THE FUNDAMENTAL GROUP OF A p-COMPACT GROUP

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1. Introduction

The notion of *p-compact group* [10] is a homotopy theoretic version of the geometric or analytic notion of *compact Lie group*, although the homotopy theory differs from the geometry is that there are parallel theories of p-compact groups, one for each prime number p. A key feature of the theory of compact Lie groups is the relationship between centers and fundamental groups; these play off against one another, at least in the semisimple case, in that the center of the simply connected form is the fundamental group of the adjoint form. There are explicit ways to compute the center or fundamental group of a compact Lie group in terms of the normalizer of the maximal torus [1, 5.47]. For some time there has in fact been a corresponding formula for the center of a p-compact group [11, 7.5], but in general the fundamental group has eluded analysis. The purpose of the present paper is to remedy this deficit.

For any space Y, we let $\mathrm{H}_i^{\mathbb{Z}_p}(Y)$ denotes $\lim_n \mathrm{H}_i(Y;\mathbb{Z}/p^n)$. Suppose that X is a connected p-compact group, with maximal torus T and torus normalizer $\mathrm{N}T$ [10, §8]. It is known that the map $\pi_1(T) \to \pi_1(X)$ is surjective [12, 6.11] [21, 5.6], or equivalently that the map $\mathrm{H}_2^{\mathbb{Z}_p}(\mathrm{B}T) \to \mathrm{H}_2^{\mathbb{Z}_p}(\mathrm{B}X)$ is surjective. We prove the following statement.

1.1. **Theorem.** *If X is a connected p-compact group, then the kernel of the map $\mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}T \to \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{BN}T$ is the same as the kernel of the map $\mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}T \to \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}X$. Equivalently, the image of the map $\mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}T \to \mathrm{H}_2^{\mathbb{Z}_p}(\mathrm{BN}T)$ is (naturally) isomorphic to $\pi_1 X$.*

1.2. *Remark.* There is a proof of the corresponding statement for compact Lie groups which relies on the Feshbach double coset formula

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(3.1). Our proof of 1.1 uses a transfer calculation (2.4) that in practice (3.4) amounts to a weak homological reflection of the double coset formula; we can get away with this because we have a splitting (3.5) of $\mathrm{H}_2^{\mathbb{Z}_p}(\mathrm{BN}T)$.

1.3. *Remark.* It is possible to derive from 1.1 a more explicit formula for $\pi_1 X$; this formula is known for p odd [4, 1.7]. Let W denote the Weyl group of X. If p is odd, then $\pi_1 X$ is naturally isomorphic to the module of coinvariants $\mathrm{H}_0(W; \mathrm{H}_2^{\mathbb{Z}_p}(\mathrm{B}T))$ (see 3.6). If $p = 2$, then up to factors which do not contribute to $\pi_1 X$, the normalizer of the torus in X is derived by $\mathbb{F}_2$-completion from the normalizer $\mathrm{N}T_G$ of a maximal torus T_G in a connected compact Lie group G [14, 9.13]. The image of the map $\mathrm{H}_2(\mathrm{B}T_G; \mathbb{Z}) \to \mathrm{H}_2(\mathrm{BN}T_G; \mathbb{Z})$ is isomorphic to $\pi_1 G$ (3.2), and so by 1.1 the tensor product of this image with $\mathbb{Z}_2$ is $\pi_1 X$. This image can be computed from the marked reflection lattice $(\pi_1 T_G, \{b_\sigma, \beta_\sigma\})$ corresponding to the root system of G [14, 2.17, 5.5] or, after tensoring with $\mathbb{Z}_2$, from the marked complete reflection lattice $(\pi_1 T, \{b_\sigma, \beta_\sigma\})$ associated to X [14, §6, §9]. The upshot is that $\pi_1 X$ is the quotient of $\pi_1 T = \pi_2\, \mathrm{B}T = \mathrm{H}_2^{\mathbb{Z}_2}\, \mathrm{B}T$ by the $\mathbb{Z}_2$–submodule generated by the elements $\{b_\sigma\}$.

Another way to describe this calculation is the following. For each reflection s_α in the Weyl group W, let u_α be a generator over $\mathbb{Z}_p$ of the rank 1 submodule of $\pi_1 T$ given by the image of $(1 - s_\alpha)$. If p is odd let $v_\alpha = u_\alpha$; if $p = 2$, let $v_\alpha = u_\alpha$ or $u_\alpha/2$, according to whether the marking of s_α is trivial or non-trivial. Then $\pi_1 X$ is the quotient of $\pi_1 T$ by the $\mathbb{Z}_p$-span of the elements v_α. See [3] for more details.

Organization of the paper. In §2 we describe a transfer map in cohomology and develop some of its properties; in §3 we use these properties to prove 1.1.

2. The transfer

In this section we set up the transfer machinery we need.

2.1. **Definition.** A fibration $f : E \to B$ with fibre F is *admissible* if B is connected, $\mathrm{H}_*(B; \mathbb{F}_p)$ is of finite type, $\mathrm{H}_*(F; \mathbb{F}_p)$ is finite dimensional, and $\pi_1 B$ is a finite p-group.

