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Homotopy theory and simplicial groupoids*

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§1. INTRODUCTION

1.1. SUMMARY. The homotopy theory of simplicial groups is well known [2, Ch. VI] to be equivalent to the pointed homotopy theory of reduced (i.e. only one vertex) simplicial sets (by means of a pair of adjoint functors G and $\bar{W}$).

The aim of this note is to show that similarly, *the homotopy theory of simplicial groupoids is equivalent to the* (*unpointed*) *homotopy theory of* (*all*) *simplicial sets.* This we do by

(i) showing that *the category of simplicial groupoids admits a closed model catagory structure* in the sense of Quillen [3], and

(ii) extending the functors G and $\bar{W}$ to pair of *adjoint functors*

$$G : (\textit{simplicial sets}) \leftrightarrow (\textit{simplicial groupoids}) : \bar{W}$$

which induce the desired equivalence of homotopy theories.

We also show that *the category of simplicial groupoids admits a simplicial structure* which produces "function complexes" and "simplicial monoids of self homotopy equivalences" of the correct homotopy types.

1.2. NOTATION AND TERMINOLOGY. We will freely use the results of [2] and [3]. Furthermore:

(i) We recall that a *groupoid* is a small category in which all maps are invertible.

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(ii) A *simplicial category* (and in particular a *simplicial groupoid*) will always be assumed to have the *same* objects in each dimension.

(iii) For any two objects A and B in a (simplicial) category X, $X(A,B)$ will denote the (simplicial) set of maps $A \to B \in X$.

(iv) We denote by $\boldsymbol{S}$ the category of simplicial sets, by $\boldsymbol{G}$ the category of simplicial groups, and by $\boldsymbol{Gd}$ the category of simplicial groupoids.

§2. A HOMOTOPY THEORY FOR SIMPLICIAL GROUPOIDS

In this section we turn the category $\boldsymbol{Gd}$ of simplicial groupoids (1.2) into a *closed model category* in the sense of Quillen, i.e. [4, p. 233] we define notions of *weak equivalence, fibration* and *cofibration* such that the following five axioms are satisfied:

CM1. *The category is closed under finite direct and inverse limits.*

CM2. *If f and g are maps such that gf is defined and if two of f, g and gf are weak equivalences, then so is the third.*

CM3. *If f is a retract of g* (i.e. if there are, in the category of maps, maps $a: f \to g$ and $b: g \to f$ such that $ba = id_f$) *and g is a weak equivalence, a fibration or a cofibration, then so is f.*

CM4. *Given a solid arrow diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

in which i is a cofibration, p is a fibration and either p or i is a weak equivalence, then the dotted arrow exists.

CM5. *Any map f can be factored in two ways:*

(i) *$f = pi$, where i is a cofibration and p is a trivial fibration* (i.e. a fibration as well as a weak equivalence), *and*

(ii) *$f = pi$, where p is a fibration and i is a trivial cofibration* (i.e. a cofibration as well as a weak equivalence).

2.1. WEAK EQUIVALENCES IN $\boldsymbol{Gd}$. A map $f: X \to Y \in \boldsymbol{Gd}$ is a *weak equivalence* if

(i) f induces a 1-1 correspondence between the components of X and those of Y (two objects A, $B \in X$ are in the same component iff there exists a map $A \to B \in X$), and

(ii) for every object $A \in X$, the induced map $X(A,A) \to Y(fA,fA)$ is a weak equivalence (in $\boldsymbol{S}$ or, equivalently, in $\boldsymbol{G}$).

2.2. FIBRATIONS IN $\boldsymbol{Gd}$. A map $f: X \to Y \in \boldsymbol{Gd}$ is a *fibration* if

(i) for every object $A \in X$ and map $b: fA \to B \in Y_0$, there is a map $a: A \to A' \in X_0$ such that $fa = b$, and

(ii) for every object $A \in X$, the induced map $X(A,A) \to Y(fA,fA)$ is a fibration (in S or, equivalently, in G).

Clearly *every object $X \in Gd$ is fibrant* (i.e. the map $X \to * \in Gd$ onto the terminal object is a fibration).

