EXOTIC CONVERGENCE OF THE EILENBERG-MOORE SPECTRAL SEQUENCE

BY

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Let $p: E \to B$ be a fibration over a connected space B with fiber F. The Eilenberg-Moore spectral sequence of p is a second quadrant spectral sequence which tries and sometimes fails to converge strongly to the homology of F (see [5]). The purpose of this paper is to determine what the spectral sequence *does* converge to. An abstract answer (Theorem 1.1) is that the spectral sequence almost always converges to the homology of the fiber of the nilpotent completion of the map p. A concrete answer (Theorem 2.1) is that under certain natural conditions on B and certain finiteness hypotheses the spectral sequence converges weakly to the homology of F with the $\pi_1(B)$ filtration.

As an aid to understanding these theorems, recall the similar behavior of the mod q Adams spectral sequence of a spectrum X. In absolute generality the spectral sequence converges only to the homotopy groups of some completion of X [1]. However, if X is connected and suitable finiteness conditions are satisfied, the spectral sequence converges to the actual homotopy groups of X with the "power of q" filtration. Note also that the spectral sequence converges *strongly* to the homotopy of X only in the rare case that each $\pi_i X$ is a q-group of finite exponent.

Sections 3 and 4 are devoted to applications of the convergence theorems in Section 1 and Section 2. In Section 3 we compute, in a certain sense, the homology of the universal cover of the nilpotent completion of a space X. In Section 4 we show that the cohomology of certain nilpotent groups is generated, in the sense of matric Massey products, by classes of degree one.

Throughout Section 1 and Section 2 we will work with the fixed fibration p described above. R will be a ring of the form $\mathbf{Z}/q\mathbf{Z}$ (q prime) or a subring of the rationals, and A will be a fixed R-module. We will freely use the ideas and conventions of [5]. In particular, we will associate to the fibration p a certain augmented cosimplicial space $F \to \mathbf{F}$, called the *Eilenberg-Moore object* of p. The mod A Eilenberg-Moore spectral sequence of p is understood to be the homotopy spectral sequence of the augmented tower of fibrations $A \otimes F \to \{Tot_s A \otimes \mathbf{F}\}_s$.

1. An abstract computation

Recall that for any space X, and any ring such as R, Bousfield and Kan ([2]) construct a functorial augmented tower of fibrations $X \to \{R_s X\}_s$. According to Dror ([4]), it is natural to think of this *tower* as the R-completion of X. For

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$s \geq 0$, let F_s be the ordinary fiber [2] of $R_s p\colon R_s E \to R_s B$. The spaces F_s form a tower $\{F_s\}_s$, which is called the fiber of the R-completion of the map p.

THEOREM 1.1. *If E and F are connected, the* mod A *Eilenberg-Moore spectral sequence of p converges to the* mod A *homology of the fiber of the R-completion of p.*

Remark. The proof requires only that E be connected and that p induce an epimorphism $H_1(E, R) \to H_1(B, R)$.

The meaning of this theorem is as follows. Let $\mathbf{F}_s$ be the Eilenberg-Moore object of $R_s(p)$. By naturality there are maps

$$\mathbf{F} \to \{\mathbf{F}_s\}_s \leftarrow \{F_s\}_s.$$

The middle object here is of course a tower of cosimplicial spaces. For each $i \geq 0$ this diagram gives rise to another pair of tower maps

$$\{\pi_i Tot_s A \otimes \mathbf{F}\}_s \to \{\pi_i Tot_s(A \otimes \mathbf{F}_s)\}_s \leftarrow \{\pi_i A \otimes F_s\}_s.$$

By definition the conclusion of (1.1) means that, for each $i \geq 0$, both of these maps are pro-isomorphisms. This is called convergence of the spectral sequence by analogy with [5: 3.3].

The proof of (1.1) consists in piecing together lemmas to show separately that each of the required arrows is a pro-isomorphism.

LEMMA 1.2. *For every $i \geq 0$ and $t \geq 0$, the map*

$$\pi_i(A \otimes F_t) \to \{\pi_i Tot_s(A \otimes \mathbf{F}_t)\}_s$$

induced by the augmentation $F_t \to \mathbf{F}_t$ is a pro-isomorphism.