Given an admissible fibration $f : E \to B$, we construct a transfer map $f_\# : \mathrm{H}^*(E; \mathbb{F}_p) \to \mathrm{H}^*(B; \mathbb{F}_p)$. This is a map of modules over $H^*(B; \mathbb{F}_p)$ which has several key properties, enumerated in the statements below.

2.2. *Remark.* It seems virtually certain that the transfer map we construct can be derived by duality from the homological transfer of [7, 1.1]. However, we need a property (2.4) of the transfer that is not easy to check from the point of view of [7], and so for the sake of economy of exposition we develop an *ad hoc* transfer that serves our immediate purposes.

2.3. **Proposition.** *Suppose that*

$$\begin{array}{ccc} E' & \xrightarrow{h} & E \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ B' & \xrightarrow{g} & B \end{array}$$

is a homotopy fibre square in which the vertical maps are admissible fibrations. Then $g^* f_\# = f'_\# h^*$.

A *map* $(E' \xrightarrow{f'} B') \to (E \xrightarrow{f} B)$ between fibrations over B is a map $h : E' \to E$ such that $fh = f'$. In the case $B = \mathrm{B}\mathbb{Z}/p$, the next property will sometimes allow the transfer to be calculated by "localizing to the homotopy fixed point set".

2.4. **Proposition.** *Suppose that* h *is a map* $(E' \xrightarrow{f'} B) \to (E \xrightarrow{f} B)$ *between admissible fibrations over* B*, and that* S *is a multiplicative subset of* $\mathrm{H}^*(B;\mathbb{F}_p)$ *such that* $S^{-1}h^* \colon S^{-1}\,\mathrm{H}^*(E;\mathbb{F}_p) \to S^{-1}\,\mathrm{H}^*(E';\mathbb{F}_p)$ *is an isomorpism. Then the following diagram commutes:*

$$\begin{array}{ccc} S^{-1}\,\mathrm{H}^*(E;\mathbb{F}_p) & \xrightarrow{S^{-1}h^*} & S^{-1}\,\mathrm{H}^*(E';\mathbb{F}_p) \\ {\scriptstyle S^{-1}f_\#}\downarrow & & \downarrow{\scriptstyle S^{-1}f'_\#} \\ S^{-1}\,\mathrm{H}^*(B;\mathbb{F}_p) & \xrightarrow{=} & S^{-1}\,\mathrm{H}^*(B;\mathbb{F}_p)\,. \end{array}$$

Next, we want to calculate the transfer in a trivial product case. If Y is a space and $y \in Y$, let Y_y be the component of Y containing y, and $\chi(Y)$ the Euler characteristic of Y. For our purposes, the Euler characteristic is the alternating sum of the ranks of the $\mathbb{F}_p$-homology groups.

2.5. **Proposition.** *Suppose that the projection map* $f : Y \times B \to B$ *is an admissible fibration. If* $y \in Y$*, let* $i_y : B \to Y \times B$ *be given by* $i_y(b) = (y, b)$*. Then for any* $x \in \mathrm{H}^*(Y \times B;\mathbb{F}_p)$*,*

$$f_\#(x) = \sum_{\{y\}\in\pi_0 Y} \chi(Y_y)\, i_y^*(x)$$

2.6. **Proposition.** *Suppose that* $f : E \to B$ *is an admissible fibration with fibre* F*. Then the composite map* $f_\# f^* : \mathrm{H}^*(B;\mathbb{F}_p) \to \mathrm{H}^*(B;\mathbb{F}_p)$ *is multiplication by* $\chi(F)$*.*

2.7. *Stable context.* In order to construct the transfer, we work in the context of $\mathbb{S}$-algebras, sometimes known as structured ring spectra or A_∞ ring spectra, and commutative $\mathbb{S}$-algebras, sometimes known as E_∞ ring spectra. (The symbol $\mathbb{S}$ stands for the sphere spectrum.) See [15] or [18] for details and examples. If k is a commutative $\mathbb{S}$-algebra, we refer to algebra spectra over k as *k-algebras*; a reference to a k-algebra carries with it the implication that k is a commutative $\mathbb{S}$-algebra. The sphere $\mathbb{S}$ is itself a commutative $\mathbb{S}$-algebra, and, as the term "$\mathbb{S}$-algebra" suggests, any ring spectrum is an algebra spectrum over $\mathbb{S}$. We refer to a module spectrum over an $\mathbb{S}$-algebra R as simple a module over R. We write $\otimes_R$ for the smash product over R of two R-modules, and we always take this smash product in the appropriate derived sense. Similarly, we use Hom_R for the derived spectrum of maps between two R-modules.

Any ring R gives rise to an $\mathbb{S}$-algebra (whose homotopy is R, concentrated in degree 0). The category of modules over this $\mathbb{S}$-algebra is then equivalent in an appropriate homotopy theoretic sense to the cagtegory of chain complexes over R. If R is commutative in the usual sense then its corresponding $\mathbb{S}$-algebra is also commutative; the category of algebras over this $\mathbb{S}$-algebra is then closely related to the category of differential graded algebras over R [22].

