

EXOTIC COHOMOLOGY FOR $\mathrm{GL}_n(\mathbf{Z}[1/2])$

W. G. DWYER

§1. INTRODUCTION

Let Λ denote the ring $\mathbf{Z}[1/2]$, G_n the group $\mathrm{GL}_n(\Lambda)$ of invertible $n \times n$ matrices over Λ , and D_n the group of diagonal matrices in G_n . The inclusion $D_n \rightarrow G_n$ induces a classifying space map $\iota_n : \mathrm{BD}_n \rightarrow \mathrm{BG}_n$ and a cohomology homomorphism

$$\iota_n^* : \mathrm{H}^*(\mathrm{BG}_n; \mathbf{F}_2) \rightarrow \mathrm{H}^*(\mathrm{BD}_n; \mathbf{F}_2) .$$

Say that the mod 2 cohomology of G_n is *detected on diagonal matrices* if ι_n^* is injective. In [15, 14.7] Quillen made a conjecture which specializes in the case of the ring Λ to the following statement (see [9, p. 51]):

1.1 Conjecture. *For any $n \geq 1$ the mod 2 cohomology of G_n is detected on diagonal matrices.*

There is some evidence for this conjecture. Mitchell [14] and Henn [10] have proved it for $n \leq 3$. Voevodsky has announced a proof of the mod 2 Quillen-Lichtenbaum Conjecture for \mathbf{Z} , and from [5] and [14] it follows that ι_n^* is injective on the image of $\mathrm{H}^*(\mathrm{BGL}(\Lambda); \mathbf{F}_2) \rightarrow \mathrm{H}^*(\mathrm{BG}_n; \mathbf{F}_2)$. In particular, 1.1 is true in the stable range.

The aim of this paper, though, is to give a *disproof* of Conjecture 1.1.

1.2 Theorem. *The mod 2 cohomology of G_{32} is **not** detected on diagonal matrices.*

Given the remarks above, it is a consequence of 1.2 that there exists an element in the cohomology of G_{32} which is not in the image of $\mathrm{H}^*(\mathrm{BGL}(\Lambda); \mathbf{F}_2)$. In fact, 1.2 is proved by a technique distantly related to the one which Quillen uses in [15, p. 592] to show that for various other number rings S and primes p the restriction map $\mathrm{H}^*(\mathrm{BGL}(S); \mathbf{F}_p) \rightarrow \mathrm{H}^*(\mathrm{BGL}_n(S); \mathbf{F}_p)$ is not surjective.

Further developments. Very recently, Henn and Lannes have improved upon 1.2 by showing that the mod 2 cohomology of G_{14} is not detected on diagonal matrices. Moreover, Henn proves in [11, 0.6] that the Poincaré series $p_n(t)$ of the kernel of ι_n^* has a pole at $t = 1$ of order $n - n_0 + 1$, where n_0 is the smallest natural number such that $\iota_{n_0}^*$ is not injective. Combining these results shows that for *any* $n \geq 14$ the cohomology of G_n is far from being detected on diagonal matrices.

This research was partially supported by National Science Foundation Grant DMS95-05024

A generalization. Theorem 1.2 is a consequence of a result which is slightly more general. Recall that if P and G are groups and $\alpha, \beta : P \rightarrow G$ are homomorphisms, then α is said to be *conjugate* to β if there is an element $g \in G$ such that $g\alpha g^{-1} = \beta$. Let $\rho_{\mathbf{R}} : G_n \rightarrow \mathrm{GL}_n(\mathbf{R})$ and $\rho_{\mathbf{F}_3} : G_n \rightarrow \mathrm{GL}_n(\mathbf{F}_3)$ be the obvious homomorphisms. Two homomorphisms $\alpha, \beta : P \rightarrow G_n$ are said to *become conjugate over \mathbf{R}* (resp. *become conjugate over \mathbf{F}_3*) if $\rho_{\mathbf{R}}\alpha$ and $\rho_{\mathbf{R}}\beta$ (resp. $\rho_{\mathbf{F}_3}\alpha$ and $\rho_{\mathbf{F}_3}\beta$) are conjugate.

1.3 Theorem. *Suppose that the mod 2 cohomology of G_n is detected on diagonal matrices. Let P be a finite 2-group with homomorphisms $\alpha, \beta : P \rightarrow G_n$. Then α is conjugate to β if and only if α becomes conjugate to β over \mathbf{R} and over \mathbf{F}_3 .*

To obtain 1.2 from 1.3, let μ_n denote the group of 2^n -th roots of unity. The smallest n with the property that the ideal class group of $\Lambda(\mu_n)$ is nontrivial is 6, and the degree of $\Lambda(\mu_6)$ over Λ (equivalently, the rank of $\Lambda(\mu_6)$ as a Λ -module) is $\phi(2^6) = 2^5 = 32$. Let $P = \mu_6$, considered as a subgroup of $\Lambda(\mu_6)^\times$, and let \mathcal{I} be a nonprincipal ideal in $\Lambda(\mu_6)$. It is then not hard to use the multiplicative actions of P on \mathcal{I} and on $\Lambda(\mu_n)$ itself to construct two nonconjugate homomorphisms $P \rightarrow G_{32}$ which become conjugate over \mathbf{R} and over \mathbf{F}_3 .