Proof. The lemma claims that the mod A Eilenberg-Moore spectral sequence of $R_t p$ converges strongly to $H_*(F_t, A)$ (see [5]). By the ordinary convergence theorem, this is true iff $\pi_1 R_t B$ acts nilpotently on each $H_i(F_t, A)$. However, $R_t E$ and $R_t B$ are both nilpotent spaces, and by [2; I, 6.2] the fiber F_t of $R_t p\colon R_t E \to R_t B$ is connected, so the argument of [2; II, 5.4] gives the desired result.

LEMMA 1.3. *For each $i \geq 0$ the map of towers*

$$\{\pi_i(A \otimes F_s)\}_s \to \{\pi_i Tot_s(A \otimes \mathbf{F}_s)\}_s$$

is a pro-isomorphism.

Proof. This follows from (1.2) and the Diagonal Lemma below.

The other arrow of (1.1) requires two technical lemmas.

LEMMA 1.4. *Let $\mathbf{X} \to \{\mathbf{Y}_t\}_t$ be a map from the fibrant cosimplicial space $\mathbf{X}$* ([2]) *into the tower $\{\mathbf{Y}_t\}_t$ of fibrant cosimplicial spaces. Suppose that for each pair (i, s) of nonnegative integers, the induced map*

$$\pi_i \mathbf{X}^s \to \{\pi \mathbf{Y}_t^s\}_t$$

is a pro-isomorphism. (Here $\mathbf{X}^s$ *is the "codimension s" part of the cosimplicial space* $\mathbf{X}$, *and* $\mathbf{Y}_t^s$ *bears the same relation to* $\mathbf{Y}_t$.) *Then the induced maps*

$$\pi_i Tot_s \mathbf{X} \to \{\pi_i Tot_s \mathbf{Y}_t\}_t$$

are also pro-isomorphisms.

Proof. The proof is by induction on s, using [2; X, 6.3] and the five lemma for pro-isomorphisms.

For the statement of the next lemma, let $\prod^n (X)$ denote the n-fold cartesian product of the space X with itself.

LEMMA 1.5. *For any nonnegative integers i and n, the map*

$$H_i\left(\prod^n (E) \times B, A\right) \to \left\{H_i\left(\prod^n (R_s E) \times R_s B, A\right)\right\}_s,$$

induced by $E \to \{R_s E\}_s$ *and* $B \to \{R_s B\}_s$, *is a pro-isomorphism.*

Proof. This follows from [2; III, 6.5] and the finite product lemma [2; I, 7.2] The proof of (1.1) is completed by:

LEMMA 1.6. *For each* $i \geq 0$ *the natural map*

$$\{\pi_i Tot_s(A \otimes \mathbf{F})\}_s \to \{\pi_i Tot_s(A \otimes \mathbf{F}_s)\}_s$$

is a pro-isomorphism.

Proof. By (1.5) and [2; X, 4.9], Lemma 1.4 can be applied to the map

$$A \otimes \mathbf{F} \to \{A \otimes \mathbf{F}_s\}_s.$$

An application of the Diagonal Lemma finishes the proof.

LEMMA 1.7. (Diagonal Lemma). *Let* $\{G_{i,j}\}_{i,j} \to \{G'_{i,j}\}_{i,j}$ *be a map of double towers of abelian groups. Suppose that the induced tower map* $\{G_{i,j}\}_j \to \{G'_{i,j}\}_j$ *is a pro-isomorphism for each fixed* $i \geq 0$. *Then the diagonal tower map* $\{G_{i,i}\}_i \to \{G'_{i,i}\}_i$ *is also a pro-isomorphism.*

Proof. The lemma is easy to obtain from the fact that a map of towers is a pro-isomorphism iff both its kernal and cokernal are pro-trivial.

Remark. In both applications of the Diagonal Lemma above, the double tower $\{G_{i,j}\}_{i,j}$ is constant in the second variable.