For the rest of this section, k will denote the commutative $\mathbb{S}$-algebra corresponding to $\mathbb{F}_p$.

2.8. *Cochain algebras.* We will need the following observations from [19] and [20]. For any space X, the the mapping spectrum $k^X = \mathrm{Hom}_{\mathbb{S}}(\Sigma^\infty X_+, k)$ has the natural structure of a commutative k-algebra; this k-algebra is a geometric form of the $\mathbb{F}_p$-cochain algebra of X, and in particuar $\pi_i(k^X) = \mathrm{H}^{-i}(X;\mathbb{F}_p)$. Suppose that

$$\begin{array}{ccc} X & \longrightarrow & E \\ \downarrow & & \downarrow \\ Y & \longrightarrow & B \end{array}$$

is a homotopy fibre square in which B is connected, and let F be the homotopy fibre of $E \to B$. Then if B, F, and Y have $\mathbb{F}_p$-cohomology rings of finite type and B is simply connected (more generally, $\pi_1 B$ acts nilpotently on each $\mathbb{F}_p$-homology group of F) the natural map

$$k^Y \otimes_{k^B} k^E \to k^X$$

is a weak equivalence of commutative k-algebras.

Suppose that R is a commutative $\mathbb{S}$-algebra and that N is an R-module.

2.9. **Definition.** An R-module M is *finitely built* from N if it is in the smallest thick subcategory of R-modules which contains N and its (de)suspensions. An R-algebra A is *finite* if A is finitely built from R as an R-module.

More explicitly, M is finitely built from N if, up to equivalence and retracts, M can be built via cofibration sequences from finitely many copies of (de)suspensions of N.

Recall that k denotes the $\mathbb{S}$-algebra derived from $\mathbb{F}_p$.

2.10. **Lemma.** *If $E \to B$ is an admissible fibration with fibre F, then k^E is a finite k^B-algebra.*

Proof. If X is a space, let $k[X] = k \otimes_{\mathbb{S}} \Sigma^\infty(X_+)$; this is a geometric form of the $\mathbb{F}_p$-chains on X, in that $\pi_i k[X] = \mathrm{H}_i(X; \mathbb{F}_p)$. Let G be the loop group ΩB; then $k[G]$ is a k-algebra and monodromy in the fibration $E \to B$ makes $k[F]$ a module over $k[G]$. The Rothenberg–Steenrod construction produces an equivalence $k \otimes_{k[G]} k[F] \sim k[E]$. Dualizing gives

$$k^E \sim \mathrm{Hom}_k(k[E], k) \sim \mathrm{Hom}_k(k \otimes_{k[G]} k[F], k) \sim \mathrm{Hom}_{k[G]}(k[F], k)\,.$$

Now $k[F]$ has only a finite number of nonzero homotopy groups, each one of which is a finite dimensional $\mathbb{F}_p$-vector space which up to filtration is trivial as a module over $\pi_0 k[G] = \mathbb{F}_p[\pi_1 B]$. (This last follows from the fact that the augmentation ideal in $\mathbb{F}_p[\pi_1 G]$ is nilpotent.) As in [8, 3.2, 3.9], it follows that $k[F]$ is finitely built from the trivial module k as a module over $k[G]$. Applying the construction $U(M) = \mathrm{Hom}_{k[G]}(M, k)$, which preserves cofibration sequences in M, shows that $k^E \sim U(k[F])$ is finitely built from $k^B \sim U(k) \sim \mathrm{End}_{k[G]}(k)$ as a module over k^B.

The reader might be worried that there are two potential module structures above for k^E as a module over k^B, one coming from the natural k-algebra map $k^B \to k^E$, and the other from the expressions $k^E = \mathrm{Hom}_{k[G]}(k[F], k)$, $k^B = \mathrm{Hom}_{k[G]}(k, k)$. We have proved that in the second module structure, k^E is finitely built from k^B. However, the two module structures are the same. We indicate very briefly how to show this; for simplicity we assume that E is connected. Let P be the path fibration over B. There is a fibrewise action of G on P which by naturality gives a map $k^B \to \mathrm{End}_{k[G]}(k^P) \sim \mathrm{End}_{k[G]}(k)$. This is the equivalence mentioned above; note that it is evidently a map of

