KÄHLER DIFFERENTIALS, THE T-FUNCTOR, AND A THEOREM OF STEINBERG

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Abstract. Let W be a finite group acting on a finite dimensional vector space V over the field k, U a subset of V, and W_U the subgroup of W which fixes U pointwise. Suppose that the algebra $\mathrm{Sym}(V^\#)^W$ is a polynomial algebra. Steinberg has shown that if k has characteristic zero the algebra $\mathrm{Sym}(V^\#)^{W_U}$ is also a polynomial algebra. We extend this result to the case $k = \mathbf{F}_p$ (see also [14]) by proving that Lannes's functor "T" preserves polynomial algebras. We give examples to show that if $k = \mathbf{F}_p$ and W is generated by reflections, it is not necessarily the case that W_U is generated by reflections.

§1. Introduction

Let k be a field and V a finite-dimensional vector space over k. We let $V^\#$ denote the dual of V, and $\mathrm{Sym}(V^\#)$ the symmetric algebra on $V^\#$. If k is infinite, $\mathrm{Sym}(V^\#)$ is the algebra of polynomial functions on V; in any case, $\mathrm{Sym}(V^\#)$ is a polynomial algebra over k isomorphic to $k[x_1, \ldots, x_d]$, where $d = \dim_k(V)$. Any group W of automorphisms of V acts on $\mathrm{Sym}(V^\#)$ in a natural way and has an associated fixed subalgebra $\mathrm{Sym}(V^\#)^W$. Our main algebraic theorem is the following one.

1.1 Theorem. *(cf.* [14, 2.1.2]*) Let V be a finite dimensional vector space over the field $\mathbf{F}_p$ and W a subgroup of* $\mathrm{Aut}(V)$*. Suppose that U is a subset of V, and let W_U be the subgroup of W consisting of elements which pointwise fix U. Then if $\mathrm{Sym}(V^\#)^W$ is a polynomial algebra over $\mathbf{F}_p$, so is $\mathrm{Sym}(V^\#)^{W_U}$.*

This is actually a corollary of a theorem with a more topological flavor. Fix the prime number p, and let $\mathcal{K}$ be the category of algebras

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over the mod p Steenrod algebra $\mathcal{A}_p$ [15, §1]. Recall that an *elementary abelian p-group* is a finite abelian group which is a vector space over $\mathbf{F}_p$. Given an object R of $\mathcal{K}$ and an elementary abelian p-group E, there is an associated "function object" $T_E(R)$ constructed by Lannes; if X is a space and $R = \mathrm{H}^*(X;\mathbf{F}_p)$, then $T_E(R)$ is an algebraic approximation to the mod p cohomology of the space of maps $\mathrm{B}E \to X$. More generally, given a $\mathcal{K}$-map $f : R \to \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$ there is an associated component $T_{E,f}(R)$ of $T_E(R)$, which for $R = \mathrm{H}^*(X;\mathbf{F}_p)$ is an algebraic approximation to the mod p cohomology of an appropriate component of the space of maps $\mathrm{B}E \to X$.

1.2 Theorem. *Suppose that R is an object of $\mathcal{K}$ which as an algebra is a finitely generated polynomial algebra over $\mathbf{F}_p$. Then for any elementary abelian p-group E and $\mathcal{K}$-map $R \to \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$, $T_{E,f}(R)$ is also a finitely generated polynomial algebra over $\mathbf{F}_p$.*

1.3 Remark. Theorem 1.2 implies that if E is an elementary abelian p-group and X is a p-complete space such that the mod p cohomology of X is a finitely generated polynomial algebra over $\mathbf{F}_p$, then the mod p cohomology of any path component of $\mathrm{Map}(\mathrm{B}E, X)$ is also a finitely generated polynomial algebra over $\mathbf{F}_p$ [12, 3.4.2].

The proof of 1.2 proceeds by studying how the functor T_E interacts with the construction of Kähler differentials, and then using a characterization of graded polynomial algebras in terms of Kähler differentials. We thank R. Shank for pointing out that earlier work of Nakajima [14] implies Theorem 1.1. Nakajima's arguments involve some non-trivial local algebra and the reader is invited to compare his approach with ours.

Reflection groups and Steinberg's theorem. Steinberg has proved an analogue of Theorem 1.1 for the case in which V is a vector space over a field k of characteristic zero. His theorem is usually stated in a different form. Recall that if V is a finitely generated free module over a domain k, an element w of finite order in $\mathrm{Aut}_k(V)$ is said to be a *reflection* if image$(w - Id_V)$ has rank 1. (Sometimes such a w is called a *generalized reflection* or a *pseudoreflection*, and the word reflection is reserved for the case in which the order of w is 2. We will use the word reflection in the wider sense.) A subgroup $W \subset \mathrm{Aut}_k(V)$ is said to be *generated by reflections* (or said to be a *reflection group*) if it is generated by the reflections that it contains.

1.4 Theorem. *(Steinberg,* [18, Thm. 1.5]*) Let k be a field of characteristic zero and let $W \subset GL(n,k)$ be a finite subgroup generated by* *reflections. Let U be a subset of k^n and W_U the subgroup of W consisting of elements which fix U pointwise. Then W_U is also generated by reflections.*

The connection between 1.4 and 1.1 is given by the following classical result.

1.5 Theorem. [7] [17] [6] [16] [4] [1] *Let V be a finite dimensional vector space over a field k and $W \subset \mathrm{Aut}(V)$ a finite subgroup. If $\mathrm{Sym}(V^{\#})^W$ is a polynomial algebra over k, then W is generated by reflections. If the characteristic of k is zero or prime to the order of W, then the converse holds.*

In other words, in the situation of 1.5 there are two alternative conditions the group W might satisfy:

(1) W is generated by reflections, or
(2) $\mathrm{Sym}(V^{\#})^W$ is a polynomial algebra.

In characteristic zero the conditions are equivalent, but in finite characteristic the second condition is potentially stronger. Theorem 1.1 is an extension of 1.4 to finite characteristic which substitutes (2) for (1). The following example illustrates that (2) is strictly stronger than (1) in finite characteristic, and shows that there is no direct generalization of 1.4 to finite characteristic which uses alternative (1).

1.6 Examples. *There is a rank 3 free abelian subgroup Σ of $SL(5, \mathbb{Z})$ and an element $u \in (\mathbb{Z})^5$ such that for any prime p the following condition holds: the mod p reduction $\bar{\Sigma}$ of Σ acts faithfully on $(\mathbf{F}_p)^5$ as a group generated by reflections, but the stabilizer in $\bar{\Sigma}$ of the mod p reduction of u is a subgroup of order p not generated by reflections.*

One lesson to carry away from 1.1 and 1.6 is that for a linear group in finite characteristic the property of having polynomial invariants has somewhat better inheritance properties than the more geometrically appealing condition of being generated by reflections.