2. A concrete computation

The purpose of this section is to show that, in some situations not covered by [5], the Eilenberg-Moore spectral sequence converges to a limit closely related to the homology of the fiber F. The main theorem is:

THEOREM 2.1. *Suppose that B is a nilpotent space. Then, under suitable* *finiteness hypotheses, the* mod A *Eilenberg-Moore spectral sequence of* p *converges to* $H_*(F, A)$ *with the* $\pi_1(B)$ *filtration.*

Remark. The connected space B is *nilpotent* ([2]) if
(a) $\pi_1(B)$ is a nilpotent group and
(b) the natural action of $\pi_1(B)$ on each of the higher homotopy groups of B is nilpotent.

Remark. Theorem 2.1 will be proved under the assumptions that
(2.2a) each $\pi_i(B)$ ($i \geq 1$) is a finitely generated group and
(2.2b) each $H_i(F, A)$ ($i \geq 0$) is a finitely generated module over the integral group ring of $\pi_1(B)$.

Somewhat weaker finiteness hypotheses will suffice. For instance, (2.2b) can be replaced by the assumption that

(2.2b)′ for each $i \geq 0$ there is a subring S of $\mathbf{Q}$ such that $H_i(F, A)$ is a finitely generated $S[\pi_i(B)]$-module.

The technical meaning of Theorem 2.1 is as follows. The family $\Phi = (\Phi_i^s)$ is called a *filtration of* $H_*(F, A)$ if, for each $i \geq 0$,

$$H_i(F, A) = \Phi_i^0 \supseteq \Phi_i^1 \supseteq \cdots \supseteq \Phi_i^s \supseteq \cdots$$

is a descending chain of subgroups inside $H_i(F, A)$. Let $\mathbf{Z}[\pi]$ be the integral group ring of $\pi = \pi_1(B)$, and let $I \subseteq \mathbf{Z}[\pi]$ be the augmentation ideal. Each group $H_i(F, A)$ is a natural $\mathbf{Z}[\pi]$-module, so a canonical filtration Φ of $H_*(F, A)$ can be defined by

$$\Phi_i^s = I^s \cdot H_i(F, A).$$

(Here I^s ($s > 0$) stands for the s'th power of I; I^0 is $\mathbf{Z}[\pi]$.) This is the $\pi_1(B)$ filtration referred to in (2.1). A given filtration Ψ of $H_*(F, A)$ is said to be *equivalent* to the $\pi_1(B)$ filtration if for every pair (s, i) of nonnegative integers there is a $k \geq 0$ such that

$$\Phi_i^{s+k} \subseteq \Psi_i^s \quad \text{and} \quad \Psi_i^{s+k} \subseteq \Phi_i^s.$$

Let $E_{i,j}^r$ denote the Eilenberg-Moore spectral sequence of p. Then, by definition, the conclusion of Theorem 2.1 means that

(a) for each pair (i, j) such that $i + j \geq 0$, $i \leq 0$, there is an N with the property that $E_{i,j}^N = E_{i,j}^\infty$ and
(b) there is some filtration Ψ of $H_*(F, A)$, equivalent to the $\pi_1(B)$ filtration, such that the augmentation of the spectral sequence induces natural isomorphisms

$$E_{-s,s+i}^\infty \approx \Psi_i^s/\Psi_i^{s+1}, \quad s \geq 0,\ i \geq 0.$$

Remark. The $\pi_1(B)$ filtration Φ is the most rapidly descending $\pi_1(B)$-equivariant filtration of $H_*(F, A)$ with the property that each quotient group Φ_i^s/Φ_i^{s+1} is a trivial $\pi_1(B)$-module.

Before proving Theorem 2.1 it is convenient to reformulate the convergence claim in the language of towers.

As before, let I be the augmentation ideal of the integral group ring of $\pi = \pi_1(B)$. For any π-module M and positive integer s, let Q_sM be the π-module $M/I^{s+1}M$. The obvious projections $Q_{s+1}M \to Q_sM$ provide structure maps that make $\{Q_sM\}_s$ into a tower of π-modules, and the quotient maps $M \to Q_sM$ furnish this tower with a natural augmentation by M.