$\mathbb{S}$-algebras. If $L = \Omega E$, there is a similar equivalence $k^E \to \mathrm{End}_{k[L]}(k)$, as well as a commutative diagram

$$\begin{array}{ccc} k^B & \longrightarrow & \mathrm{End}_{k[G]}(k) \\ \downarrow & & \downarrow \\ k^E & \longrightarrow & \mathrm{End}_{k[L]}(k) \end{array}$$

in which the left vertical map is the natural one and the right vertical map is induced by $L \to G$. By a form of Shapiro's lemma, there are equivalences

$$k^E \sim \mathrm{End}_{k[L]}(k) \sim \mathrm{Hom}_{k[G]}(k[G/L], k) \sim \mathrm{Hom}_{k[G]}(k[F], k)$$

of modules over $\mathrm{End}_{k[G]}(k) \sim k^B$. □

If R is a commutative $\mathbb{S}$-algebra, and M is a R-module, write $D_R M$ for the R-module $\mathrm{Hom}_R(M, R)$. For any R-module N, composition gives a map

$$N \otimes_R D_R(M) \sim \mathrm{Hom}_R(R, N) \otimes_R \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R(M, N)$$

2.11. **Lemma.** *If M is finitely built from R, then for any R-module N the natural map $N \otimes_R D_R M \to \mathrm{Hom}_R(M, N)$ is an equivalence.*

Proof. The statement is clearly true if M is a suspension of R and follows in general by an induction on the number of cofibration sequences needed to construct M from R. The key point is that both of the constructions $N \otimes_R D_R M$ and $\mathrm{Hom}_R(M, N)$ preserve cofibration sequences in M. □

Suppose that M is an R-module which is finitely built from R. There is a map $\eta = \eta_M : R \to M \otimes_R D_R M$ corresponding to the identity map in $\mathrm{Hom}_R(M, M)$, as well as an evaluation map $\epsilon = \epsilon_M : M \otimes_R D_R M \to R$. Both of these are R-module maps.

2.12. **Definition.** Suppose that A is a finite R-algebra. The *transfer associated to* A is the R-module map $\mathrm{tr}_{A/R} : A \to R$ given by the following composite

$$A \sim A \otimes_R R \xrightarrow{\mathrm{id} \otimes \eta_A} A \otimes_R A \otimes_R D_R A \xrightarrow{\mathrm{mult} \otimes \mathrm{id}} A \otimes_R D_R A \xrightarrow{\epsilon_A} R.$$

2.13. *Remark.* If R is a commutative ring and A is an R-algebra which is finitely generated and free as an R-module, then $\mathrm{tr}_{A/R}$ assigns to each element $a \in A$ the trace over R of multiplication by a on A.

2.14. **Definition.** Suppose that $f : E \to B$ is an admissible fibration, so that k^E is a finite k^B-algebra (2.10). The *transfer map* $f_\# :$

$\mathrm{H}^*(E;\mathbb{F}_p) \to \mathrm{H}^*(B;\mathbb{F}_p)$ is defined to be the map on homotopy groups induced by

$$\mathrm{tr}_{k^E/k^B} \colon k^E \to k^B \, .$$

2.15. **Proposition.** *Suppose that A is a finite R-algebra, $R \to R'$ is a map of commutative $\mathbb{S}$-algebras, and $A' = R' \otimes_R A$. Then A' is a finite R'-algebra, and the following diagram commutes up to homotopy:*

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{tr}_{A/R}} & R \\ \downarrow & & \downarrow \\ A' & \xrightarrow{\mathrm{tr}_{A'/R'}} & R' \, . \end{array}$$

Proof. The finiteness condition is easy to check. The rest follows from the fact that chain of maps (2.12) which determines $\mathrm{tr}_{A'/R'}$ is obtained from the chain of maps which determines $\mathrm{tr}_{A/R}$ by applying the functor $- \otimes_R R'$. □

Proof of 2.3. This is a consequence of 2.15; since f is admissible, it follows from 2.8 that $k^{E'}$ is equivalent to $k^{B'} \otimes_{k^B} k^E$. □

Proof of 2.4. According to [16], for any commutative $\mathbb{S}$-algebra R and subset S of $\pi_* R$, there is a localized commutative $\mathbb{S}$-algebra $S^{-1}R$ together with a map $R \to S^{-1}R$ inducing an isomorphism $S^{-1}\pi_* R \cong \pi_*(S^{-1}R)$. Moreover, if M is an R-module and $S^{-1}M$ is defined as $(S^{-1}R) \otimes_R M$, then $\pi_*(S^{-1}M) \cong S^{-1}\pi_* M$. The hypotheses imply that $S^{-1}k^E \to S^{-1}k^{E'}$ is an equivalence, and so the proposition can be proved by two applications of 2.15. (The first application, for instance, identifies $S^{-1}f_\#$ with the transfer associated to the $S^{-1}(k^B)$-algebra given by $S^{-1}(k^E)$.) □

Proof of 2.5. We leave it to the reader to check this in the case in which B is a point; it amounts to calcuating a composite

$$\begin{aligned} \mathrm{H}^*(Y;\mathbb{F}_p) &\to \mathrm{H}^*(Y;\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{H}^*(Y;\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{H}_*(Y;\mathbb{F}_p) \\ &\to \mathrm{H}^*(Y;\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{H}_*(Y;\mathbb{F}_p) \to \mathbb{F}_p \end{aligned}$$

in which the only nonzero component is in degree 0. Since $f_\#$ is a map of modules over $\mathrm{H}^*(B;\mathbb{F}_p)$, the general case can be obtained by applying 2.3 to the fibre square

$$\begin{array}{ccc} Y \times B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B & \longrightarrow & * \end{array} .$$

Proof of 2.6. Since $f_\# f^*$ is a map of modules over $\mathrm{H}^*(B;\mathbb{F}_p)$, it is enough to calculate its effect in degree 0. This can be done by applying 2.5 to the fibre square

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \\ * & \longrightarrow & B \end{array} .$$

3. Using the transfer

We first sketch the proof of a result parallel to 1.1 for compact Lie groups. We will have to make a few adjustments to the Lie group argument in order to compensate for missing components of the homotopy theoretic machinery, but this proof is the prototype for our approach.