The proof of 1.3 is homotopy theoretic. If X and Y are spaces, let $[X, Y]$ denote the set of (unpointed) homotopy classes of maps $X \rightarrow Y$; if P and G are groups, let $\{P, G\}$ denote the set of conjugacy classes of homomorphisms $P \rightarrow G$. The classifying space functor gives a bijection $\{P, G\} \cong [BP, BG]$. For each $n \geq 1$ we construct a space X_n together with a map

$$\chi_n : BG_n \rightarrow X_n$$

such that the following three statements hold.

1.4 Proposition. *If the mod 2 cohomology of G_n is detected on diagonal matrices, then $\chi_n^* : H^*(X_n; \mathbf{F}_2) \rightarrow H^*(BG_n; \mathbf{F}_2)$ is an isomorphism.*

1.5 Proposition. *If χ_n^* is an isomorphism, then for any finite 2-group P the map $\chi_n \cdot (-) : [BP, BG_n] \rightarrow [BP, X_n]$ is a bijection.*

1.6 Proposition. *Let P be a finite 2-group with homomorphisms $\alpha, \beta : P \rightarrow G_n$. Then $\chi_n \cdot (B\alpha)$ is homotopic to $\chi_n \cdot (B\beta)$ if and only if α becomes conjugate to β over \mathbf{R} and over \mathbf{F}_3 .*

Together these imply 1.3.

Section 2 contains a description of the machinery which is used to construct the space X_n , §3 has proofs of the above three propositions, and §4 contains the derivation of 1.2 from 1.3. Throughout the paper, unspecified homology and cohomology is to be taken with mod 2 coefficients. The symbol $\hat{B}G$ denotes the 2-completion [1] of the classifying space of G . The usual topological groups $\mathrm{GL}_n(\mathbf{R})$ and $\mathrm{GL}_n(\mathbf{C})$ are denoted $\mathrm{GL}_n^{\mathrm{top}}(\mathbf{R})$ and $\mathrm{GL}_n^{\mathrm{top}}(\mathbf{C})$.

The idea for this paper originated in conversations with S. Mitchell, and the approach depends heavily on his calculations from [14]. The construction of X_n goes back to work with E. Friedlander [4]. Some of the arguments below can be generalized, but we have decided to concentrate on proving 1.2.

§2. ÉTALE HOMOTOPY THEORY

The space X_n promised in §1 is constructed using étale homotopy theory [8], which gives a covariant mechanism for assigning a space (more accurately a pro-space) $A_{\acute{e}t}$ to any reasonable scheme or simplicial scheme A . We build X_n as a certain space of maps between two étale homotopy types (2.6), in imitation of the way in which $\mathrm{GL}_n(\Lambda)$ can be described as a certain set of maps between schemes.

Étale homotopy types. Here are some examples of étale homotopy types. In the examples, the symbol ℓ stands for a prime number.

2.1 Fields and complete local rings. If k is a field, then $\mathrm{Spec}(k)_{\acute{e}t}$ is a pro-space of type $K(\pi, 1)$, where π is the Galois group over k of the separable algebraic closure of k . If S is a complete local ring with residue class field k , then the map $\mathrm{Spec}(k)_{\acute{e}t} \rightarrow \mathrm{Spec}(S)_{\acute{e}t}$ induced by $S \rightarrow k$ is an equivalence. For instance, $\mathrm{Spec}(\mathbf{C})_{\acute{e}t}$ is contractible, $\mathrm{Spec}(\mathbf{R})_{\acute{e}t}$ is equivalent to $\mathbf{B}\mathbf{Z}/2$, and both $\mathrm{Spec}(\mathbf{F}_3)_{\acute{e}t}$ and $\mathrm{Spec}(\mathbf{Z}_3)_{\acute{e}t}$ are equivalent to the profinite completion of a circle.

If S is a commutative ring, let $\mathrm{GL}_{n,S}$ denote the rank n general linear group scheme over S . Applying the usual bar construction to $\mathrm{GL}_{n,S}$ gives a classifying object $\mathrm{BGL}_{n,S}$ [8, 1.2], which is a simplicial scheme. See [8, §8] for the following three examples.

2.2 General linear groups over algebraically closed fields. If k is an algebraically closed field of characteristic zero, the pro-space $(\mathrm{BGL}_{n,k})_{\acute{e}t}$ is equivalent to the profinite completion of the space $\mathrm{BGL}_n^{\mathrm{top}}(\mathbf{C})$. If k is the algebraic closure of a finite field, then the ℓ -completion of $(\mathrm{BGL}_{n,k})_{\acute{e}t}$ at any prime ℓ not equal to $\mathrm{char}(k)$ is equivalent to the Bousfield-Kan ℓ -completion tower $\{(\mathbf{Z}/\ell)_s \mathrm{BGL}_n^{\mathrm{top}}(\mathbf{C})\}_s$.