Consider the π-equivariant Eilenberg-Moore augmentations

$$\pi_i(A \otimes F) \to \{\pi_i Tot_s A \otimes \mathbf{F}\}_s, \quad i \geq 0.$$

The argument of [5; Section 2] shows that $\pi_i Tot_s(A \otimes \mathbf{F})$ can be constructed from trivial π-modules by making no more than s extensions. (In other words, $\pi_i Tot_s A \otimes \mathbf{F}$ is a π-module of nilpotency class $\leq s + 1$.) This implies that the Eilenberg-Moore augmentations factor through natural tower maps

$$\{Q_s\pi_i(A \otimes F)\}_s \to \{\pi_i Tot_s A \otimes \mathbf{F}\}_s.$$

LEMMA 2.3. *The* mod A *Eilenberg-Moore spectral sequence of* p *converges to* $H_*(F, A)$ *with the* $\pi_1(B)$ *filtration iff each of the tower maps*

$$\{Q_s\pi_i(A \otimes F)\}_s \to \{\pi_i Tot_s A \otimes \mathbf{F}\}_s, \quad i \geq 0,$$

is a pro-isomorphism.

Proof. This is an algebraic consequence of the definitions.

The next lemma is the key to (2.1).

LEMMA 2.4. *Suppose that* B *is a nilpotent space which satisfies* (2.2a), *and that* M *is a finitely generated module over the integral group ring of* $\pi_1(B)$. Then *the augmentation* $M \to \{Q_sM\}_s$ *induces pro-isomorphisms*

$$H_i(B, M) \to \{H_i(B, Q_sM)\}_s$$

for each $i \geq 0$.

The homology groups of B which appear in the lemma are, of course, twisted homology groups. Lemma 2.4 will be proved below.

The proof of Theorem 2.1 now goes along the lines of the proof of the main result in [5]. Recall from [5; Section 4] that there is a tower of first-quadrant spectral sequences

$$\{E_s^2(i, j) = H_i(B, \pi_j Tot_s A \otimes \mathbf{F})\}_s$$

which is augmented by the Serre spectral sequence of p:

$$E^2(i, j) = H_i(B, \pi_j(A \otimes F)).$$

On the E^2-level this augmentation is induced by the maps

$$\pi_j(A \otimes F) \to \{\pi_j Tot_s A \otimes \mathbf{F}\}_s, \quad j \geq 0.$$

At "E^∞" this augmentation is covered by pro-isomorphisms in each dimension between the limit of the Serre spectral sequence and the limit of the tower.

Assume by induction that the tower map

$$\{Q_s\pi_j(A \otimes F)\}_s \to \{\pi_j Tot_s A \otimes \mathbf{F}\}_s$$

is a pro-isomorphism for each $j < n$. (This is certainly true for $n = 0$.) We must show that the map is also a pro-isomorphism in the case $j = n$. By the induction hypothesis, and Lemma 2.4, it is clear that the maps

$$E^2(i, j) \to \{E_s^2(i, j)\}_s$$

are pro-isomorphisms for $j < n$ and all $i \geq 0$. The comparison theorem [5; 4.3] implies that

$$E^2(0, n) \to \{E_s^2(0, n)\}_s$$

is a pro-isomorphism and

$$E^2(1, n) \to \{E_s^2(1, n)\}_s$$

is a pro-epimorphism. From this it is easy to deduce that

$$\{H_0(\pi, Q_s\pi_n(A \otimes F)\}_s \to \{H_0(\pi, \pi_n Tot_s A \otimes \mathbf{F})\}_s$$

is a pro-isomorphism, and

$$\{H_1(\pi, Q_s\pi_n(A \otimes F)\}_s \to \{H_1(\pi, \pi_n Tot_s A \otimes \mathbf{F})\}_s$$

is a pro-epimorphism, where $\pi = \pi_1(B)$. However, both

$$\{Q_s\pi_n(A \otimes F)\}_s \quad \text{and} \quad \{\pi_n Tot_s A \otimes \mathbf{F}\}_s$$

are towers of nilpotent π-modules, so by [5; 4.4] the map between these two towers is a pro-isomorphism. This completes the inductive step and concludes the proof of (2.1).