2.3. FREE OBJECTS AND MAPS IN Gd. A map $f: X \to Y \in Gd$ is *free* if

(i) f is 1-1 on objects and maps, and

(ii) there is a set $\Gamma \subset Y$ of non-identity maps (called a *basis*), which is closed under the degeneracy operators and which has the property that every non-identity map of Y can uniquely be written as a *reduced* composition of maps in Γ, their inverses and non-identity maps in the image of f. Here *reduced* means that no map of Γ appears next to its inverse and that no two non-identity maps in the image of f appear next to each other.

Similarly, an object $X \in Gd$ is *free* if the map $\phi \to X \in Gd$ from the initial object is free.

2.4. COFIBRATIONS IN Gd. A map $f: X \to Y \in Gd$ is a *cofibration* if it is a retract of a free map.

This definition readily implies that *the cofibrant objects of Gd are the free ones* (just as in G).

Then we finally have

2.5. THEOREM. *The category Gd with weak equivalences, fibrations and cofibrations defined as above, is a closed model category. Moreover, the factorizations* CM5 *can be done functorially.*

In preparation of the proof of this theoerm we first list several results which are not difficult to verify.

2.6. PROPOSITION. *Let $X \in Gd$ be connected* (i.e. only one component), *let $A \in X$ be an object and let $X_A \subset X$ denote the simplicial subgroupoid consisting of A and all maps $A \to A \in X$. Then*

(i) *the inclusion $X_A \to X$ is a free map as well as a weak equivalence, and*

(ii) *any retraction $X \to X_A$ is a trivial fibration.*

2.7. PROPOSITION. *Let $f: X \to Y \in Gd$ be a trivial cofibration. Then so is, for every object $A \in X$, the induced map $X(A,A) \to Y(fA,fA) \in G$.*

2.8. THE FUNCTOR $F: S \to Gd$. For every object $K \in S$ we denote by FK the simplicial groupoid with

(i) two objects 0 and 1

(ii) in each dimension $n \geq 0$, the free groupoid which has one generator $0 \to 1$ for every n-simplex of K, and

(iii) face and degeneracy operators induced by those of K.

Furthermore, in order to simplify the formulation of the following two propositions, we denote by $FV[0,0]$ (resp. $F\dot{\Delta}[0]$) the simplicial groupoid consisting of the object 0 (resp. the objects 0 and 1) and identity maps only.

2.9. PROPOSITION. *For every integer $n \geq 0$, the inclusion* [3, Ch. II, §2] $F\dot{\Delta}[n] \to F\Delta[n]$ *is free and so is any of its cobase extensions. Moreover a map $f: X \to Y \in Gd$ is a trivial fibration if and only if it has the "right lifting property"* [3, Ch. I, §5] *with respect to these maps $F\dot{\Delta}[n] \to F\Delta[n]$, $n \geq 0$.*

2.10. PROPOSITION. *For every pair of integers (k, n) with $0 \leq k \leq n$, the inclusion* [3, Ch. II, §2] $FV[n,k] \to F\Delta[n]$ *is free as well as a weak equivalence, and so is any of its cobase extensions. Moreover a map $f: X \to Y \in Gd$ is a fibration if and only if it has the "right lifting property" with respect to these maps $FV[n,k] \to F\Delta[n]$, $0 \leq k \leq n$.*

PROOF OF THEOREM 2.5. Verification of CM1, CM2, CM3 and the first part of CM4 is easy and 2.7 readily implies the second half of CM4. To prove CM5(i) one combines 2.9 with the small object argument of [3, Ch. II, §3], while the proof of CM5(ii) is similar, using 2.10 instead.

§3 COMPARISON WITH THE HOMOTOPY THEORY OF SIMPLICIAL SETS

In order to prove that the homotopy theory of simplicial groupoids of §2 is equivalent to the usual homotopy theory of simplicial sets, we need the following generalizations of the loop group functor G and the classifying complex functor $\bar{W}$ [2, Ch. VI].