3.1. **Theorem.** *Suppose that G is a connected compact Lie group with maximal torus T and torus normalizer* $\mathrm{N}T$. *Then the kernel of the map* $\mathrm{H}_*(\mathrm{B}T;\mathbb{Z}) \to \mathrm{H}_*(\mathrm{B}G;\mathbb{Z})$ *is the same as the kernel of the map* $\mathrm{H}_*(\mathrm{B}T;\mathbb{Z}) \to \mathrm{H}_*(\mathrm{BN}T;\mathbb{Z})$.

3.2. *Remark.* Since $\mathrm{H}_2(\mathrm{B}T;\mathbb{Z}) \to \mathrm{H}_2(\mathrm{B}G;\mathbb{Z})$ is surjective [13, 9.3], this theorem calculates $\pi_1 G = \mathrm{H}_2(\mathrm{B}G;\mathbb{Z})$ in terms of the image of the map $\mathrm{H}_2(\mathrm{B}T;\mathbb{Z}) \to \mathrm{H}_2(\mathrm{BN}T;\mathbb{Z})$.

3.3. *Proof of 3.1 (sketch).* If $f : E \to B$ is a fibre bundle with a compact manifold as fibre, we let $f_! : \mathrm{H}_*(B;\mathbb{Z}) \to \mathrm{H}_*(E;\mathbb{Z})$ denote the associated Becker-Gottlieb-Dold transfer map [5] [6] (this is very much the same kind of map as the $\mathbb{F}_p$–cohomology transfer discussed in §2). Consider the following commutative diagram of vertical fibre sequences. Here the space V is defined so that the lower right square is a homotopy fibre square; equivalently, V is the Borel construction of the left action of T on $G/\mathrm{N}T$.

$$\begin{array}{ccccc} (G/\mathrm{N}T)^T & \longrightarrow & G/\mathrm{N}T & \xrightarrow{=} & G/\mathrm{N}T \\ \downarrow & & \downarrow & & \downarrow \\ (G/\mathrm{N}T)^T \times \mathrm{B}T & \xrightarrow{a} & V & \xrightarrow{b} & \mathrm{BN}T \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle w}\downarrow \\ \mathrm{B}T & \xrightarrow{=} & \mathrm{B}T & \xrightarrow{c} & \mathrm{B}G \end{array}$$

By a basic property (cf. 2.3) of the transfer, $b_* v_! = w_! c_*$. The space $G/\mathrm{N}T$ has Euler characteristic one, and so $w_* w_!$ is the identity map of $\mathrm{H}_*(\mathrm{B}G;\mathbb{Z})$ (cf. 2.6); in particular $w_!$ is injective, and so we conclude that the kernel of c_* is the same as the kernel of $b_* v_!$.

For any closed subgroup $K \subset T$, let $p(K)$ be the projection map in the fibration

$$T/K \to \mathrm{B}K \xrightarrow{p(K)} \mathrm{B}T\,.$$

Since T is abelian, the connected compact Lie group T/K acts freely on $\mathrm{B}K$; in particular, if $K \neq T$ the fibre bundle $\mathrm{B}K \to \mathrm{B}T$ admits a fibrewise self–map which is fixed–point free but fibrewise homotopic to the identity. It follows from a theorem of Dold [6] that if $K \neq T$ the transfer map $p(K)_!$ is trivial. By the Feshbach formula [17], then, $b_*v_! = b_*a_*u_!$. But the action of T on $G/\mathrm{N}T$ has only a single fixed point; this follows from the fact that any element of G which conjugates T into $\mathrm{N}T$ must conjugate T into the identity component of $\mathrm{N}T$, i.e., into T itself, and hence must lie in $\mathrm{N}T$. It is now clear that the composite $b_*a_*u_!$ is just the homology homomorphism i_* induced by the usual map $i\colon \mathrm{B}T \to \mathrm{BN}T$, and so $\ker(i_*) = \ker(b_*a_*u_!) = \ker(b_*v_!)$ and as above this last is the same as $\ker(c_*)$. □

For the rest of this section, X by default denotes a connected p-compact group with maximal torus T, torus normalizer $\mathrm{N}T$, and Weyl group W. In addition, N_pT denote the p-compact group whose classifying space is the cover of $\mathrm{BN}T$ corresponding to a Sylow p-subgroup W_p of W [10, 9.8].