2.3 General linear groups over other fields. If k is a field of characteristic zero with algebraic closure \bar{k} , the natural sequence

$$(\mathrm{BGL}_{n,\bar{k}})_{\acute{e}t} \rightarrow (\mathrm{BGL}_{n,k})_{\acute{e}t} \rightarrow \mathrm{Spec}(k)_{\acute{e}t}$$

is a fibration sequence of pro-spaces. In effect, if π is the Galois group of \bar{k} over k , $(\mathrm{BGL}_{n,k})_{\acute{e}t}$ is the Borel construction of the natural action of π on $(\mathrm{BGL}_{n,\bar{k}})_{\acute{e}t}$. If k is a finite field with algebraic closure \bar{k} and $\ell \neq \mathrm{char}(k)$, there is a similar fibration sequence

$$\{(\mathbf{Z}/\ell)_s (\mathrm{BGL}_{n,\bar{k}})_{\acute{e}t}\}_s \rightarrow \{(\mathbf{Z}/\ell)_s^\bullet (\mathrm{BGL}_{n,k})_{\acute{e}t}\}_s \rightarrow \mathrm{Spec}(k)_{\acute{e}t} ,$$

where $\{(\mathbf{Z}/\ell)_s^\bullet(-)\}_s$ denotes fibrewise ℓ -completion over $\mathrm{Spec}(k)_{\acute{e}t}$.

2.4 General linear groups over number rings. Suppose that R is a number ring, and that S is the ring $R[1/\ell]$. Let \bar{k} denote either the algebraic closure of the quotient field of S , or the algebraic closure of one of the residue class fields of S (note that none of these residue fields have characteristic ℓ). Then the natural sequence

$$\{(\mathbf{Z}/\ell)_s (\mathrm{BGL}_{n,\bar{k}})_{\acute{e}t}\}_s \rightarrow \{(\mathbf{Z}/\ell)_s^\bullet (\mathrm{BGL}_{n,S})_{\acute{e}t}\}_s \rightarrow \mathrm{Spec}(S)_{\acute{e}t} .$$

is a fibration sequence. There are identical fibration sequences if S is replaced by a complete local ring of residue characteristic different from ℓ .

2.5 Number rings. The étale homotopy type of a number ring is not easy to pin down; see [5, 2.1] for a description of its untwisted “integral” homology. We will settle for a partial description of $\mathrm{Spec}(\Lambda)_{\acute{e}t}$, where, as usual, $\Lambda = \mathbf{Z}[1/2]$. Choose an embedding $\mathbf{Z}_3 \rightarrow \mathbf{C}$. There are induced commutative diagrams of rings and of pro-spaces

$$\begin{array}{ccc} \mathbf{C} & \longleftarrow & \mathbf{R} & & \mathrm{Spec}(\mathbf{C})_{\acute{e}t} & \longrightarrow & \mathrm{Spec}(\mathbf{R})_{\acute{e}t} \\ \uparrow & & \uparrow & & \downarrow & & \downarrow \\ \mathbf{Z}_3 & \longleftarrow & \Lambda & & \mathrm{Spec}(\mathbf{Z}_3)_{\acute{e}t} & \longrightarrow & \mathrm{Spec}(\Lambda)_{\acute{e}t} \end{array} .$$

Since $\mathrm{Spec}(\mathbf{C})_{\acute{e}t}$ is contractible, the diagram induces a map

$$\mathrm{Spec}(\mathbf{Z}_3)_{\acute{e}t} \vee \mathrm{Spec}(\mathbf{R})_{\acute{e}t} \rightarrow \mathrm{Spec}(\Lambda)_{\acute{e}t} .$$

Pick an equivalence $\mathbf{BZ}/2 \rightarrow \mathrm{Spec}(\mathbf{R})_{\acute{e}t}$, and a map $S^1 \rightarrow \mathrm{Spec}(\mathbf{Z}_3)$ which sends the generator of $\pi_1 S^1$ to the Frobenius automorphism of $\bar{\mathbf{F}}_3$ over \mathbf{F}_3 . What results is a map

$$S^1 \vee \mathbf{BZ}/2 \rightarrow \mathrm{Spec}(\Lambda)_{\acute{e}t} .$$

By [5], this map induces an isomorphism on mod 2 cohomology (cf. [6, §2]).

2.6 Étale approximations to $\mathrm{BGL}_n(S)$. Let S be an algebra over Λ . The homomorphism $\Lambda \rightarrow S$ induces a map $\mathrm{Spec}(S)_{\acute{e}t} \rightarrow \mathrm{Spec}(\Lambda)_{\acute{e}t}$. We set $\ell = 2$ and let $\mathrm{BGL}_n^{\acute{e}t}(S)$ denote the basepoint component of the space which in [4, 2.3] is called $\mathrm{Hom}_\ell(\mathrm{Spec}(S)_{\acute{e}t}, (\mathrm{BGL}_{n,\Lambda})_{\acute{e}t})_\Lambda$; in other words, $\mathrm{BGL}_n^{\acute{e}t}(S)$ is the basepoint component of the function space of maps over $\mathrm{Spec}(\Lambda)_{\acute{e}t}$ from $\mathrm{Spec}(S)_{\acute{e}t}$ to the fibrewise 2-completion of $(\mathrm{BGL}_{n,\Lambda})_{\acute{e}t}$. (The function space is pointed because the map $\mathrm{BGL}_{n,\Lambda} \rightarrow \mathrm{Spec}(\Lambda)$ has a natural section provided by the basepoint of the fibrewise bar construction.) Since $\mathrm{BGL}_n(S)$ can be identified as the basepoint component of the space of maps $\mathrm{Spec}(S) \rightarrow \mathrm{BGL}_{n,\Lambda}$ over $\mathrm{Spec}(\Lambda)$ [4, 4.2], functoriality gives a map

$$\chi_{n,S} : \mathrm{BGL}_n(S) \rightarrow \mathrm{BGL}_n^{\acute{e}t}(S)$$

(see [4, 2.5, pf. of 4.2]). We will describe this map below in some particular cases.