3.1. THE LOOP GROUPOID FUNCTOR G. The *loop groupoid* of a simplicial set K is the simplicial groupoid GK which has

(i) an object tx for every vertex $x \in X_0$,

(ii) in dimension n, the groupoid with a generator $tx: td_0d_2 \cdots d_{n+1}x \to$ $\to td_1d_2 \cdots d_{n+1}x$ for every $x \in K_{n+1}$ and a relation $ts_0y = id_{td_1 \cdots d_n y}$ for every $y \in K_n$, and

(iii) face and degeneracy operators given by

$$\begin{aligned} d_0 tx &= (td_1 x)^{-1} \\ d_i tx &= td_{i+1} x \quad \text{for } i > 0 \\ s_i tx &= ts_{i+1} x \quad \text{for } i \geq 0. \end{aligned}$$

Thus *GK is a free simplicial groupoid* with a basis consisting of the maps of the form tx where $\dim x > 0$ and $x \neq s_0 y$ for some y.

If $K \in S$ is reduced, then the above definition clearly reduces to the old one [2, Ch. VI]. Furthermore, if $K \in S$ and $EK \in S$ denotes its (*unreduced*) *suspension* (i.e. [2, Ch. VI] EK has two vertices and its n-simplices ($n > 0$) which are not in the image of s_0 are in 1-1 correspondence with the $(n-1)$-simplices of K), then it is not difficult to see that (2.8) $GEK \approx FK$.

3.2. THE CLASSIFYING COMPLEX FUNCTOR $\bar{W}$. The *classfying complex* of a simplicial groupoid X is the simplicial set $\bar{W}X$ which has as vertices the object of X and as n-simplices ($n>0$) the sequences of maps in X

$$A_n \xrightarrow{g_{n-1}} \cdots \longrightarrow A_1 \xrightarrow{g_0} A_0 \quad \text{where } g_i \in X_i \text{ for all } i,$$

with faces and degeneracies given by the formulas

$$d_i(g_0, \dots, g_{n-1}) = (g_0, \dots, g_{n-i-1}, g_{n-i} \cdot d_0 g_{n-i+1}, \dots, d_{i-1} g_{n-1})$$
$$s_i(g_0, \dots, g_{n-1}) = (g_0, \dots, g_{n-i}, id, s_0 g_{n-i+1}, \dots, s_{i-1} g_{n-1})$$

If X has only one object, then this definition clearly reduces to the old one [2, Ch. IV].

One then has

3.3. THEOREM.

(i) *The functor $G: S \to Gd$ preserves cofibrations and weak equivalences.*

(ii) *The functor $\bar{W}: Gd \to S$ preserves fibrations and weak equivalences.*

(iii) *The functor $G: S \to Gd$ is left adjoint to the functor $\bar{W}: Gd \to S$ and for every pair of objects $K \in S$ and $X \in Gd$, a map $K \to \bar{W}X \in S$ is a weak equivalence if and only if its adjoint $GK \to X \in Gd$ is so.*

PROOF. A lengthy but essentially straightforward calculation yields that the function which assigns to a map $f: GK \to X \in Gd$ the map $g: K \to \bar{W}X \in S$ given by

$$gx = (ftd_0^{n-1}x, \dots, ftd_0 x, ftx) \quad \text{for } x \in K_n$$

is the desired adjunction. The rest of the proof is easy.

If L^H denotes the (*simplicial*) *hamock localization* of [1, §3] with respect to the weak equivalences, then [1, §5] theorem 3.3 implies

3.4. COROLLARY. *The functors G and $\bar{W}$ induce,* in the sense of [1, 2.5], *homotopy equivalences of simplicial categories*

$$L^H S \underset{L^H \bar{W}}{\overset{L^H G}{\rightleftarrows}} L^H Gd$$

which are homotopy inverses of each other.

This means that *the categories S and Gd have equivalent homotopy theories* in the sense that the functors G and $\bar{W}$ not only induce a 1-1 correspondence on homotopy types and homotopy clases of maps, but also on homotopy types of (abstract) function complexes. In order to give some more concrete meaning to this last statement we need a suitable notion of

3.5. FUNCTION COMPLEXES FOR Gd. We construct function complexes for Gd by enlarging Gd to a *simplicial category* (1.2) Gd_* such that $Gd_0 = Gd$ and then defining, for every pair of objects $X, Y \in \mathbf{Gd}$, the *function complex of maps from X to Y* as $\mathbf{Gd}_*(X, Y)$.