3.4. *Remark.* There are two technical issues which make our proof of 1.1 slightly more complicated than the sketch above. First of all, fibrations with fibre $X/\mathrm{N}T$ are awkward to handle, because $\mathrm{N}T$ is not a p-compact group. Instead we work with X/N_pT, a change which introduces some additional bookkeeping. Secondly, in the p-compact case we do not have a Feshbach transfer formula for the fibration over $\mathrm{B}T$ with X/N_pT as the fibre, much less a Dold–type result which would focus the transfer calculation on the (homotopy) fixed points of the action of T on X/N_pT. To compensate, we pull the fibration back over various maps $\mathrm{B}\mathbb{Z}/p \subset \mathrm{B}T$ and use 2.4, which in light of Smith theory [9] does in fact allow the transfer to be calculated by restricting to $(X/\mathrm{N}_pT)^{\mathrm{h}\mathbb{Z}/p}$. This requires dealing with $\mathbb{F}_p$-(co)homology instead of with $\mathrm{H}_*^{\mathbb{Z}_p}$, but there is an *a priori* splitting theorem (3.5) which permits this.

We first handle the splitting.

3.5. **Lemma.** *The image of the map* $\mathrm{H}_2^{\mathbb{Z}_p}\mathrm{B}T \to \mathrm{H}_2^{\mathbb{Z}_p}\mathrm{BN}T$ *is a split summand of* $\mathrm{H}_2^{\mathbb{Z}_p}\mathrm{BN}T$.

3.6. *Proof.* Let W be the Weyl group of X. For p odd, the lemma is a consequence of the fact that the fibration

$$\mathrm{B}T \to \mathrm{BN}T \to \mathrm{B}W$$

has a section [2]; the Serre spectral sequence then gives isomorphisms

$$\begin{aligned}\mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{BN}T &\cong \mathrm{H}_2^{\mathbb{Z}_p}(\mathrm{B}W) \oplus \mathrm{im}(\mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}T \to \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{BN}T)\\ &\cong \mathrm{H}_2^{\mathbb{Z}_p}(\mathrm{B}W) \oplus \mathrm{H}_0(W; \mathrm{H}_2^{\mathbb{Z}_p}(\mathrm{B}T))\,.\end{aligned}$$

For $p = 2$, note that by [14, 7.4] there are are splittings $T \cong T_1 \times T_2$, $\mathrm{N}T \cong \mathrm{N}T_1 \times \mathrm{N}T_2$, $W \cong W_1 \times W_2$, where $\mathrm{N}T_1$ is obtained from the normalizer of the maximal torus in a connected compact Lie group T by $\mathbb{F}_2$-completion of the maximal torus [14, 8.1, 9.1], and NT_2 is the product of a number of copies of the normalizer of the torus in the exotic 2-compact group $DI(4)$. As is clear from [14, 7.2], the group $\mathrm{H}_0(W_2; \mathrm{H}_2^{\mathbb{Z}_2}\,\mathrm{B}T_2)$ vanishes, and so the image of the map $\mathrm{H}_2^{\mathbb{Z}_2}\,\mathrm{B}T_2 \to \mathrm{H}_2^{\mathbb{Z}_2}\,\mathrm{BN}T_2$ is trivial. It thus suffices to consider the image of the map $\mathrm{H}_2^{\mathbb{Z}_2}\,\mathrm{B}T_1 \to \mathrm{H}_2^{\mathbb{Z}_2}\,\mathrm{BN}T_1$, or, equivalently, to show that if G is a connected compact Lie group with maximal torus T_G and torus normalizer $\mathrm{N}T_G$, the image of the map $\mathrm{H}_2(\mathrm{B}T_G; \mathbb{Z}_2) \to \mathrm{H}_2(\mathrm{BN}T_G; \mathbb{Z}_2)$ is a summand of $\mathrm{H}_2(\mathrm{BN}T_G; \mathbb{Z}_2)$. This follows from 3.3.

Recall that a monomorphism $Y \to X$ of p-compact groups [10, 3.2] is said to be of *maximal rank* [11, 4.1] if for some (equivalently, any) maximal torus T of Y the composite $T \to Y \to X$ is a maximal torus for X.

3.7. **Lemma.** *If $f : Y \to X$ is a monomorphism of p-compact groups, then f has maximal rank if and only if $\chi(X/Y) \neq 0$.*

Proof. Let T be a maximal torus for Y. There is a fibration sequence $Y/T \to X/T \to X/Y$ which by [11, 10.6] gives a product formula $\chi(X/T) = \chi(Y/T)\chi(X/Y)$ (we leave it to the reader to check the nilpotent action condition). The lemma follows from the fact that a maximal torus T of a p-compact group Z is characterized by the fact that $\chi(Z/T) \neq 0$ [11, 2.15]. □

The following proposition is the 1-connected case of 1.1.