Centralizers of subgroups of compact Lie groups. Steinberg's theorem (1.4) was originally motivated by a theorem of Hopf about centralizers of tori in connected compact Lie groups. Given 1.4, it is possible to state a variant of Hopf's theorem which deals with the rational cohomology of classifying spaces of centralizers of tori and applies to some disconnected groups. Theorem 1.2 leads similarly to a variant of Hopf's theorem which deals with mod p cohomology of centralizers of elementary abelian p-subgroups and applies to some disconnected groups. We will now describe these results.

Suppose that G is a compact Lie group with maximal torus $T = (S^1)^r$. Hopf's original theorem has to do with centralizers in G of subtori of T.

1.7 Theorem. *(Hopf,* [10, Hilfsatz 23]*, Borel,* [2, Prop. 18.4]*) Let G be a connected compact Lie group and T' a sub-torus of a maximal torus T of G. Then the centralizer $C_G(T')$ of T' is connected.*

On the face of it this theorem has nothing to do with reflection groups and polynomial rings of invariants, but in fact it is closely connected with these things. Recall that if G is a compact Lie group and T is a maximal torus of G, the *Weyl group* $W(G)$ is the quotient $N_G(T)/T$, where $N_G(T)$ is the normalizer of T. The group $W(G)$ is finite and acts on $V(G) = \mathbf{Q} \otimes \pi_1(T) \cong \mathbf{Q}^r$ by conjugation; if G is connected this action is faithful and the image of $W(G) \to \mathrm{Aut}(V(G))$ is generated by reflections. If T' is a subtorus of T, then T is a maximal torus of the centralizer $C_G(T')$ and the Weyl group of $C_G(T')$ can be identified with the subgroup of $W(G)$ which pointwise fixes $\pi_1 T' \subset V(G)$. If $C_G(T')$ is connected then of course the image of its Weyl group in $\mathrm{Aut}(V(G))$ is generated by reflections, so Hopf's Theorem has the following corollary.

1.8 Corollary. *Suppose that G is a connected compact Lie group and that T' is a sub-torus of a maximal torus T for G. Let $W = W(G)$ be the Weyl group of G (considered as a group of automorphisms of $V(G)$) and $U \subset V(G)$ the image of $\pi_1 T'$. Then the subgroup W_U of W consisting of elements which fix U pointwise is generated by reflections.*

Steinberg's theorem (1.4) generalizes 1.8 to arbitrary reflection groups W and arbitrary subsets U of the reflection representation space. It also leads to the following cohomological variant of 1.8. Recall that for any compact Lie group G (connected or not) there are isomorphisms

$$\mathrm{H}^*(\mathrm{B}G; \mathbf{Q}) \cong \mathrm{H}^*(\mathrm{B}T; \mathbf{Q})^W \cong \mathrm{Sym}(V^\#)^W \ , \qquad (1.9)$$

where T is a maximal torus for G, $V = V(G)$, and $W = W(G)$ (which acts on T and on V by conjugation).

1.10 Proposition. *Suppose that G is a compact Lie group such that $\mathrm{H}^*(\mathrm{B}G; \mathbf{Q})$ is a polynomial algebra over $\mathbf{Q}$. Then for any sub-torus T' of G, $\mathrm{H}^*(\mathrm{B}C_G(T'); \mathbf{Q})$ is also a polynomial algebra over $\mathbf{Q}$.*

Proof. Let T be a maximal torus of G; by adjusting T' up to conjugacy we can assume that $T' \subset T$. Let $V = V(G)$; by 1.9 and 1.5, the image of the map $W(G) \to \mathrm{Aut}(V)$ is generated by reflections. Let $U \subset V$ be the image of $\pi_1 T'$ and let $W_U \subset W(G)$ denote the subgroup of elements which fix U pointwise. By 1.4, the image of the map $W_U \to \mathrm{Aut}(V)$

is generated by reflections. Since W_U is the Weyl group of $C_G(T')$ it follows from another application of 1.9 and 1.5 that $\mathrm{H}^*(\mathrm{B}C_G(T');\mathbf{Q}) = \mathrm{Sym}(V^{\#})^{W_U}$ is a polynomial algebra. □

Proposition 1.10 has a direct analogue for mod p cohomology.

1.11 Proposition. *Suppose that G is a compact Lie group such that $\mathrm{H}^*(\mathrm{B}G;\mathbf{F}_p)$ is a polynomial algebra over $\mathbf{F}_p$. Then for any elementary abelian p-subgroup E of G, $\mathrm{H}^*(\mathrm{B}C_G(E);\mathbf{F}_p)$ is also a polynomial algebra over $\mathbf{F}_p$.*

Proof. Suppose that E is an elementary abelian p-subgroup of G and that $f : \mathrm{H}^*(\mathrm{B}G;\mathbf{F}_p) \to \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$ is the map induced by the inclusion $E \to G$. By [11], $\mathrm{H}^*(\mathrm{B}C_G(E);\mathbf{F}_p)$ is isomorphic as an object of $\mathcal{K}$ to $T_{E,f}(\mathrm{H}^*(\mathrm{B}G;\mathbf{F}_p))$. The desired result follows from 1.2. □

Organization of the paper. For the rest of the paper, p denotes a fixed prime number. Section 2 describes a way to recognize polynomial algebras over $\mathbf{F}_p$ by looking at modules of Kähler differentials. Section 3 shows that Lannes's functor T preserves Kähler differentials, and §4 shows that T preserves the properties of these differentials which characterize polynomial algebras. The proof of 1.2 is immediate. In §5 there is an explanation of how to obtain 1.1 from 1.2. Finally, §6 gives a construction of Example 1.6, and §7 shows that similar examples show up in practice when it comes to looking at centralizers of elementary abelian p-subgroups of connected compact Lie groups.

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§2. Kähler Differentials and Polynomial Algebras

We first need a recognition principle for polynomial algebras. If R is a commutative algebra over a commutative base ring k, there is a short exact sequence of $R \otimes_k R$-modules

$$0 \to J \to R \otimes_k R \xrightarrow{\mu} R \to 0$$

where the map μ is multiplication on R. The two inclusions $R \to R \otimes_k 1$ and $R \to 1 \otimes_k R$ give possibly different left R-module structures on J.

2.1 Definition. The *module of Kähler differentials of R relative to k*, denoted $\Omega(R|k)$, is $J/J^2 = J/\operatorname{image}(J \otimes_k J \xrightarrow{\mu} J)$.

2.2 Remark. The action of $R \otimes_k R$ on $\Omega(R|k)$ factors through the multiplication map μ, and so the above two R-module structures on J induce the same R-module structure on $\Omega(R|k)$. Let $S = R \otimes_k R$. The module $\Omega(R|k)$ can be identified with $J \otimes_S R$ or equivalently, by a long exact sequence argument, with $\mathrm{Tor}_1^S(R, R)$.

