$, namely, $(n+1)$. This implies that the order of the center of A_n is $(n+1)$. In fact this center is a cyclic group ($A_n = \mathrm{SU}_{n+1}$) but our numerical methods don't give this directly.

A similar calculation shows that for the Weyl groups of G_2, F_4, and E_8, $|\det(I - w_{\text{Cox}})| = 1$. Hence these Lie groups have trivial center. The corresponding determinants for E_6 and E_7 have absolute values 3 and 2, respectively. Thus the center of E_6 is cyclic of order 3, and the center of E_7 is cyclic of order 2. For B_n and C_n the determinants have absolute value 2 and so these groups have center $\mathbb{Z}/2$.

The groups D_n are left; the determinant $\det(1 - w_{\mathrm{Cox}})$ has absolute value 4 in all cases. The degrees associated to the Weyl group of D_{2k} are even, so this Weyl group contains a central element of order 2 (2.7) and the center of D_{2k} is an elementary abelian 2-group (1.8). Since this center has order 4, we conclude that for n even the center of D_n is $(\mathbb{Z}/2)^2$. Our numerical methods do not determine the group structure of center of D_{2k+1}; for that purpose it is necessary to look more closely at the explicit reflection representation, or to examine the group Spin_{4k+2}. It turns out that the center of D_{2k+1} is isomorphic to $\mathbb{Z}/4$.

7. Centers of p-compact groups

Suppose that X is a p-compact group. Associated to the maximal torus T_X and the center $\mathcal{Z}_X$ are certain discrete groups $\breve{T}_X$ and $\breve{\mathcal{Z}}_X$ called the *discrete torus* and *discrete center* respectively. The group $\breve{T}_X$ is isomorphic to $(\mathbb{Z}_{p^\infty})^{\ell_X}$, while $\breve{\mathcal{Z}}_X \subset \breve{T}_X$ is isomorphic to the product of a finite abelian p-group and a group of the form $(\mathbb{Z}_{p^\infty})^k$, where $k \leq \ell_X$ and $k = 0$ if $\pi_1(X)$ is finite. The group $\breve{\mathcal{Z}}_X$ effectively determines the p-compact group $\mathcal{Z}_X$, since $\mathrm{B}\mathcal{Z}_X$ is the p-completion of $\mathrm{B}\breve{\mathcal{Z}}_X$ [11].

In this section we write down various estimates for the discrete center of a p-compact group X. These are somewhat different from the Lie group estimates in §5, for two reasons: first of all, we do not have a version of Theorem 5.3 for p-compact groups; secondly, we do not have a theory of Coxeter elements for non-rational p-adic reflection groups.

Suppose that X is a connected p-compact group. The Weyl group W_X acts on the discrete torus $\breve{T}_X$; in fact, $\breve{T}_X$ is isomorphic to V_X/L_X as a W_X-module. By [11] there is an inclusion

$$\breve{\mathcal{Z}}_X \subset (\breve{T}_X)^{W_X}$$

analogous to 5.2.

7.1. Theorem. *Suppose that X is a connected p-compact group. If p is odd then $\breve{\mathcal{Z}}_X = (\breve{T}_X)^{W_X}$. If $p = 2$ and reduction mod 2 gives a mononomorphism*

$$W_X \to \mathrm{GL}_\ell(\mathbb{Z}_2) \to \mathrm{GL}_\ell(\mathbb{Z}/2) \tag{7.2}$$

then the same conclusion holds.

7.3. *Remark.* The first statement of 7.1 parallel to 1.4 but a bit stronger in that it does not require simple connectivity. The second statement is parallel to 1.10. Since every finite subgroup of the kernel of the reduction map $\mathrm{GL}_\ell(\mathbb{Z}_2) \to \mathrm{GL}_\ell(\mathbb{Z}/2)$ is an elementary abelian 2-group, the conclusion of 7.1 holds for any connected 2-compact group X with the property that the Weyl group of X has no normal elementary abelian 2-subgroups.

Proof of 7.1. The statement for p odd is in the remark after [11, 7.7]. The second statement is proved by the arguments in the proof of 1.10, but with references to [13, §8] replaced by references to [11, §7]. □

7.4. Proposition. *Suppose that X is a connected p-compact group with a generalized Coxeter element $w \in W_X$. Then $\breve{\mathcal{Z}}_X$ is finite, and $\#(\breve{\mathcal{Z}}_X)$ divides $|\det(1-w)|$ in $\mathbb{Z}_p$. If the order of w is prime to p, then $\breve{\mathcal{Z}}_X$ is trivial.*

Proof. This is very similar to the proof of 1.7. □

7.5. Proposition. *Suppose that X is a connected p-compact group of rank ℓ such that W_X is an irreducible reflection group with a nontrivial center. Then if p is odd, $\breve{\mathcal{Z}}_X$ is trivial, while if $p = 2$, $\breve{\mathcal{Z}}_X$ is isomorphic to $(\mathbb{Z}/2)^s$ for some $s \leq \ell$.*