2.7 Remark. Suppose that B is a space (i.e. a trivial pro-space) together with a map $B \rightarrow \mathrm{Spec}(\Lambda)_{\acute{e}t}$. Let “holim” denote the homotopy inverse limit functor from pro-spaces to spaces [8, §6], and E be the fibration over B obtained by the following homotopy pullback diagram

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{holim}\{(\mathbf{Z}/2)_s^\bullet(\mathrm{BGL}_{n,\Lambda})_{\acute{e}t}\}_s \\ \downarrow & & \downarrow \\ B & \longrightarrow & \mathrm{holim}\mathrm{Spec}(\Lambda)_{\acute{e}t} \end{array} .$$

The homotopy fibre of $E \rightarrow B$ is $\hat{\mathrm{BGL}}_n^{\mathrm{top}}(\mathbf{C})$. Essentially by definition, the space $\mathrm{Hom}_\ell(B, (\mathrm{BGL}_{n,\Lambda})_{\acute{e}t})_\Lambda$ is equivalent to the space of sections of the map $E \rightarrow B$.

It follows from a spectral sequence argument [4, 2.11] that if S is a Λ -algebra and $f : B \rightarrow \mathrm{Spec}(S)_{\acute{e}t}$ is a map of pro-spaces which induces an isomorphism on mod 2 cohomology, then f induces an equivalence

$$\mathrm{BGL}_n^{\acute{e}t}(S) \xrightarrow{\simeq} \mathrm{Hom}_\ell(B, (\mathrm{BGL}_{n,\Lambda})_{\acute{e}t})_\Lambda .$$

2.8 The complex numbers. Since $\mathrm{Spec}(\mathbf{C})_{\acute{e}t}$ is contractible, it follows from 2.7 that $\mathrm{BGL}_n^{\acute{e}t}(\mathbf{C})$ is equivalent to $\hat{\mathrm{BGL}}_n^{\mathrm{top}}(\mathbf{C})$. The same calculation works for any separably closed field of characteristic not 2. The map $\chi_{n,\mathbf{C}}$ is essentially the composite of the usual map $\mathrm{BGL}_n(\mathbf{C}) \rightarrow \mathrm{BGL}_n^{\mathrm{top}}(\mathbf{C})$ with the 2-completion map.

2.9 The real numbers. Since $\mathrm{Spec}(\mathbf{R})_{\acute{e}t}$ is equivalent to $\mathbf{BZ}/2$, it follows from 2.7 that $\mathrm{BGL}_n^{\acute{e}t}(\mathbf{R})$ is equivalent to the basepoint component of the space of sections of a fibration over $\mathbf{BZ}/2$ with $\hat{\mathrm{BGL}}_n^{\mathrm{top}}(\mathbf{C})$ as the fibre. By naturality, this fibration is the one associated to the action of $\mathbf{Z}/2$ on $\hat{\mathrm{BGL}}_n^{\mathrm{top}}(\mathbf{C})$ by complex conjugation. It follows from [7] that the natural map from $\hat{\mathrm{BGL}}_n^{\mathrm{top}}(\mathbf{R})$ to this space of sections is an equivalence. The map $\chi_{n,\mathbf{R}}$ is essentially the composite of the usual map $\mathrm{BGL}_n(\mathbf{R}) \rightarrow \mathrm{BGL}_n^{\mathrm{top}}(\mathbf{R})$ with the 2-completion map.

2.10 The field $\bar{\mathbf{F}}_3$. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{BGL}_n(\bar{\mathbf{F}}_3) & \longrightarrow & \mathrm{colim}_n \mathrm{BGL}_n(\bar{\mathbf{F}}_3) = \mathrm{BGL}(\bar{\mathbf{F}}_3) \\ \chi_{n,\bar{\mathbf{F}}_3} \downarrow & & \downarrow \mathrm{colim}_n(\chi_{n,\bar{\mathbf{F}}_3}) \\ \mathrm{BGL}_n^{\acute{e}t}(\bar{\mathbf{F}}_3) & \longrightarrow & \mathrm{colim}_n \mathrm{BGL}_n^{\acute{e}t}(\bar{\mathbf{F}}_3) \end{array}$$

in which the colimit maps are induced by the usual matrix block inclusions. The right hand vertical arrow induces an isomorphism on mod 2 cohomology (see [4, 4.5, 8.6]). The horizontal maps induce surjections on mod 2 cohomology; for the upper one see [16] and for the lower one 2.7. By diagram chasing, $\chi_{n,\bar{\mathbf{F}}_3}$ induces a surjection on mod 2 cohomology. Since $\mathrm{BGL}_n(\bar{\mathbf{F}}_3)$ and $\mathrm{BGL}_n^{\acute{e}t}(\bar{\mathbf{F}}_3)$ have mod 2 cohomology rings which are abstractly isomorphic ([16], 2.7) and finite in each dimension, $\chi_{n,\bar{\mathbf{F}}_3}$ must be an isomorphism on mod 2 cohomology. It follows that $\mathrm{BGL}_n^{\acute{e}t}(\bar{\mathbf{F}}_3)$ is equivalent to $\hat{\mathrm{BGL}}_n(\mathbf{F}_3)$.