There is a map $d : R \to \Omega(R|k)$ which is a universal k-linear derivation and sends $x \in R$ to the equivalence class of $x \otimes 1 - 1 \otimes x$. If R has a set $\{x_1, \dots, x_n\}$ of k-algebra generators, $\Omega(R|k)$ is generated as an R-module by $\{dx_1, \dots , dx_n\}$; if R is a polynomial algebra $k[x_1, \dots, x_n]$, then $\Omega(R|k)$ is the free R-module on the classes dx_i. Under some conditions a converse of this last observation holds.

2.3 Theorem. *Let k be a perfect field of characteristic p and R a finitely generated connected graded algebra over. Suppose that R has no nilpotent elements and that $\Omega(R|k)$ is a free R-module. Then R is isomorphic to the polynomial algebra $k[x_1, \dots, x_n]$ where $n = \mathrm{rank}_R(\Omega(R|k))$.*

Remark. The assumption that R has no nilpotent elements cannot be removed from 2.3. To see this, note that if R is a connected graded bicommutative Hopf algebra over k, a standard untwisting argument gives an isomorphism

$$\mathrm{Tor}_*^{R\otimes_k R}(R, R) \cong R \otimes_k \mathrm{Tor}_*^R(k, k)\,.$$

In particular, $\Omega(R|k)$ is free as an R-module. Now take R to be, say, a primitively generated Hopf algebra with one even dimensional generator x truncated at height p. Then $\Omega(R|k)$ is free as an R-module but R is not a polynomial algebra.

Proof of 2.3. This follows from a general form of the Jacobian Criteria for smoothness for complete local rings, see for example [13, Theorem 30.3]. However, the particular case we have is simpler than the general one and we include a proof sketched to us by L. Avramov. Since R is finitely generated and connected, there exists a surjection $\phi : S = k[x_1, \dots, x_N] \to R$, where the x_i are polynomial variables of positive grading. Choose ϕ in a minimal way so that the induced map on indecomposable quotients is a monomorphism. Let I be the kernel of ϕ and select a minimal collection $\{f_1, \dots, f_r\}$ of homogeneous S-generators for I, ordered so that f_1 has the smallest grading. Let S_+ and R_+ denote the elements of positive degree in S and R respectively. The goal is to show that $I = 0$. From the minimality of the presentation and the connectivity assumption, it is enough to show that I/I^2 is zero. There is a natural map $\delta : I/I^2 \to R \otimes_S \Omega(S|k)$, defined by $f \to 1 \otimes_S df$. The first step is to show that the image of δ is zero.

Let $x_i' = \phi(x_i)$. The exact sequence [13, p. 194]

$$I/I^2 \xrightarrow{\delta} R \otimes_S \Omega(S|k) \xrightarrow{\alpha} \Omega(R|k) \to 0.$$

presents $\Omega(R|k)$ as an R-module with generators $\{dx_i'\}$ and relations

$$\{r_j = \sum_i (\partial f_j / \partial x_i) dx_i'\},\ j = 1, \ldots, r.$$

Since each f_i lies in $(S_+)^2$, these relations are contained in

$$R_+(R \otimes_S \Omega(S|k))\ ;$$

hence $k \otimes_R \alpha$ is an isomorphism. On the other hand, $R \otimes_S \Omega(S|k)$ is free over R by construction and $\Omega(R|k)$ is free by the hypothesis. It follows from the connectivity assumption that α is an isomorphism, and hence that $r_j = 0$ for all j. In other words, for each j, we have

$$\sum_i (\partial f_j / \partial x_i) dx_i \in I\Omega(S|k).$$

The linear independence of $dx_1, \ldots, dx_N$ yields

$$(\partial f_j / \partial x_i) \in (f_1, \ldots, f_r)S$$

for all i and j. Since f_1 has minimal degree among the chosen generators of I, this implies that

$$\partial f_1 / \partial x_1 = \ldots = \partial f_1 / \partial x_N = 0.$$

Since k is perfect and S is a polynomial algebra, there exists a polynomial $g \in S$ such that $f_1 = (g)^p$, where p is the characteristic of k. The residue class of g in R is nilpotent and hence, by assumption, 0. Therefore $g = 0$ and $f_1 = 0$. By induction I is the zero ideal and $S \cong R$. □

§3. Kähler Differentials and T

As in [15, §1], let $\mathcal{A}_p$ be the mod p Steenrod algebra, $\mathcal{U}$ the category of unstable modules over $\mathcal{A}_p$, and $\mathcal{K}$ the category of unstable algebras over $\mathcal{A}_p$. An object of $\mathcal{U}$ or of $\mathcal{K}$ is a nonnegatively graded $\mathbf{F}_p$-vector space with an action of $\mathcal{A}_p$ and, in the case of $\mathcal{K}$, a graded commutative multiplication which obeys the Cartan formula and a p'th power condition [15, 1.3, 1.4]. There is a forgetful functor $\mathcal{K} \to \mathcal{U}$. If X is a space, the cohomology algebra $\mathrm{H}^*(X;\mathbf{F}_p)$, with its usual cup product structure, belongs to $\mathcal{K}$.

Let E be an elementary abelian p-group and $\mathrm{B}E$ its classifying space, Lannes has studied the functor $T_E : \mathcal{U} \to \mathcal{U}$ which is left adjoint to the functor which sends $M \in \mathcal{U}$ to $M \otimes_{\mathbf{F}_p} \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$ (note that the action of $\mathcal{A}_p$ on such a tensor product is given by the Cartan formula). The functor T_E has some remarkable algebraic properties; in particular, it is exact, preserves tensor products over $\mathbf{F}_p$ up to natural isomorphism, and lifts to an identically named functor $\mathcal{K} \to \mathcal{K}$ [15, §3]. The topological significance of T_E has to do with its usefulness for computing the cohomology of function spaces, but we are interested in it from an algebraic point of view.

There is a slight refinement of T_E which has topological applications in computing the cohomology of individual components of function spaces. Suppose that $R \in \mathcal{K}$, and that $f : R \to \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$ is a $\mathcal{K}$-map. By adjointness f corresponds to a $\mathcal{K}$-map $f^\flat : T_E(R) \to \mathbf{F}_p$. Since the range of $f^\flat$ is concentrated in degree 0, $f^\flat$ amounts to an ordinary ring homomorphism $f^\flat : T_E^0(R) \to \mathbf{F}_p$; this homomorphism makes $\mathbf{F}_p$ into a module over $T_E^0(R)$. In fact, it follows from the fact that $T_E^0(R)$ is a p-Boolean algebra [15, 3.8] that $\mathbf{F}_p$ is a flat module over $T_E^0(R)$. We define $T_{E,f}(R) = \mathbf{F}_p \otimes_{T_E^0 R} T_E(R)$. More generally, let $\mathcal{U}(R)$ be the category in which an object is an element $M \in \mathcal{U}$ together with a $\mathcal{U}$-map $R \otimes_{\mathbf{F}_p} M \to M$ which makes M into a module over R. If $M \in \mathcal{U}(R)$ then $T_E(M) \in \mathcal{U}(T_E(R))$ (this follows from the fact that T_E preserves tensor products) and we define $T_{E,f}M = \mathbf{F}_p \otimes_{T_E^0 R} T_E M \cong T_{E,f}(R) \otimes_{T_E R} T_E M$. The construction $T_{E,f}$ gives an exact functor $\mathcal{U}(R) \to \mathcal{U}(T_{E,f}R)$. In particular, $T_{E,f}(R)$ is flat as a module over $T_E R$.