Proof. This is very similar to the proof of 1.8. The difference between $p = 2$ and p odd is discussed in 2.14. □

8. Irreducible p-adic reflection groups

The Weyl group of a connected p-compact group is naturally presented as a finite p-adic reflection group. Any such group is a product of finite irreducible p-adic reflection groups, and the list of such finite irreducible p-adic reflection groups has been known for some time. This list (which depends on p) was given by Clark and Ewing [5]; they started with the list of irreducible finite *complex* reflection groups compiled by Shephard and Todd [19] and then determined, for each p, which of the groups in the Shephard-Todd list have realizations over the p-adic integers.

We are motivated by 7.4 to search for generalized Coxeter elements in Weyl groups of p-compact groups. In the case that the reflection group can be generated by exactly n reflections, 5.4 [19] verifies that there is a choice of product of the generating reflections with properities similar to the classical Coxeter elements. Unfortunately, there is a rather substantial list of complex reflection groups which are not generated by n reflections. On the other hand, the theorem of Solomon [20] gives the existence of a large number of elements with no eigenvalue $+1$ if the action is essential. The *regular* elements of Springer [21] are a related generalization of Coxeter elements.

However, for our needs, it is most convenient to analyse directly four cases :

1. Rational reflection groups (Weyl groups of Lie groups).
2. Sporadic non-rational reflection groups over $\mathbb{Q}_p$, p odd.
3. Infinite families of non-rational reflection group over $\mathbb{Q}_p$, p odd.
4. Non-rational reflection groups over $\mathbb{Q}_2$.

Clark and Ewing list these groups in these various classes and their degrees (which were determined by Shephard and Todd). Hardly any of our work directly depends on the field of definition.

8.1. Rational reflection groups. These were already treated in §6. We give the list of ordered pairs (G, d) where G is the standard designation for the a compact simply connected almost simple compact Lie group and $d = |\det(1 - w_{\text{Cox}})|$: $(A_n, n+1)$, $(B_n, 2)$, $(C_n, 2)$, $(D_n, 4)$, $(G_2, 1)$, $(F_4, 1)$, $(E_6, 3)$, $(E_7, 2)$, $(E_8, 1)$. Note that the Weyl groups of B_n and C_n are isomorphic as rational reflection groups.

8.2. Sporadic non-rational reflection groups. If W is one of these reflection groups, then W contains either a central element of order 2, or in the case of Type 25 [5] a central element of order 3. This follows from glancing over the degrees of these groups and applying 2.7. One can also calculate the order of $\mathcal{Z}_W$ as the ratio of the columns labeled g and g' in Table VII, page 302 of [19]. Note that if the prime $p > 2$ divides the order of $\mathcal{Z}_W$, then the representation is not realizable over $\mathbb{Q}_p$.

8.3. Infinite familes of non-rational reflection groups. The calculations are along the lines of [1, §3] but because of the nature of what has to be proved, they are simpler. There are three infinite families which contain non-rational complex reflection groups: Family $2a$, Family $2b$, and Family 3. We treat these families in reverse order.

If W belongs to Family 3 and is non-rational, then $\#(W) = m > 2$ and W is cyclic. Any generator of W is a central generalized Coxeter element w of order m, with $\det(1-w) = 1-m$. If W is realizable over $\mathbb{Q}_p$ with p odd, then m divides $(p-1)$, and m is prime to p.

If W belongs to Family $2b$, then W is a dihedral group of order $2m$, and contains a generalized Coxeter element w of order m. If W is non-rational and realizable over $\mathbb{Q}_p$, then $m > 1$ and m divides either $(p-1)$ or $(p+1)$ and hence, since $p > 2$, m is relatively prime to p.

If W belongs to Family $2a$ and has rank n, then W is one of the groups $G(m, r, n)$ of Shephard-Todd. Here r must divide m. Given a basis $\{e_j\}$ of $\mathbb{C}^\ell$ and $\theta = e^{2\pi i/m}$, $G(m,r,n)$ is generated by the operations $e_i \mapsto \theta^{\nu_i} e_{\sigma(i)}$, $\sigma \in \Sigma_n$, subject to the constraint that $\sum_i \nu_i = 0 \bmod r$. If $m > 1$, the action is *essential.* $G(m,r,n)$ has degrees $\{m, \ldots, (n-1)m, (m/r)n\}$ and order $(m/r)m^{n-1}n!$. We show below that in all cases for which $m > 1$, there is a generalized Coxeter element in $G(m,r,n)$ with order dividing $2m$.