Proof of Lemma 2.4. Let I be the augmentation ideal of the integral group ring of $\pi = \pi_1(B)$, and let $M_s = I^{s+1}M$. The family $\{M_s\}_s$ forms a tower, which lies in a short exact sequence:

$$0 \to \{M_s\}_s \to M \to \{Q_s M\}_s \to 0.$$

To prove Lemma 2.4 it is enough to show that $\{H_i(B, M_s)\}_s$ is pro-trivial for each $i \geq 0$. Only two properties of the tower $\{M_s\}_s$ are relevant:

(2.5a) $\{H_0(\pi, M_s)\}_s$ is pro-trivial and
(2.5b) each M_s is a finitely generated $\mathbf{Z}[\pi]$ module.

Property (a) is easy, while (b) follows from the fact that $\mathbf{Z}[\pi]$ is left and right noetherian [9].

Remark 2.6. Properties (2.5a) and (2.5b) surprisingly imply that the towers $\{H_i(\pi, M_s)\}_s$ are pro-trivial for *all* $i \geq 0$ (see [6]).

Let $\tilde{B}$ be the universal cover of B. For each s there is a first quadrant Serre spectral sequence (cf. [11]):

$$E_s^2(i, j) = H_i(\pi, H_j(\tilde{B}, M_s)) \to H_{i+j}(B, M_s).$$

The action of π on $H_j(\tilde{B}, M_s)$ which figures here in E_s^2 is a diagonal action which is built from the topological action of π on $\tilde{B}$ and the algebraic action of π on M_s. These spectral sequences stack together into a tower $\{E_s^r(i, j)\}_s$ of spectral sequences, which converges to the graded tower $\{H_*(B, M_s)\}_s$.

In the light of the lemma below, the universal coefficient theorem, and Remark 2.6, it is not hard to see that all of the towers

$$\{H_j(\tilde{B}, M_s)\}_s, \quad j \geq 0,$$

have the strong acyclicity property of (2.6): $\{H_i(\pi, H_j(\tilde{B}, M_s))\}_s$ is pro-trivial for all $i \geq 0$. This is exactly the statement that the whole E^2-term of the above spectral sequence tower is pro-trivial. Evidently the limit of the spectral sequence tower is pro-trivial too. This completes the proof of (2.4).

For the duration of this final lemma, the respective symbols $\otimes$ and $*$ will denote tensor and torsion product over the integers. The tensor or torsion product of two π-modules is again a π-module in a natural diagonal way.

LEMMA 2.7. *Let π be a finitely generated nilpotent group, and let N be a finitely generated nilpotent π-module. Suppose that $\{M_s\}_s$ is a tower of π-modules which satisfies* (2.5a) *and* (2.5b). *Then both of the towers $\{M_s \otimes N\}_s$ and $\{M_s * N\}_s$ also satisfy* (2.5a) *and* (2.5b).

Remark. If π is as above and N is a nilpotent π-module, then N is finitely generated as a $\mathbf{Z}[\pi]$-module iff N is finitely generated as an abelian group.

Proof of 2.7. Suppose to begin with that N is a trivial π-module. Choose finitely generated free abelian groups F_1 and F_2 such that the sequence

$$0 \to F_1 \to F_2 \to N \to 0$$

is exact. Give F_1 and F_2 the trivial π-module structure, so that the above becomes a short exact sequence of π-modules. There is an induced exact sequence of towers of π-modules:

$$0 \to \{M_s * N\}_s \to \{M_s \otimes F_1\}_s \to \{M_s \otimes F_2\}_s \to \{M_s \otimes N\}_s \to 0.$$

Each of the middle towers is a direct sum of a finite number of copies of the original tower $\{M_s\}_s$; thus both of these middle towers satisfy (2.5a) and (2.5b).