2.11 The field \mathbf{F}_3 . There is a map $S^1 \rightarrow \mathrm{Spec}(\mathbf{F}_3)_{\acute{e}t}$ which sends a generator of $\pi_1(S^1)$ to Frobenius automorphism ψ of $\bar{\mathbf{F}}_3$ over \mathbf{F}_3 . This map is an isomorphism on cohomology with finite coefficients, in particular, on mod 2 cohomology. It follows from 2.7 that $\mathrm{BGL}_n^{\acute{e}t}(\mathbf{F}_3)$ is equivalent to the space of sections of a fibration over S^1 with fibre $\hat{\mathrm{BGL}}_n^{\mathrm{top}}(\mathbf{C}) \simeq \mathrm{BGL}_n^{\acute{e}t}(\bar{\mathbf{F}}_3)$ (2.8). By naturality, this is the fibration associated to the action of ψ on $\mathrm{BGL}_n^{\acute{e}t}(\bar{\mathbf{F}}_3)$ and so its space of sections is (by definition) the homotopy fixed point set $(\mathrm{BGL}_n^{\acute{e}t}(\bar{\mathbf{F}}_3))^{\mathrm{h}\psi}$. Consider the commutative diagram of spaces with an action of ψ (the action in the left hand column is trivial):

$$\begin{array}{ccc} \mathrm{BGL}_n(\mathbf{F}_3) & \xrightarrow{u} & \mathrm{BGL}_n(\bar{\mathbf{F}}_3) \\ \chi_{n,\mathbf{F}_3} \downarrow & & \downarrow \chi_{n,\bar{\mathbf{F}}_3} \\ \mathrm{BGL}_n^{\acute{e}t}(\mathbf{F}_3) & \xrightarrow{v} & \mathrm{BGL}_n^{\acute{e}t}(\bar{\mathbf{F}}_3) \end{array} .$$

The space $\mathrm{BGL}_n^{\acute{e}t}(\bar{\mathbf{F}}_3)$ is 2-complete (2.7). The map $\chi_{n,\bar{\mathbf{F}}_3}$ induces an equivalence $\hat{\mathrm{BGL}}_n(\bar{\mathbf{F}}_3) \rightarrow \mathrm{BGL}_n^{\acute{e}t}(\bar{\mathbf{F}}_3)$ (2.10). Quillen [16] shows that the map $\mathrm{BGL}_n(\mathbf{F}_3) \rightarrow (\hat{\mathrm{BGL}}_n(\bar{\mathbf{F}}_3))^{\mathrm{h}\psi}$ induced by the map u gives an isomorphism on mod 2 cohomology. It follows that the map

$$\mathrm{BGL}_n(\mathbf{F}_3) \xrightarrow{\chi_{n,\mathbf{F}_3}} \mathrm{BGL}_n^{\acute{e}t}(\mathbf{F}_3) \xrightarrow{\simeq} (\mathrm{BGL}_n^{\acute{e}t}(\bar{\mathbf{F}}_3))^{\mathrm{h}\psi}$$

also induces an isomorphism on mod 2 cohomology, and that $\mathrm{BGL}_n^{\acute{e}t}(\mathbf{F}_3)$ is equivalent to $\hat{\mathrm{BGL}}_n(\mathbf{F}_3)$.

Remark. The conjecture that $\chi_{n,S}$ induces an isomorphism on mod 2 cohomology is a very strong unstable analogue of the mod 2 Quillen-Lichtenbaum Conjecture for the ring S . This conjecture is true for a finite field (cf. 2.11) or the algebraic closure of a finite field (cf. 2.10). It is unknown whether or not it is true for the fields \mathbf{R} and \mathbf{C} . The results in this paper show that it is *false* for the ring Λ .

§3. THE SPACE X_n AND ITS PROPERTIES

We define X_n to be the space $\mathrm{BGL}_n^{\acute{e}t}(\Lambda)$ from 2.6, and $\chi_n : \mathrm{BG}_n \rightarrow X_n$ the map $\chi_{n,\Lambda} : \mathrm{BGL}_n(\Lambda) \rightarrow \mathrm{BGL}_n^{\acute{e}t}(\Lambda)$. In this section we prove 1.4, 1.5 and 1.6.

(Co)homology of X_n . Let BD_\bullet , BG_\bullet and X_\bullet denote respectively the spaces $\coprod \mathrm{BD}_n$, $\coprod \mathrm{BG}_n$ and $\coprod X_n$. The index n in these coproducts runs over all non-negative integers, where for $n = 0$ the spaces involved are contractible. There are maps

$$\mathrm{BD}_\bullet \xrightarrow{\iota} \mathrm{BG}_\bullet \xrightarrow{\chi} X_\bullet .$$

Under matrix block sum all three of these spaces are homotopy associative H-spaces; BG_\bullet and X_\bullet are homotopy commutative. The maps ι and χ respect the multiplications.