Remark. Suppose that $\mathrm{Hom}_{\mathcal{K}}(R, \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p))$ is a finite set; this is always the case if R is finitely generated as an algebra (since $\mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$ is finite in each dimension). In this situation $T_E^0(R)$ is isomorphic as a ring to a direct product of copies of $\mathbf{F}_p$, indexed by the $\mathcal{K}$-maps $f : R \to \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$. There is a corresponding product decomposition [15, 3.8.6]

$$T_E R \cong \prod_f T_{E,f} R \ .$$

For the rest of this section we assume that E is a fixed elementary abelian p-group and write T for T_E and T_f for $T_{E,f}$.The goal of this section is to prove the following two propositions.

3.1 Proposition. *Let R be an object of $\mathcal{K}$ and $f : R \to \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$ a*

$\mathcal{K}$-map. Then there is a natural isomorphism of T_fR-modules

$$T_f\Omega(R|\mathbf{F}_p) \cong \Omega(T_fR|\mathbf{F}_p) \ .$$

3.2 Remark. An object R of $\mathcal{K}$ need not be commutative as a ring (in general it is graded commutative), but we follow 2.2 in defining $\Omega(R|\mathbf{F}_p)$ as $\mathrm{Tor}_i^S(R,R)$, where $S = R\otimes_{\mathbf{F}_p} R$. It will become clear below how to identify this as a object of $\mathcal{U}(R)$.

Note that if $R \in \mathcal{K}$ is commutative for the transparent reason that $p = 2$ or that R is concentrated in even degrees, then $T(R)$ and hence any $T_f(R)$ are commutative for the same reason [15, 3.6].

An algebra R is said to be nilpotent free if the only nilpotent element of R is the zero element.

3.3 Proposition. *Let R be an object of $\mathcal{K}$ which is nilpotent free. Then for any $\mathcal{K}$-map $f : R \to \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$, T_fR is nilpotent free.*

The proof of 3.1 depends on two lemmas.

3.4 Lemma. *Let R and S be objects of $\mathcal{K}$, $g : S \to R$ and $f : R \to \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$ a pair of $\mathcal{K}$-maps with g surjective, and M an object of $\mathcal{U}(R)$. Let $h = gf$. Then T_hM is naturally isomorphic to T_fM as an object of $\mathcal{U}(T_hS)$.*

Proof. Observe that in the statement of 3.4, M is considered to be an S-module via the homomorphism $g : S \to R$, and T_fM is a T_hS-module via the induced homomorphism $T_hS \to T_fR$.

The adjoint $h^\flat$ of h is the composite $T(S) \xrightarrow{T(g)} T(R) \xrightarrow{f^\flat} \mathbf{F}_p$. Since g is surjective and T is exact, $T(g)$ is also surjective and in particular the map $T^0(S) \to T^0(R)$ induced by $T(g)$ is surjective. It is now clear that for any $T^0(R)$-module N (such as $T(M)$) the map $\mathbf{F}_p \otimes_{T^0S} N \to \mathbf{F}_p \otimes_{T^0R} N$ is an isomorphism, and the lemma follows. □

Note that if R is an object of $\mathcal{K}$ and M, N are objects of $\mathcal{U}(R)$, then for any $i \geq 0$ the R-module $\mathrm{Tor}_i^R(M,N)$ is also in a natural way an object of $\mathcal{U}(R)$; the $\mathcal{A}_p$ action on these R-modules can be obtained, for instance, by letting $\mathcal{A}_p$ act via the Cartan formula on the bar construction [15, 6.4].

3.5 Lemma. *Suppose that R is an object of $\mathcal{K}$, $f : R \to \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$ is a $\mathcal{K}$-map, and M, N are objects of $\mathcal{U}(R)$. Then there are natural isomorphisms in $\mathcal{U}(T_fR)$:*

$$T_f(\mathrm{Tor}_i^R(M,N)) \cong \mathrm{Tor}_i^{T_fR}(T_fM, T_fN) \qquad i \geq 0 \ .$$

Proof. Since T is exact and preserves tensor products, it follows as in [15, 6.4.2] that there are natural isomorphisms $T(\mathrm{Tor}_i^R(M,N)) \cong \mathrm{Tor}_i^{TR}(TM,TN)$ in $\mathcal{U}(TR)$. Since $T_f(R)$ is flat (in fact projective [15, pf. of 6.4.3]) as a $T(R)$-module, there are natural isomorphisms

$$T_f(R) \otimes_{TR} Tor_i^{TR}(TM,TN) \cong \mathrm{Tor}_i^{TR}(T_fM,TN) \ .$$

Again by flatness there is an isomorphism between $\mathrm{Tor}_i^{TR}(T_fM,TN)$ and $\mathrm{Tor}_i^{T_fR}(T_fM,T_fN)$ (cf. [15, pf. of 6.4.3]). Combining these isomorphisms gives the desired result. □

Proof of 3.1. Let $S = R \otimes_{\mathbf{F}_p} R$ and let $h : S \to \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$ be the composite of f with the multiplication map $S \to R$. By 3.5, there is a natural isomorphism $T_h(\mathrm{Tor}_1^S(R,R)) \cong \mathrm{Tor}_1^{T_hS}(T_hR,T_hR)$. By 3.4 there are natural isomorphisms $T_h(\mathrm{Tor}_1^S(R,R)) \cong T_f(\mathrm{Tor}_1^S(R,R)$ and $T_h(R) \cong T_f(R)$. Finally, by using the fact that T preserves tensor products over $\mathbf{F}_p$ we obtain an isomorphism

$$T_h(S) = T(S) \otimes_{T^0S} \mathbf{F}_p \cong T_f(R) \otimes_{\mathbf{F}_p} T_f(R)$$

To finish up, identify $\Omega(R|\mathbf{F}_p)$ with $\mathrm{Tor}_1^S(R,R)$ (3.2). □

Proof of 3.3. Suppose that R is an object of $\mathcal{K}$ which is nilpotent free. We first show that TR is nilpotent free. Since R is graded commutative, it is clear that either R is concentrated in even degrees, or $p=2$. Let $\Phi(R)$ be the object of $\mathcal{U}$ constructed as in [15, 1.7.2] by multiplying the degrees of elements in R by a factor of p. There is a natural $\mathcal{U}$-map $\lambda_R : \Phi(R) \to R$ which loosely speaking sends each element of R to its p'th power. Since R is nilpotent free, the map λ_R is a monomorphism. By [15, 3.4], the map λ_{TR} can be identified with $T(\lambda_R)$. Since T is exact, it follows that λ_{TR} is a monomorphism and hence that TR is nilpotent free.