The noetherian property of $\mathbf{Z}[\pi]$ implies that $\{M_s \otimes N\}_s$ and $\{M_s * N\}_s$ both satisfy (2.5b). Moreover, the surjectivity of the arrow $\{M_s \otimes F_2\}_s \to \{M_s \otimes N\}_s$ immediately gives that the tower $\{M_s \otimes N\}_s$ has property (2.5a). Thus all of the towers in this sequence have the property of Remark 2.6, except perhaps $\{M_s * N\}_s$. An easy long exact sequence argument shows that $\{M_s * N\}_s$ also has this strong acyclicity property and therefore, *a fortiori*, property (2.5a).

The proof of (2.7) can be finished by using similar arguments in an induction on the nilpotency class of N.

3. Applications: Homology of the universal cover of the completion

In this section X will be any connected space and R will be the ring of integers. The tower $\{R_sX\}_s$ is then the Bousfield-Kan integral completion tower of the space X. Let $\sim$ denote the universal covering space functor. We are interested in computing the towers $\{H_i(\widetilde{R_sX}, R)\}_s$. According to the conventions of Section 1, these towers give the homology of the universal cover of the completion of X.

PROPOSITION 3.1. *Suppose that $\pi = \pi_1(X)$ is a finitely generated group. Then, for all $i \geq 0$,*

$$\{H_i(\widetilde{R_sX}, R)\}_s \sim \{H_i(X, R[\pi]/I^s)\}_s.$$

Remark. Here $R[\pi]$ is the integral group ring of π with augmentation ideal $I = I[\pi]$. The symbol $\sim$ denotes the equivalence relation between towers of groups which is generated by pro-isomorphisms.

Remark. This is completely parallel to the classical formula $H_i(\tilde{X}, R) \approx H_i(X, R[\pi])$.

The first step in the proof of (3.1) is to determine the behavior of Eilenberg-Moore spectral sequences over certain nilpotent $K(G, 1)$'s.

LEMMA 3.2. *Suppose that ν is a finitely generated nilpotent group, and that $p: E \to K(\nu, 1)$ is a fibration. Let $\mathbf{F}$ be the Eilenberg-Moore object of p. Then, for all $i \geq 0$,*

$$\{\pi_i Tot_s R \otimes \mathbf{F}\}_s \sim \{H_i(E, R[\nu]/I^s[\nu])\}_s.$$

Remark. The homology groups in the right-hand tower are twisted by the map $\pi_1(E) \to \nu$ induced by p.

Proof. Let $\mathbf{P}$ be the Eilenberg-Moore object of the path fibration over $K(\nu, 1)$. There is a projection map from $\mathbf{F}$ into the constant cosimplicial space E which has (essentially) $\mathbf{P}$ as the inverse image of the basepoint. Using this map, and the techniques of [5; Section 5], it is possible to construct a tower of first quadrant spectral sequences

$$\{E_s^2(i, j) = H_i(E, \pi_j Tot_s R \otimes \mathbf{P})\}_s \to \{\pi_{i+j} Tot_s R \otimes \mathbf{F}\}_s.$$

By Theorem 2.1, the tower $\{\pi_j Tot_s R \otimes \mathbf{P}\}_s$ is pro-trivial if $j > 0$ and pro-isomorphic to $\{R[\nu]/I^s[\nu]\}_s$ if $j = 0$. This implies that the tower of spectral sequences collapses into the isomorphism claimed by the lemma.

To prove (3.1), let $\Gamma_t(\pi)$ $(t \geq 1)$ be the tth lower central series subgroup of π ([2]). Let $\nu_t = \pi/\Gamma_t$, and let $p_t \colon X \to K(\nu_t, 1)$ be the natural map. Note that ν_t is a finitely generated nilpotent group. Let $F_{s,t}$ be the fiber of the map

$$R_s E \to R_s(K(\nu_t, 1)).$$

The assembly $\{F_{s,t}\}_{s,t}$ is a double tower, and, according to the argument below,

$$\{H_i(\widetilde{R_s E}, R)\}_s \sim \{H_i(F_{s,s}, R)\}_s, \quad i \geq 0. \tag{3.3}$$