For every pair of objects $X \in \mathbf{Gd}$ and $K \in \mathbf{S}$, let $X \otimes K$ denote the simplicial groupoid which has the same objects as X and which, in dimension k, consists of the free product with amalgamation of the objects of as many copies of X_k as K has k-simplices. Next, for every integer $n \geq 0$, let $[n]$ denote the groupoid with the integers $0, \ldots, n$ as objects and exactly one map $i \to j$ for every pair of integers (i, j) with $0 \leq i, j \leq n$. Then we define the simplicial category $\mathbf{Gd}_*$ by the formula

$$\mathbf{Gd}_n(X, Y) = \mathbf{Gd}((X \times [n]) \otimes \Delta[n], Y).$$

Of course one has to verify that composition is well defined and associative, but that is straightforward.

If $\mathbf{S}_*$ denotes the category of simplicial sets with its usual simplicial structure [3], $\mathbf{S}_*^f \subset \mathbf{S}_*$ its full simplicial subcategory spanned by the fibrant objects (which are automatically cofibrant) and $\mathbf{Gd}_*^c \subset \mathbf{Gd}_*$ the full simplicial subcategory spanned by the cofibrant objects (which are automatically fibrant), then one has

3.6. THEOREM. *There is a sequence of simplicial categories and functors between them*

$$\mathbf{S}_*^f \longrightarrow \operatorname{diag} L^H\mathbf{S}_* \longleftarrow L^H\mathbf{S} \underset{L^H\bar{W}}{\overset{L^HG}{\rightleftarrows}} L^H\mathbf{Gd} \longrightarrow \operatorname{diag} L^H\mathbf{Gd}_* \longleftarrow \mathbf{Gd}_*^c$$

in which all functors are homotopy equivalences (in the sense of [1, 2.5]).

3.7. COROLLARY. *For every two objects $K \in \mathbf{S}$ and $L \in \mathbf{S}^f$, the function complexes*

$$\mathbf{S}_*(K, L) \quad \text{and} \quad \mathbf{Gd}_*(GK, GL)$$

have the same homotopy type.

Similarly one has, if, for $L \in \mathbf{S}^f$ and $X \in \mathbf{Gd}^c$

$$\operatorname{haut} L \subseteq \mathbf{S}_*(L, L) \quad \text{and} \quad \operatorname{haut} X \subset \mathbf{Gd}_*(X, X)$$

denote the simplicial submonoids consisting of those simplices

$$L \times \Delta[n] \to L \quad \text{and} \quad (X \times [n] \otimes \Delta[n] \to X$$

which are weak equivalences:

3.8. COROLLARY. *For every object $L \in \mathbf{S}^f$, the classifying complexes*

$$\bar{W} \operatorname{haut} L \quad \text{and} \quad \bar{W} \operatorname{haut} GL$$

of the simplicial monoids of self weak equivalences, have the same homotopy type.

PROOF OF THEOREM 3.6. The functors in the middle are homotopy equivalences in view of corollary 3.4 and those on the left are so in view of [1, §5] and the fact that S_* is a closed simplicial model category [3, Ch. II, §3]. Moreover the proofs of the results of [1, §5] also apply to the functors on the right, even though $\mathbf{Gd}_*$ is *not* a closed simplicial model category. One only needs that

(i) for every cofibrant object $X \in \mathbf{Gd}$, the objects $(X \times [n]) \otimes \Delta[n]$ together with the obvious maps between them, form a *cosimplicial resolution* in the sense of [1, §4], and

(ii) for every integer $n \geq 0$, the functor

$$(-x[n]) \otimes \Delta[n] : \mathbf{Gd} \to \mathbf{Gd}$$

preserves weak equivalences and cofibrations and has a right adjoint.

The verification of these statements is straightforward.

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