The final step is to show that $T_f(R)$ is nilpotent free. The functors T and T_f commute with colimits (because T is a left adjoint and T_f is obtained from T by a tensor product). This implies that it is enough to treat the case in which R is a finitely generated object of $\mathcal{K}$, i.e., generated by a finite number of elements under product, sum, and the operation of $\mathcal{A}_p$. In this case there are only a finite number of $\mathcal{K}$-maps $R \to \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$ and $T^0(R)$ is isomorphic as a ring to a direct product of copies of $\mathbf{F}_p$, one for each such $\mathcal{K}$-map [15, 3.8.6]. It follows that $T_f(R)$ is a direct factor, as a ring, of $T(R)$, and so $T_f(R)$ is nilpotent free if $T(R)$ is. □

4. Free R modules and T

We continue using the notation of the previous section in letting E stand for an elementary abelian p-group and writing T and T_f for T_E and $T_{E,f}$ respectively. Our goal is to complete the proof of 1.2 by showing that T_f preserves the freeness of modules, in the following sense.

4.1 Proposition. *Suppose that R is an object of $\mathcal{K}$ and that M is object of $\mathcal{U}(R)$ which is free as an R-module. Then for any $\mathcal{K}$-map $f : R \to \mathrm{H}^*(\mathrm{B}E; \mathbf{F}_p)$, $T_f(M)$ is free as a $T_f(R)$-module.*

Proof of 1.2. This consists of stringing together 3.1, 4.13.3 and 2.3. Note that $T_f(R)$ is finitely generated as a polynomial algebra because in general T preserves finite generation of algebras [9, 1.4]. $\square$

Proof of 4.1. Since $T_f(R)$ is a connected graded algebra, in order to show that $T_f(M)$ is a free module over $T_f(R)$ it is enough to show that the groups $\mathrm{Tor}_i^{T_f R}(\mathbf{F}_p, T_f M)$ vanish for $i > 0$ (actually, vanishing of Tor_1 would be enough). Let $H = \mathrm{H}^*(\mathrm{B}E; \mathbf{F}_p)$. The map $f^\flat : T_f(R) \to \mathbf{F}_p$ that is used in computing Tor is adjoint to $f : R \to H$. It follows from naturality that $f^\flat$ is the composite of $T_f(f) : T_f(R) \to T_f(H)$ with the map $\iota^\flat : T_f(H) \to \mathbf{F}_p$ adjoint to the identity map $\iota : H \to H$. In particular, for any $T_f(R)$-module N there is a natural isomorphism

$$\mathbf{F}_p \otimes_{T_f R} N \cong \mathbf{F}_p \otimes_{T_f H} (T_f H \otimes_{T_f R} N) \ .$$

This gives rise to a composition of functors spectral sequence

$$E^2_{i,j} = \mathrm{Tor}_i^{T_f H}(\mathbf{F}_p, \mathrm{Tor}_j^{T_f R}(T_f H, N)) \Rightarrow \mathrm{Tor}_{i+j}^{T_f R}(\mathbf{F}_p, N) \ .$$

We will show that for $N = T_f(M)$ this spectral sequence has $E^2_{i,j} = 0$ for $(i,j) \neq (0,0)$.

Note first of all that the groups $\mathrm{Tor}_j^R(H, M)$ vanish for $j > 0$ because M is free as an R-module (here R acts on H via f). By 3.5, $\mathrm{Tor}_j^{T_f R}(T_f H, T_f M) = 0$ for $j > 0$. Now consider the object

$$U = \mathrm{Tor}_0^{T_f R}(T_f H, T_f M) = T_f(H \otimes_R M)$$

as a module over $T_f(H)$. By [15, 3.8.6] and [15, 3.9], $T_f(H)$ is isomorphic to a direct product $\prod_\alpha H(\alpha)$ of copies of H indexed by the (finite) collection of $\mathcal{K}$-maps $\alpha : H \to H$ with $\alpha \cdot f = f$. Under this isomorphism, the map $\iota^\flat : T_f(H) \to \mathbf{F}_p$ is obtained by composing projection on the component $H(\iota)$ with the unique ring homomorphism $H(\iota) = H \to \mathbf{F}_p$.

Since $H(\iota) = H$ is flat (in fact projective) over $T_f(H)$, there are natural isomorphisms

$$\mathrm{Tor}_i^{T_f H}(\mathbf{F}_p, U) \cong \mathrm{Tor}_i^H(\mathbf{F}_p, H \otimes_{T_f H} U) \quad i \geq 0 \ .$$

The projection $T_f(H) \to H(\iota)$ can alternatively be interpreted as the natural map $T_f(H) \to T_\iota(H)$. This implies that there are isomorphisms

$$\begin{aligned} H \otimes_{T_f H} U &\cong T_\iota H \otimes_{T_f H} T_f H \otimes_{T_f R} T_f R \otimes_{TR} TM \\ &\cong T_\iota H \otimes_{TH} (TH \otimes_{TR} TM) \\ &\cong T_{\iota H} \otimes_{TH} T(H \otimes_R M) \\ &\cong T_\iota (H \otimes_R M) \end{aligned}$$

where we have used the fact that T is exact and preserves tensor products to give the isomorphism $TH \otimes_{TR} TM \cong T(H \otimes_R M)$ (cf. 3.5). The desired vanishing now follows from the fact that if N is *any* object of $\mathcal{U}(H)$, in particular $H \otimes_R M$, then $T_\iota(N)$ is free as a module over $T_\iota(H) \cong H$ [9, 2.4] [12, 4.5]. □

§5. Rings of invariants and T

In order to deduce Theorem 1.1 from Theorem 1.2 we have to find a connection between rings of invariants, algebras over $\mathcal{A}_p$, and the functor T. This is provided by the following construction.

5.1 Definition. Suppose that V is a finite dimensional vector space over $\mathbf{F}_p$. The *enhanced symmetric algebra* $\mathbf{S}(V^\#)$ is the unstable algebra over $\mathcal{A}_p$ freely generated as a commutative algebra by the elements of $V^\#$, which are considered to have grading two. The action of $\mathcal{A}_p$ on $\mathbf{S}(V^\#)$ is the unique one allowed by the usual unstable algebra conditions.