There is a natural identification

$$\mathrm{BD}_1 \simeq \mathrm{B}(\mathbf{Z} \times \mathbf{Z}/2) \simeq S^1 \times \mathrm{B}\mathbf{Z}/2 .$$

Let $e \in \mathrm{H}_1 S^1$ and $\beta_k \in \mathrm{H}_k \mathrm{B}\mathbf{Z}/2$ be generators. We denote the classes $e \otimes \beta_{k-1}$ and $1 \otimes \beta_k$ in $\mathrm{H}_k \mathrm{BD}_1$ by a_k and b_k respectively. Let $a_k^G = \iota_*(a_k)$, $b_k^G = \iota_*(b_k)$, $a_k^X = (\chi\iota)_*(a_k)$, $b_k^X = (\chi\iota)_*(b_k)$. Since $\mathrm{H}_* \mathrm{BD}_\bullet$ is the free \mathbf{F}_2 -algebra on the elements a_k ($k \geq 1$) and b_k ($k \geq 0$), the image of ι_* is the subalgebra of $\mathrm{H}_* \mathrm{BG}_\bullet$ generated by the classes a_k^G and b_k^G , while the image of $(\chi\iota)_*$ is generated by the classes a_k^X and b_k^X .

3.1 Proposition. *The algebra $\mathrm{H}_* X_\bullet$ is the free commutative \mathbf{F}_2 -algebra on the classes a_k^X ($k \geq 1$) and b_k^X ($k \geq 0$) subject to the following relations:*

- (1) $(a_k^X)^2 = 0$ for k odd, and
- (2) $a_k^X b_0^X + a_{k-1}^X b_1^X + \cdots + a_1^X b_{k-1}^X = 0$ for k even.

3.2 Lemma. *There is a homotopy fibre square*

$$\begin{array}{ccc} X_n & \longrightarrow & \mathrm{BGL}_n^{\acute{e}t}(\mathbf{R}) \\ \downarrow & & \downarrow \\ \mathrm{BGL}_n^{\acute{e}t}(\mathbf{Z}_3) & \longrightarrow & \mathrm{BGL}_n^{\acute{e}t}(\mathbf{C}) \end{array} .$$

Proof. This is a consequence of 2.5 and 2.7. \square

3.3 Remark. The square from 3.2 can up to homotopy be rewritten in the following way:

$$\begin{array}{ccc} X_n & \longrightarrow & \hat{\mathrm{BGL}}_n^{\mathrm{top}}(\mathbf{R}) \\ \downarrow & & \downarrow u \\ \hat{\mathrm{BGL}}_n(\mathbf{F}_3) & \xrightarrow{v} & \hat{\mathrm{BGL}}_n^{\mathrm{top}}(\mathbf{C}) \end{array} .$$

The rewriting is justified by 2.9, 2.8, 2.1 and 2.7. By naturality the map u is the usual one induced by the map $\mathbf{R} \rightarrow \mathbf{C}$. The map v is a little more problematical, but it is clear from 2.5 that the restriction of v to the diagonal matrices in $GL_n(\mathbf{F}_3)$ is induced by the ordinary diagonal inclusion $\{\pm 1\}^n \rightarrow \mathrm{BGL}_n(\mathbf{C})$. The fact that the mod 2 cohomology of $\mathrm{BGL}_n(\mathbf{F}_3)$ is detected on diagonal matrices [14, §3] [16] means that it is easy to compute the map on mod 2 homology or cohomology induced by v .

Proof of 3.1. This is essentially given by Mitchell in [14, 4.6]; his space $JK(\mathbf{Z})$ can be identified with $\mathrm{colim}_n X_n$, where the maps in the colimit are the block inclusions given by product with a point in X_1 . The role of diagram [14, 4.1] is played by the homotopy fibre square from 3.3 above. \square

3.4 Proposition. *The classes a_k^G , $k \geq 1$ and b_k^G , $k \geq 0$ in $H_* \mathrm{BG}_\bullet$ satisfy the analogs of relations (1) and (2) from 3.1.*

Proof. See [14, pf. of 7.1]. The place to look for these relations is in $H_* \mathrm{BGL}_2(\Lambda)$, and Mitchell computes this homology explicitly [14, §6]. \square

Proof of 1.4. By 3.1 and 3.4, the homology map $(\chi_n \iota_n)_*$ is surjective and its kernel is equal to the kernel of $(\iota_n)_*$. By duality, the cohomology map $(\chi_n \iota_n)^*$ is injective and its image is equal to the image of ι_n^* . This implies that ι_n^* is injective if and only if χ_n^* is an isomorphism. \square

Maps into X_n . Let P be a finite 2-group. We are interested in studying the set $[BP, X_n]$ of unbased homotopy classes of maps from BP to X_n . The homotopy fibre square from 3.3 gives a map

$$(3.5) \quad [BP, X_n] \rightarrow [BP, \hat{\mathrm{BGL}}_n(\mathbf{F}_3)] \times [BP, \hat{\mathrm{BGL}}_n^{\mathrm{top}}(\mathbf{R})] .$$

3.6 Lemma. *The map 3.5 is injective.*

Proof. It follows from 3.3 that there is a homotopy fibre square of mapping spaces

$$\begin{array}{ccc} \mathrm{Map}(BP, X_n) & \longrightarrow & \mathrm{Map}(BP, \hat{\mathrm{BGL}}_n^{\mathrm{top}}(\mathbf{R})) \\ \downarrow & & \downarrow \\ \mathrm{Map}(BP, \hat{\mathrm{BGL}}_n(\mathbf{F}_3)) & \longrightarrow & \mathrm{Map}(BP, \hat{\mathrm{BGL}}_n^{\mathrm{top}}(\mathbf{C})) \end{array} .$$