By (3.2) and (1.1),

$$\{H_i(F_{s,t}, R)\}_s \sim \{H_i(E, R[\nu_t]/I^s[\nu_t])\}_s, \quad i \geq 0.$$

Thus, by the Diagonal Lemma 1.7,

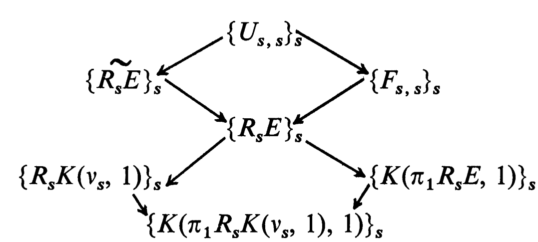
$$\{H_i(F_{s,s}, R)\}_s \sim \{H_i(E, R[\nu_s]/I^s[\nu_s])\}_s, \quad i \geq 0.$$

It is an easy algebraic exercise to show that the map $\pi \to \nu_s$ induces an isomorphism $R[\pi]/I^s[\pi] \to R[\nu_s]/I^s[\nu_s]$. This proves the proposition.

To prove (3.3), let $q_{s,t} \colon R_s E \to K(\pi_1 R_s K(\nu_t, 1), 1)$ be the composite of $p_{s,t}$ with the natural projection, and let

$$r_s \colon R_s E \to K(\pi_1 R_s E, 1)$$

be the natural map. Let $U_{s,t}$ be the fiber of $q_{s,t}$. There is a commutative diagram



Using the techniques of [2; IV: 2.4, 5.1] it is straightforward to show that both of the extreme lower arrows are weak pro-homotopy equivalences, i.e., induce pro-isomorphisms of all homotopy groups. From fibration long exact homotopy sequences it immediately follows that the two topmost arrows are also weak pro-homotopy equivalences. Equation (3.3) is a direct consequence.

4. Applications: The sufficiency of Massey products

Gugenheim and May have proven in [8; 5.17] that the algebraic cohomology of many connected Hopf algebras is generated, in the sense of matric Massey products, by classes of cohomological degree 1. Theorem 2.1 allows their technique to be applied to certain nonconnected algebras.

Suppose that R is a field of the form $\mathbf{Q}$ or $\mathbf{Z}/p\mathbf{Z}$, p prime.

Proposition 4.1. *Let π be a finitely generated nilpotent group. Then $H^*(\pi, R)$ is generated by $H^1(\pi, R)$ in the sense of matric Massey products.*

Remark. Note that $H^*(\pi, R)$ is the same as the algebraic cohomology ring of the discrete Hopf algebra $R[\pi]$. The conclusion of (4.1) fails if either of the two conditions on π is removed. To see this, let $R = \mathbf{Z}/p\mathbf{Z}$, and let π be either a large alternating group or the abelian group $\mathbf{Q}/\mathbf{Z}$.

To prove (4.1) it is necessary to check that the "Eilenberg-Moore" spectral sequence of [5] is correctly named.

Lemma 4.2. *The Eilenberg-Moore spectral sequence of* [5] *coincides with the cobar construction spectral sequence of* [7], *at least when coefficients are taken in R.*

Proof of 4.1. Consider the mod R Eilenberg-Moore spectral sequence of the path space fibration over $K(\pi, 1)$. By (2.1) this spectral sequence converges to a limit which is concentrated in degree zero. Therefore the cohomology bar construction spectral sequence of this fibration, which is dual to the cobar construction spectral sequence, also converges to a degree zero limit. The argument of [8] shows that this implies the conclusion of the proposition.

Proof of 4.2. Let $p: E \to B$ be a fibration with Eilenberg-Moore object $\mathbf{F}$. There are two distinct ways of exploiting $\mathbf{F}$ to get a homology spectral sequence: one of these is the method of [5], the other is the homology version of the method of Rector ([10]). By [3], both methods give exactly the same spectral sequence. Rector, however, shows that the end product of his technique is isomorphic to the (co-)bar construction spectral sequence.

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