5.2 Remark. The action of $\mathcal{A}_p$ on $\mathbf{S}(V^\#)$ can be described explicitly as follows. If p is odd and $x \in V^\# = \mathbf{S}(V^\#)^2$, then $\mathcal{P}^1(x) = x^p \in \mathbf{S}(V^\#)^{2p}$, $\beta(x) = 0$, and $\mathcal{P}^i(x) = 0$ for $i > 1$; $\mathcal{A}_p$ acts on products of two-dimensional classes by the Cartan formula. The same formulas work for $p = 2$ if β is interpreted as Sq^1 and $\mathcal{P}^i$ as Sq^{2i}. The algebra $\mathbf{S}(V^\#)$ is isomorphic as an element of $\mathcal{K}$ to the cohomology ring of a product of copies of CP^∞, where the number of factors in the product is the dimension of V; $\mathbf{S}(V^\#)$ is also isomorphic to the subalgebra of $\mathrm{H}^*(\mathrm{B}V; \mathbf{F}_p)$ generated by the Bocksteins of one-dimensional cohomology classes. In particular $\mathbf{S}(V^\#)$ is functorial in V, and is isomorphic as an algebra to $\mathrm{Sym}(V^\#)$; it differs from $\mathrm{Sym}(V^\#)$ only in that it is explicitly graded and has a specified action of $\mathcal{A}_p$.

5.3 Lemma. *Suppose that E and V are finite-dimensional vector spaces over $\mathbf{F}_p$. Then there is a natural bijection*

$$\mathrm{Hom}_{\mathcal{K}}(\mathbf{S}(V^{\#}), \mathrm{H}^*(\mathrm{B}E; \mathbf{F}_p)) \cong \mathrm{Hom}(E, V)$$

and a natural $\mathcal{K}$-isomorphism between $T_E\mathbf{S}(V^{\#})$ and $\mathbf{S}(V^{\#})^{\mathrm{Hom}(E,V)}$.

Remark. The notation $\mathbf{S}(V^{\#})^{\mathrm{Hom}(E,V)}$ from 5.3 denotes the collection of set-maps from $\mathrm{Hom}(E, V)$ to $\mathbf{S}(V^{\#})$, with pointwise ring operations; equivalently, this is a product of copies of $\mathbf{S}(V^{\#})$ indexed by the elements of $\mathrm{Hom}(E, V)$. Under the isomorphism of 5.3 the action of $\mathrm{Aut}(V)$ on $T_E\mathbf{S}(V^{\#})$ corresponds to the diagonal action of $\mathrm{Aut}(V)$ on $\mathbf{S}(V^{\#})^{\mathrm{Hom}(E,V)}$, i.e., the action which given $g \in \mathrm{Aut}(V)$ and $\alpha : \mathrm{Hom}(E, V) \to \mathbf{S}(V^{\#})$ has $(g \cdot \alpha(\phi) = g\alpha(g^{-1}\phi)$ (where $\phi \in \mathrm{Hom}(E, V)$).

Proof of 5.3. Let $H = H^*$ denote $\mathrm{H}^*(\mathrm{B}E; \mathbf{F}_p)$. The calculation of $\mathrm{Hom}_K(\mathbf{S}(V^{\#}), H)$ is as follows. Any $f \in \mathrm{Hom}_K(\mathbf{S}(V^{\#}), H)$ is an algebra map and so is determined by its effect on the copy of $V^{\#}$ in dimension 2. Clearly $f(V^{\#})$ lies in the kernel of the Bockstein map $\beta : H^2 \to H^3$. Since $\tilde{\mathrm{H}}^*(\mathrm{B}E; \mathbb{Z})$ is of exponent p, this kernel is isomorphic to $H^1 = E^{\#}$ via $\beta : H^1 \to H^2$ (cf. [15, 1.5]). Thus any such f gives a map $V^{\#} \to E^{\#}$ or equivalently a map $E \to V$. It is now easy to check that any homomorphism $V^{\#} \to E^{\#}$ arises from a unique f.

For any object R of $\mathcal{K}$ there is a natural map $R \to T_E(R)$ adjoint to the morphism $R \otimes_{\mathbf{F}_p} H^* \to R \otimes_{\mathbf{F}_p} \mathbf{F}_p \cong R$ induced by the unique K-map $H \to \mathbf{F}_p$. For any $f : R \to H$ there is a composite map $\epsilon_f : R \to T_E R \to T_{E,f} R$. The last statement in the lemma follows from the fact that for any $f : \mathbf{S}(V^{\#}) \to H$ the map $\epsilon_f : \mathbf{S}(V^{\#}) \to T_{E,f}\mathbf{S}(V^{\#})$ is an isomorphism. The most economical way to obtain this is from a geometric theorem of Lannes [15, 9.6] [11]. Let G denote the circle group $S^1 = SO(2)$. The algebra $\mathbf{S}(V^{\#})$ is isomorphic to $\mathrm{H}^*(\mathrm{B}G^d; \mathbf{F}_p)$, where $d = \dim V$ (cf. 5.2). By Lannes' theorem, a $\mathcal{K}$-map $f : \mathbf{S}(V^{\#}) \to H$ corresponds to a homomorphism $h(f) : E \to SG^d$, and $T_f\mathbf{S}(V^{\#})$ is then naturally isomorphic to the cohomology of the classifying space of the centralizer in G^d of the image of $h(f)$. Since G^d is abelian, this centralizer is G^d itself, and the cohomology of its classifying space is again $\mathbf{S}(V^{\#})$. $\square$

Proof of 1.1. Let $S = \mathbf{S}(V^{\#})$ and $R = S^W$. Suppose that E is an elementary abelian p-group. It follows directly from the exactness of T that there is a $\mathcal{K}$-isomorphism $T_E R = T_E(S^W) \cong (T_E S)^W$ [15, 3.9.5].

By 5.3, this gives isomorphisms

$$
\begin{aligned}
T_E(R) = T_E(S^W) &\cong \mathrm{Map}(\mathrm{Hom}(E,V),S)^W \\
&\cong \mathrm{Map}_W(\mathrm{Hom}(E,V)S) \\
&\cong \textstyle\prod_{\{\phi\}} S^{W_\phi}
\end{aligned} \quad . \tag{5.4}
$$

Here the product in the third line is taken over the set of orbits $\{\phi\}$ of the action of W on $\mathrm{Hom}(E,V)$, and W_ϕ denotes the isotropy subgroup in W of an orbit representative ϕ. To pass from the second to the third line we have used the fact that if A and B are W-sets and W acts transitively on A, there is an isomorphism $\mathrm{Map}_W(A,B) \cong B^{W_a}$, where a is any element of the orbit a.