By an elementary argument, it is enough to show that each component of the space $\mathrm{Map}(BP, \hat{\mathrm{BGL}}_n^{\mathrm{top}}(\mathbf{C}))$ is 1-connected. By [7] the space $\mathrm{Map}(BP, \hat{\mathrm{BGL}}_n^{\mathrm{top}}(\mathbf{C}))$ is equivalent to the 2-completion of the disjoint union $\coprod_{\rho} BC(\rho(P))$, where ρ runs through a set of representatives for the conjugacy classes of homomorphisms $P \rightarrow \mathrm{GL}_n(\mathbf{C})$, and $C(\rho(P))$ is the centralizer of $\rho(P)$. By elementary representation theory each one of these centralizers is isomorphic to a product of complex general linear groups, and so is connected. \square

The following theorem is derived in [13, §1] from Carlsson's work in [2] and [3]. We state it only for the prime 2, although it holds for any prime.

3.7 Theorem. *Let Γ be a group of virtually finite cohomological dimension, and let P be a finite 2-group. Then the natural map*

$$\{P, \Gamma\} \cong [BP, B\Gamma] \rightarrow [BP, \hat{B}\Gamma]$$

is a bijection.

Proof of 1.5. The space X_n is 2-complete (3.3), so the fact that χ_n^* is an isomorphism implies that X_n is equivalent to $\hat{B}G_n$. Since G_n is a group of virtually finite cohomological dimension [17, p. 124] the result follows from 3.7. \square

Proof of 1.6. Consider the commutative diagram

$$\begin{array}{ccccc} [BP, \mathrm{BGL}_n(\mathbf{R})] & \xleftarrow{B\rho_{\mathbf{R}}} & [BP, \mathrm{B}G_n] & \xrightarrow{B\rho_{\mathbf{F}_3}} & [BP, \mathrm{BGL}_n(\mathbf{F}_3)] \\ \downarrow & & \downarrow & & \downarrow \\ [BP, \mathrm{BGL}_n^{\acute{e}t}(\mathbf{R})] & \longleftarrow & [BP, X_n] & \longrightarrow & [BP, \mathrm{BGL}_n^{\acute{e}t}(\mathbf{F}_3)] \end{array} .$$

By 2.9, 2.11, 3.7 and [7], the right and left vertical arrows are bijections. The result follows from 3.6. \square

§4. EXPLOITING THE CLASS GROUP

In this section we derive 1.1 from 1.3 by showing how class groups of cyclotomic extensions of Λ can be used to construct the necessary homomorphisms from finite 2-groups into $\mathrm{GL}_{32}(\Lambda)$.

Recall that μ_n denotes the multiplicative group of 2^n -th roots of unity. We make a distinction between $\Lambda[\mu_n]$, which is the group ring of μ_n over Λ , and $\Lambda(\mu_n)$, which is the integral closure of Λ in the field obtained from \mathbf{Q} by adjoining μ_n . There is a surjection $\Lambda[\mu_n] \rightarrow \Lambda(\mu_n)$ (cf. [18, 2.6]) which is not an isomorphism unless

$n = 0$. Because of this surjection, though, two modules over $\Lambda(\mu_n)$ are isomorphic as $\Lambda(\mu_n)$ -modules if and only if they are isomorphic as $\Lambda[\mu_n]$ -modules.

4.1 Remark. If M is a Λ -module and $\alpha : \mu_n \rightarrow \mathrm{Aut}(M)$ is a homomorphism, let M_α denote the $\Lambda[\mu_n]$ -module obtained by letting μ_n act on M via α . Given two homomorphisms $\alpha, \beta : \mu_n \rightarrow \mathrm{Aut}(M)$, it is clear that α is conjugate to β if and only if the $\Lambda[\mu_n]$ -modules M_α and M_β are isomorphic.

4.2 Lemma. *Suppose that \mathcal{I} is a nonzero ideal in $\Lambda(\mu_n)$. Then there are isomorphisms of $\Lambda(\mu_n)$ -modules*

$$\begin{aligned} \mathbf{R} \otimes_\Lambda \mathcal{I} &\cong \mathbf{R} \otimes_\Lambda \Lambda(\mu_n) \\ \mathbf{F}_3 \otimes_\Lambda \mathcal{I} &\cong \mathbf{F}_3 \otimes_\Lambda \Lambda(\mu_n) \end{aligned}$$

Proof. Since the quotient $\Lambda(\mu_n)/\mathcal{I}$ is a torsion group, the first isomorphism results from taking the inclusion $\mathcal{I} \rightarrow \Lambda(\mu_n)$ and tensoring with \mathbf{R} . For the second, note that by the Chebotarev density theorem there are an infinite number of prime ideals in $\Lambda(\mu_n)$ isomorphic (as modules) to \mathcal{I} . Up to isomorphism, then, \mathcal{I} can be taken to be a prime ideal of residue characteristic different from 3. Tensoring the inclusion $\mathcal{I} \rightarrow \Lambda(\mu_n)$ with \mathbf{F}_3 again gives the required isomorphism. \square

Recall that $\Lambda(\mu_n)$ is free as a module over Λ of rank $\phi(2^n) = 2^{n-1}$ [18, 2.5]. The same is true of any nonzero ideal in $\Lambda(\mu_n)$.