3.8. **Proposition.** *If X is 1-connected, the natural map $\mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}T \to \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{BN}T$ is zero.*

Proof. By 3.5 it is enough to show that $\mathrm{H}_2(\mathrm{B}T; \mathbb{F}_p) \to \mathrm{H}_2(\mathrm{BN}T; \mathbb{F}_p)$ is zero. Since $\mathrm{H}^2(\mathrm{B}T; \mathbb{F}_p)$ is detected on maps $\mathrm{B}\mathbb{Z}/p \to \mathrm{B}T$, it is even enough to show that for any map $\mathrm{B}\mathbb{Z}/p \to \mathrm{B}T$, the induced composite $\mathrm{H}^2(\mathrm{BN}T; \mathbb{F}_p) \to \mathrm{H}^2(\mathrm{B}T; \mathbb{F}_p) \to \mathrm{H}^2(\mathrm{B}\mathbb{Z}/p; \mathbb{F}_p)$ is trivial.

Pick such a map $\mathrm{B}\mathbb{Z}/p \to \mathrm{B}T$, and consider the following commutative diagram of vertical fibration sequences. The space V is defined so that lower square is a fibre square; equivalently, V is the Borel construction of the action of T on X/N_pT. The map c is the composite $\mathrm{B}\mathbb{Z}/p \to \mathrm{B}T \to \mathrm{B}X$. The space $(X/\mathrm{N}_pT)^{\mathrm{h}\mathbb{Z}/p}$ is a homotopy fixed point set [10, §10] which in this case amounts to the space of sections of the fibration $V \to \mathrm{B}\mathbb{Z}/p$.

$$\begin{array}{ccccc}
(X/\mathrm{N}_pT)^{\mathrm{h}\mathbb{Z}/p} & \longrightarrow & X/\mathrm{N}_pT & \longrightarrow & X/\mathrm{N}_pT \\
\downarrow & & \downarrow & & \downarrow \\
(X/\mathrm{N}_pT)^{\mathrm{h}\mathbb{Z}/p} \times \mathrm{B}\mathbb{Z}/p & \xrightarrow{a} & V & \xrightarrow{b} & \mathrm{BN}_pT \\
{\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle w}\downarrow \\
\mathrm{B}\mathbb{Z}/p & \xrightarrow{=} & \mathrm{B}\mathbb{Z}/p & \xrightarrow{c} & \mathrm{B}X
\end{array}$$

If f is a map of spaces, let f^* be the induced map on $\mathbb{F}_p$-cohomology. By 2.3, $v_\# b^* = c^* w_\#$; since X is 1-connected, $v_\# b^*$ is trivial in dimension 2.

Let $S \subset \mathrm{H}^*(\mathrm{B}\mathbb{Z}/p; \mathbb{F}_p)$ be the multiplicative subset generated by a nonzero class in dimension 2. By Smith theory [10, 4.11, 5.7], $S^{-1}a^*$ is an isomorphism, and so by 2.4, $S^{-1}v_\# = S^{-1}u_\# a^*$. Since $H^*(\mathrm{B}\mathbb{Z}/p; \mathbb{F}_p) \to S^{-1}\,\mathrm{H}^*(\mathrm{B}\mathbb{Z}/p; \mathbb{F}_p)$ is a monomorphism, it even follows that $v_\# = u_\# a^*$. In particular, the map $u_\# a^* b^*$ vanishes in dimension 2. Let j be the map $\mathrm{BN}_pT \to \mathrm{BN}T$, and i the composite $\mathrm{B}\mathbb{Z}/p \xrightarrow{c} \mathrm{B}T \to \mathrm{BN}T$. To finish the proof it is enough to show that $u_\# a^* b^* j^*$ is a nonzero multiple of i^*, since it will then follow, as desired, that i^* vanishes in dimension 2.

Let $Y = (X/\mathrm{N}_pT)^{\mathrm{h}\mathbb{Z}/p}$. By [10, 4.6] the Euler characteristic of Y is congruent mod p to $\chi(X/\mathrm{N}_pT)$; this latter is nonzero mod p [10, 9.9 ff] and in fact is equal to the cardinality of W/W_p. There is a fibration

$$Y \to \mathrm{Map}(\mathrm{B}\mathbb{Z}/p, \mathrm{BN}_pT)_{\langle c\rangle} \to \mathrm{Map}(\mathrm{B}\mathbb{Z}/p, \mathrm{B}X)_c$$

where $\mathrm{Map}(\mathrm{B}\mathbb{Z}/p, \mathrm{B}X)_c$ is the component of maps homotopic to c and $\mathrm{Map}(\mathrm{B}\mathbb{Z}/p, \mathrm{BN}T)_{\langle c\rangle}$ consists of maps which cover c up to homotopy. If we interpret c as a homomorphism $\mathbb{Z}/p \to X$ then Y can be expressed as a disjoint union

$$Y = \coprod_{\{\gamma\}} \mathcal{C}_X(c)/\mathcal{C}_{\mathrm{N}_pT}(\gamma)\,,$$

where $\{\gamma\}$ runs over conjugacy classes of homomorphisms $\mathbb{Z}/p \to \mathrm{N}_pT$ which cover c up to conjugacy, and $\mathcal{C}_T(-)$ denotes the centralizer in T of the image of the indicated map [10, §5]. For each component $\{\gamma\}$ of Y, let $f_\gamma : \mathrm{B}\mathbb{Z}/p \to \mathrm{BN}_pT$ be the map obtained by using any point in that component to map $\mathrm{B}\mathbb{Z}/p$ to $Y \times \mathrm{B}\mathbb{Z}/p$, and then following through with the composite ba. According to 2.5

$$u_{\#}a^*b^* = \sum_{\{\gamma\}\in\pi_0 Y} \chi(Y_\gamma) f_\gamma^* \, . \tag{3.9}$$