It follows from the dimension 0 part of 5.4 that $\mathcal{K}$-maps $f : R \to \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$ (equivalently ring homomorphisms $T^0 R \to \mathbf{F}_p$) are in bijective correspondence with orbits $\{\phi\}$ of the action of W on $\mathrm{Hom}(E,V)$. Moreover, for any such f, $T_f(R)$ is isomorphic to S^{W_ϕ}. Let U be the chosen subset of V, E the linear span of U, and $\phi : E \to V$ the inclusion. The W_U (the subgroup of W consisting of elements which fix U pointwise) is equal to W_ϕ (the subgroup of W consisting of elements which under composition leave the map ϕ unchanged). Let $f : R \to \mathrm{H}^*(\mathrm{B}E;\mathbf{F}_p)$ correspond to $\{\phi\}$. Since R is a polynomial algebra by assumption (finitely generated for elementary transcendence degree reasons), it follows from 1.2 that $T_f(R) \cong S^{W_\phi} \cong S^{W_U} \cong \mathrm{Sym}(V^\#)^{W_U}$ is also a polynomial algebra. □

§6 A universal example with non-reflection stabilizers

We first provide a "universal" example to satisfy the promise of Example 1.6.

6.1 Example. *Let Σ be the rank 3 free abelian subgroup of $GL(5,\mathbb{Z})$ generated by the matrices*

$$
\begin{pmatrix} 1&0&1&0&0\\ 0&1&1&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \quad
\begin{pmatrix} 1&0&0&1&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \quad
\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&1\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix},
$$

denoted r, s and t respectively. The stabilizer of $(0,0,1,-1,-1)^T$ in Σ is the subgroup generated by the product $z = rst$.

$$
z = \begin{pmatrix} 1&0&1&1&0\\ 0&1&1&0&1\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}
$$

and so $z - \mathrm{Id}$ has rank 2. Let $\rho : GL(5,\mathbb{Z}) \to GL(5,\mathbf{F}_p)$ be the mod p reduction map. Then the images of $\{r,s,t\}$ are reflections of order p, and $\rho(\Sigma)$ is an elementary abelian p-group of rank 3. In this image, the stabilizer of the reduction mod p of $(0,0,1,-1,-1)^T$ is the subgroup of order p generated by $\rho(z)$ and thus contains no reflections.

Proof. The verification is a straightforward calculation. The matrices have been chosen so that multiplication of the matrices corresponds to addition of the upper right 2×3 blocks - if A and B are matrices which are zero off this block, then $AB = 0$ and $(I+A)(I+B) = (I+B)(I+A)$. Hence the group Σ is commutative and isomorphic to the free abelian group of rank 3, and $\rho(\Sigma)$ is a rank 3 elementary abelian p-group.n Second, $r^a(s^b(t^c((0,0,1,-1,-1)^T))) = (a-b, a-c, 1, -1, -1)^T$, so the stabilizer of $(0,0,1,-1,-1)^T$ consists of the elements $r^a s^b t^c$ with $a = b$ and $a = c$. This is the group generated by z^a, where $z = rst$. An identical calculation works mod p. □

§7 Some Lie examples of non-reflection group stabilizers

In this section we look for naturally occurring examples of reflection groups W acting on mod p vector spaces V such that for suitable subsets U of V, W_U is not generated by mod p reflections. It turns out that it is frequently possible to find examples like this by taking a connected compact Lie group G with p-torsion in its homology, and looking at the action of the Weyl group of G on the group of elements of exponent p in a maximal torus.

Suppose then that G is a connected compact Lie group with maximal torus $T = (S^1)^r$ and Weyl group $W = W(G)$. Let $L(G) = \pi_1 T$ and $V_p(G) = \mathbf{F}_p \otimes L(G)$. Conjugation gives actions of W on $L(G)$ and on $V_p(G)$, and the images of the maps $W \to \mathrm{Aut}(L(G))$ and $W \to \mathrm{Aut}(V_p(G))$ are generated by reflections. The group $V_p(G)$ is naturally isomorphic to the group of elements of exponent p in T, and the stabilizer subgroup in W of a subset U of $V_p(G)$ is isomorphic to the Weyl group of the centralizer in G of the elementary abelian p-subgroup $\langle U \rangle$ of T generated by U. Now for $p > 2$ such a stabilizer is generated by elements of W which are *rational reflections* (i.e. reflections in $\mathrm{Aut}(L(G))$) if and only if the centralizer of $\langle U \rangle$ is connected. Since the question of whether all such centralizers are connected is equivalent to the question of whether the integral homology of G (equivalently BG) has $p-$torsion (see Borel [3] and Dwyer-Wilkerson [8]), it is natural to examine Lie groups with homology torsion in order to find examples (W,U) such that W_U is not generated by rational reflections. We go further and look for examples in which W_U is not even generated by reflections mod p (i.e.

reflections in $\mathrm{Aut}(V_p(G))$). The natural place to start is with connected Lie groups with nontrivial finite fundamental groups.

Central quotients of the special unitary groups $\{SU(n)\}$ provide such examples for each prime. Recall that the unitary group $U(n)$ has Weyl group the symmetric group Σ_n, which acts on the rank n free abelian group $L(U(n)) = \pi_1(T)$ by permuting the elements $\{\vec{e_i}\}$ of a basis. The Weyl group of $SU(n)$ is also Σ_n, acting on the rank $n-1$ free abelian group $L(SU(n))$ spanned by the collection $\{\vec{e_i} - \vec{e_j}\}$.

Let $\vec{z}$ denote the element $\sum_{i=1}^{n-1}(\vec{e_i} - \vec{e_n})$ of $L(SU(n))$. Given an m which divides n, one can form a new Σ_n-module L_m by taking the subgroup of $L(SU(n)) \otimes \mathbf{Q}$ generated by $L(SU(n))$ and $\vec{z}/m$. The lattice L_m is $L(G)$ for $G = SU(n)/C_m$, where $C_m \cong \mathbb{Z}/m\mathbb{Z}$ is the unique central subgroup in $SU(n)$ of order m. Since $SU(n)$ is simply connected, $SU(n)/C_m$ has fundamental group C_m and so has p-torsion in its homology for all p dividing m. If $m = n$, $SU(n)/C_m$ is the projective unitary group $PSU(n)$.

We first focus on the special case in which $n = pN$ is divisible by p and $m = p$.

7.1 Example. *Let G denote $SU(pN)/C_p$. Consider the action of $W = W(G) \cong \Sigma_{pN}$ on $V_p(G)$. There is a element $v \in V_p(G)$ such that the stabilizer subgroup W_v is isomorphic to the wreath product*

$$\Sigma_N \wr \mathbb{Z}/p\mathbb{Z}.$$

For the cases $(p > 3, N \geq 1)$, $(p = 3, N > 1)$, or $(p = 2, N > 2)$, W_v is not generated by reflections mod p.

Remark. The wreath product $\Sigma_N \wr \mathbb{Z}/p\mathbb{Z}$ in 7.1 is the semidirect product of $\mathbb{Z}/p$ and $(\Sigma_N)^p$, with $\mathbb{Z}/p$ acting on $(\Sigma_N)^p$ by cyclically permuting the factors.