4.3 Lemma. *If $\Lambda(\mu_n)$ has a nonzero ideal which is not principal, then there exist two nonconjugate homomorphisms $\alpha, \beta : \mu_n \rightarrow G_{\phi(2^n)}$ which become conjugate both over \mathbf{R} and over \mathbf{F}_3 .*

Proof. Let \mathcal{I} be such a nonzero ideal. Choosing bases for \mathcal{I} and for $\Lambda(\mu_n)$ allows both to be identified (as Λ -modules) with $\Lambda^{\phi(2^n)}$. The multiplicative actions of μ_n on \mathcal{I} and on $\Lambda(\mu_n)$ thus give two homomorphisms $\alpha, \beta : \mu_n \rightarrow \mathrm{GL}_{\phi(2^n)}(\Lambda)$. These homomorphisms are not conjugate, because \mathcal{I} and $\Lambda(\mu_n)$ are not isomorphic as $\Lambda[\mu_n]$ -modules (equivalently, as $\Lambda(\mu_n)$ -modules). The homomorphisms are conjugate over \mathbf{R} or over \mathbf{F}_3 , because (4.2) the two modules become isomorphic when tensored with \mathbf{R} or with \mathbf{F}_3 . \square

Proof of 1.2. According to 4.3, it is enough to show that $\Lambda(\mu_6)$ is not a principal ideal domain. By [18, p. 353] the ideal class group of $\mathbf{Z}(\mu_6)$ is cyclic of order 17. Since the ideal class group of $\Lambda(\mu_6)$ is the quotient of the ideal class group of $\mathbf{Z}(\mu_6)$ by the subgroup generated by the prime ideals of $\mathbf{Z}(\mu_6)$ which lie above 2, it is enough to show that there is only one prime \mathcal{P} in $\mathbf{Z}(\mu_6)$ above 2, and that \mathcal{P} is principal. This is a standard calculation; see [18, 1.4] or [12, p. 73]. \square

REFERENCES

- [1] A.K. Bousfield and D.M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math. 304, Springer-Verlag, Berlin, 1972.
- [2] G. Carlsson, *Equivariant stable homotopy and Segal's Burnside ring conjecture*, Ann. of Math. **120** (1984), 189–224.
- [3] G. Carlsson, *Equivariant stable homotopy theory and the finite descent problem for unstable algebraic K-theories*, Amer. J. Math. **113** (1991), 963–973.

- [4] W. G. Dwyer and E. M. Friedlander, *Algebraic and étale K-theory*, Trans. Amer. Math. Soc. **272** (1985), 247–280.
- [5] W. G. Dwyer and E. M. Friedlander, *Conjectural calculations of general linear group homology*, Applications of Algebraic K-theory to Algebraic Geometry and Number Theory, Contemporary Mathematics Volume 55, Part I, Amer. Math. Soc., Providence, 1986, pp. 135–147.
- [6] W. G. Dwyer and E. M. Friedlander, *Topological models for arithmetic*, Topology **33** (1994), 1–24.
- [7] W. G. Dwyer and A. Zabrodsky, *Maps between classifying spaces*, Algebraic Topology, 1986 Barcelona, Lecture Notes in Math. 1298, Springer, Berlin, 1987, pp. 106–119.
- [8] E. M. Friedlander, *Étale Homotopy of Simplicial Schemes*, Ann. of Math. Studies 104, Princeton Univ. Press, Princeton, 1982.
- [9] H.-W. Henn, J. Lannes and L. Schwartz, *Localization of unstable A-modules and equivariant mod p cohomology*, Math. Ann. **301** (1995), 23–68.
- [10] H.-W. Henn, *The cohomology of $SL(3, \mathbf{Z}[1/2])$* , preprint (Universität Heidelberg) 1996.
- [11] H.-W. Henn, *Commutative algebra of unstable modules, Lannes' T-functor and equivariant cohomology*, preprint (Universität Heidelberg) 1996.
- [12] S. Lang, *Algebraic Number Theory*, Grad. Texts. in Math. 110, Springer, Berlin, 1986.
- [13] C.-N. Lee, *On the unstable homotopy type of the classifying space of a virtually torsion-free group*, preprint (Northwestern).
- [14] S. A. Mitchell, *On the plus construction for $BGLZ(1/2)$ at the prime 2*, Math. Zeit **209** (1992), 205–222.
- [15] D. G. Quillen, *The spectrum of an equivariant cohomology ring I, II*, Ann. of Math. **94** (1971), 549–602.
- [16] D. G. Quillen, *On the cohomology and K-theory of the general linear group over finite fields*, Annals of Math. **96** (1972), 552–586.
- [17] J.-P. Serre, *Cohomologie des groupes discrets*, Prospects in Mathematics, Ann. of Math. Studies 70, Princeton Univ. Press, Princeton, 1971, pp. 77–169.
- [18] L. Washington, *Introduction to Cyclotomic Fields*, Graduate Texts in Math. 83, Springer, Berlin, 1982.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556 USA

E-mail address: dwyer.1@nd.edu