There are two types of such homomorphisms γ. The first type consists of those γ with the property that $\gamma(\mathbb{Z}/p)$ is *not* contained in the torus T in N_pT. In this case $\mathcal{C}_{\mathrm{N}_pT}(\gamma)$ is not a maximal rank subgroup of $\mathcal{C}_X(c)$, and so by 3.7 the Euler characteristic of the homogeneous space $\mathcal{C}_X(c)/\mathcal{C}_{\mathrm{N}_pT}(\gamma)$ is zero. Here we are using the idea of [11, 3.2] to compute $\mathcal{C}_{\mathrm{N}_pT}(\gamma)$ and combining this with the observation that the action of $\mathrm{N}_pT/T = W_p$ on (the discrete approximation to) T is faithful [11, 2.12]. The components of Y corresponding to such maps γ contribute nothing to the sum on the right hand side of 3.9.

Consider then maps $\gamma : \mathbb{Z}/p \to \mathrm{N}_pT$ which lift c and whose image lies in T. For any such γ, the composite $j \cdot f_\gamma$ is homotopic to i; this is expressed in [12, 3.4] as the statement that any two homomorphisms $\mathbb{Z}/p \to T$ which are conjugate in X are conjugate in $\mathrm{N}T$. Write $\pi_0 Y = A \cup B$, where A is indexed by lifts of the first kind, and B by lifts of the second kind. Since $\chi(Y_\gamma) = 0$ if $\{\gamma\} \in A$, it is clear that $\sum_{\{\gamma\}\in B} \chi(Y_\gamma) = \chi(Y)$ is nonzero mod p. On the other hand, the argument above gives

$$u_{\#}a^*b^*j^* = \sum_{\{\gamma\}\in B} \chi(Y_\gamma) i^*$$

It follows that $u_{\#}a^*b^*j^*$ is a nonzero multiple of i^*, as required. □

Note that any connected covering space of a connected p-compact group is again a p-compact group [21, 3.3].

3.10. **Lemma.** *Let X be a connected p-compact group with maximal torus T, and X' the universal cover of X, with maximal torus T'. Then the cokernel of $\pi_1 T' \to \pi_1 T$* [10, 8.11] *is isomorphic to the cokernel of $\pi_1 X' \to \pi_1 X$.*

Proof. Let T'' be the identity component of the center of X. According to [21, 5.4] there is a short exact sequence

$$K \to X' \times T'' \to X$$

is which K is a finite abelian subgroup of the center of X'. This gives rise to a parallel exact sequence

$$K \to T' \times T'' \to T \, .$$

Combining the two leads to the conclusion that the map $X'/T' \to X/T$ is an equivalence. The proof is completed by taking the parallel fibration sequences

$$\begin{array}{ccccc} X'/T' & \longrightarrow & \mathrm{B}T' & \longrightarrow & \mathrm{B}X' \\ \sim\downarrow & & \downarrow & & \downarrow \\ X/T & \longrightarrow & \mathrm{B}T & \longrightarrow & \mathrm{B}X \end{array}$$

and comparing the associated long exact homotopy sequences. □

Proof of 1.1. Let X' be the universal cover of X, with maximal torus T' and torus normalizer $\mathrm{N}T'$. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}T' & \longrightarrow & \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{BN}T' & \longrightarrow & \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}X'\ (=0) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}T & \longrightarrow & \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{BN}T & \longrightarrow & \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}X \end{array}$$

The homology groups in the outside columns are isomorphic to the corresponding π_2's. By 3.8, the image of the map $\mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}T' \to \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{BN}T'$ is zero. To prove the theorem, it is enough to show that if there is an $x \in \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}T$ which does *not* map to 0 in $\mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{BN}T$ but does map to 0 in $\mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}X$, then there exists an $x' \in \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}T'$ which does not map to 0 in $\mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{BN}T'$. But by 3.10 the cokernel of $\pi_2\,\mathrm{B}T' \to \pi_2\,\mathrm{B}T$ is isomorphic to the cokernel of $\pi_2\,\mathrm{B}X' \to \pi_2\,\mathrm{B}X$, i.e., to $\pi_2\,\mathrm{B}X$, so that any such $x \in \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}T$ automatically lifts to an $x' \in \mathrm{H}_2^{\mathbb{Z}_p}\,\mathrm{B}T'$. By the commutativity of the diagram, x' has the desired (impossible) property. □

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