The order of the center is $d = (m/r)gcd(r,n)$. If $d > 1$, the center is generated by the scalar diagonal matrix $D = \theta^a I_n$, where $a = m/d$. Thus D is a generalized Coxeter element of order d dividing m. If $d = 1$ then $r = m$ and $gcd(r,n) = 1$. Hence, there exists a number b with $0 < b < r$ so that $bn = 1 \bmod r$. If $b > 1$, define the non-scalar diagonal matrix $M = (\theta^b, \ldots, \theta^b, \theta^{b-1})$ in $G(m,r,n)$. The matrix M has no $+1$ eigenvalues and has order m. If $b = 1$, take $M = (\theta, \ldots, \theta, [0, \theta; 1, 0])$, where the notation specifies a matrix which is diagonal except for a 2×2 block in the lower right-hand corner. Now M has no $+1$ eigenvalues and M has order $2m$. For example, $G(2,2,2n+1)$ is the Weyl group of type D_{2n+1}. Finally, if $G(m,r,n)$ is realizable over $\mathbb{Q}_p$, then by [5] m is a divisor of $(p-1)$. Hence in all cases $G(m,r,n)$ has a generalized Coxeter element of order prime to p.

8.4. Non-rational reflection groups over $\mathbb{Q}_2$. There is only one non-rational reflection group W over $\mathbb{Q}_2$. This is the reflection group of Type 24 on the Clark-Ewing list [5], i.e., the Weyl group of the 2-compact group $DI(4)$ constructed in [9]. The group W is abstractly isomorphic to $\mathbb{Z}/2 \times \mathrm{GL}_3(\mathbb{Z}/2)$. We show that W has a generalized Coxeter element w such that $\det(1-w)$ is a unit in $\mathbb{Z}_2$. Let $V = (\mathbb{Q}_2)^3$ be the vector space which affords the reflection representation of W. According to the constructions in [9], there is a 2-adic integral lattice $L \subset V$ such that L is preserved by W and such that the natural map

$$W \to \mathrm{Aut}(L) \to \mathrm{Aut}(L/2L) \cong \mathrm{GL}_3(\mathbb{Z}/2)$$

can be identified with the obvious surjection

$$\mathbb{Z}/2 \times \mathrm{GL}_3(\mathbb{Z}/2) \to \mathrm{GL}_3(\mathbb{Z}/2)\ .$$

Let M be a matrix in $\mathrm{GL}_3(\mathbb{Z}/2)$ such that $\det(I-M) \neq 0$, and let w be a lift of M to W. Then $\det(I-w)$ is a unit in $\mathbb{Z}_2$.

8.5. *Remark.* By using more information about the fields of definitions of the particular complex reflection groups, one can replace some of the theory of this section with calculations. For example, if the order of W is prime to p, then any p-compact group realizing W has trivial center. However, these considerations do not cover all the examples, e.g. the group denoted by Type 29 at the prime 5.

9. Centers of specific p-compact groups

The following result includes 1.15.

9.1. Theorem. *Let X be a simply connected almost simple p-compact group with Weyl group W. If W is a rational reflection roup, i.e. X is of Lie type G for some simply connected almost simple compact Lie group G, then $\breve{\mathcal{Z}}_X$ is a finite abelian p-group whose order divides the order of the center of G . If W is a non-rational reflection group, then $\breve{\mathcal{Z}}_X$ is trivial.*

9.2. *Remark.* Taken in combination with the list in 8.1, Theorem 9.1 implies that if $p > 3$ and X is a connected p-compact group with an irreducible Weyl group, then the center of X is trivial unless X is of Lie type A_n, i.e., $W_X = \Sigma_n$.

Proof of 9.1. If W is rational, the result follows from 1.6 and 7.4; the point is that any Coxeter element of W gives a formula for the order of the center of G and, as a generalized Coxeter element, a bound for the order of the center of X.

Suppose that W is non-rational. If $p > 2$, then according to the calculations in §8, W contains a generalized Coxeter element of order prime to p. By 7.2 and 4.3, the center of X is trivial. If $p = 2$ then W is the Weyl group of $DI(4)$, and W contains a generalized Coxeter element w such that $\det(1 - w)$ is a unit in $\mathbb{Z}_2$ (see 8.4). Again by 7.2 and 4.3, the center of X is trivial. □

Proof of 1.16. This follows from the results in [12], [11], and [16]. As described in 1.14, X has a finite cover $\tilde{X}$ of the form $T \times \prod_i Y_i$, where each Y_i is simply connected and almost simple, and $X = \tilde{X}/A$ for some finite subgroup of $\breve{\mathcal{Z}}_{\tilde{X}}$. Let R be the product of those Y_i's of Lie type (those with rational Weyl groups) and let N be the product of the remaining Y_i's. By 9.1, $\breve{\mathcal{Z}}_N = 0$, so that $A \subset \breve{\mathcal{Z}}_R$ and

$$X = (T \times R \times N)/A = ((T \times R)/A) \times N \ .$$

Since $(T \times R)/A$ is of Lie type, this gives the desired product decomposition of X. □

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