By a quirk, for $p = 3$ and $N = 1$, the stabilizer subgroup from 7.1 is generated by reflections mod 3 and its invariants are polynomial. Thus quotients of $SU(6)$ provide the first non-reflection group stabilizer examples in this series for $p = 3$. Likewise, for $p = 2$ and $N = 2$ the stabilizer is generated by mod 2 reflections that do not lift to reflections over the rationals. Thus quotients of $SU(6)$ provide the first interesting examples for $p = 2$.

There is a similar class of examples related to the exceptional Lie groups. Let u be a generator of Center($SU(p)$) and $\Delta(p)$ the central subgroup of $SU(p) \times SU(p)$ generated by (u, u). Form $(SU(p) \times SU(p))/\Delta(p)$. Such quotient groups arise naturally; for $p = 3$, this group is realized as the centralizer H of an element of order 3 in the simple group $G = F_4$ and for $p = 5$ as the centralizer H of an element of order 5 in the simple group $G = E_8$. In each of these cases G/H has Euler characteristic prime to p and hence (by a transfer argument) BH captures the p-torsion of $\mathrm{H}^*(\mathrm{B}G, \mathbb{Z})$.

7.2 Example. *Let G denote $(SU(p) \times SU(p))/\Delta(p)$, and suppose that $p > 2$. Consider the action of $W = W(G) \cong \Sigma_p \times \Sigma_p$ on $V_p(G)$. There is an element $v \in V_p(G)$ such that the stabilizer subgroup W_v has order p and is not generated by reflections mod p.*

We make the convention for the rest of this section that a vector $\vec{x} = (x_1, \ldots, x_n)$ of integers is to be interpreted as the element of $V_p(U(n)$ obtained by taking the reduction mod p of $\Sigma x_i \vec{e_i} \in L(U(n))$. If the coordinate sum Σx_i is zero mod p, then $\vec{x}$ can also be considered as an element of $V_p(SU(n))$.

Proof of 7.1. First assume that p is odd or N is even. Let $\vec{x}$ be the element of $V_p(SU(pN))$ given by the vector

$$\vec{x} = (0, \ldots 0, 1, \ldots, 1, \ldots, p-1, \ldots, p-1)$$

in which each integer is repeated N times. We let v be the image of $\vec{x}$ in $V_p(G)$.

First note that the stabilizer W_0 of $\vec{x}$ is exactly the p-fold product

$$\Sigma_N \times \ldots \Sigma_N$$

since any cycle that stabilizes $\vec{x}$ must work within a block of like coordinate values mod p.

But the kernel of $\phi : V_p(SU(pN)) \to V_p(G)$ is generated by $\vec{z} = (1, \ldots, 1)$ (1 repeated pN times) and $T\vec{z} = \vec{z}$ for any $T \in \Sigma_{pN}$. Hence if $T(v) = v$, then $(T-1)\vec{x} = a\vec{z}$. For such a T, $T^p\vec{x} = \vec{x}$, that is, $T^p \in W_0$. Likewise, if S and T are such permutations, $STS^{-1}T^{-1}\vec{x} = \vec{x}$. Furthermore $S\vec{x} = T^b\vec{x}$ for some b, so $W_v/W_0 = \mathbb{Z}/p\mathbb{Z}$. There is an explicit splitting of the surjection $W_v \to \mathbb{Z}/p\mathbb{Z}$. Let S be the permutation which in cycle form has the description $(1, 2, \ldots, pN)$, and let $T = S^N$. Then $T^p = 1$, T preserves the block structure of $\vec{x}$, and $T^c\vec{x} - \vec{x} = c\vec{z}$.

Finally, we need to show that W_v is not generated by reflections mod p. Since W_0 is a normal subgroup of W_v, it is enough to show that no element $w \in W_v \setminus W_0$ acts on $V_p(G)$ as a reflection. Such a permutation w switches the p blocks in $\vec{x}$ cyclically among themselves; in particular, under the action of w each element of the set $\{1, 2, \ldots, pN\}$ has an orbit of size at least p. The kernel of $(w-1)$ acting of $V_p(U(pN))$ thus has dimension at most $pN/p = N$, so the rank of $(w-1)$ acting on

$V_p(U(pN))$ is at least $N(p-1)$. This implies that the rank of $(w-1)$ acting on $V_p(SU(pN)$ is at least $N(p-1)-1$. The kernel of the map $V_p(SU(pN)) \to V_p(G)$ has dimension 1, so the action of $(w-1)$ on $V_p(SU(pN)/C_p)$ has rank at least $N(p-1)-2$. Thus, if $p>3$, w is not a mod p reflection. Similarly, if $p=3$ and $N>1$, w is not a mod p reflection.

For $p=2$ and N odd we need a different description of $V_p(G)$: it is the quotient of $V_p(U(2N))$ by the one dimensional submodule C generated by the element $\vec{z}=(1,\ldots,1)$. Let $\phi : V_p(U(2N)) \to V_p(G)$ denote the quotient map and let $v = \phi(0,\ldots,0,1,\ldots 1))$. The claim is that the stabilizer of v in W is $\Sigma_N \wr \Sigma_2$ and that no element in this stabilizer which is not in the normal subgroup $\Sigma_N \times \Sigma_N$ is a reflection mod 2. The argument is virtually identical to the one above, and we omit it □

Proof of 7.2. This is basically the same proof as 7.1, so we only sketch the calculation. Consider the vector

$$\vec{x} = ((0,1,\ldots(p-1)),(0,\ldots,(p-1)))$$

in $V_p(SU(p)\times SU(p))$, and let v be the image of $\vec{x}$ in $V_p(G)$. The subgroup of W which stabilizes $\vec{x}$ is trivial. Let $\vec{z}$ denote the element $((1,\ldots,1),(1,\ldots,1)$ of $V_p(SU(p)\times SU(p))$, so that $\vec{z}$ generates the kernel of $\phi : V_p(SU(p)\times SU(p)) \to G$. As in 7.1, if $T(v)=(v)$, then $T^p\vec{x} = \vec{x} + a\vec{z}$. This implies that $T^p = Id$ and that W_v is at most a $\mathbb{Z}/p\mathbb{Z}$. But the diagonal cyclic permutation subgroup $\mathbb{Z}/p\mathbb{Z} \subset \Sigma_p \times \Sigma_p$ is in W_v, so $W_v \cong \mathring{A}/p$. Let T be any generator of this diagonal cyclic permutation subgroup of W. The rank of $T-1$ on $V_p(SU(p)\times SU(p))$ is $(2p-2)$. This implies that the rank of $T-1$ on the image of ϕ is at least $(2p-2)-1$. Hence for $p>2$, T is not a reflection mod p. □

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