Localization and Cellularization of Principal Fibrations

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Abstract. We prove that any cellularization of a principal fibration is again a principal fibration, and that any localization of a principal fibration with respect to a suspended map is again a principal fibration. The structure group of the new fibration is not necessarily the cellularization (localization) of the original structure group; however, they share the same cellularization (localization).

1. Introduction

Let Cell_A denote the A-cellular approximation functor associated to a pointed space A, and let L_f denote the functor given by localization with respect to a map f between pointed spaces (see [**1**], [**5**], [**7**] or 1.1). Given a principal fibration $E \to X$ over a connected space X, we show in this note that the induced maps $\mathrm{Cell}_A E \to \mathrm{Cell}_A X$ and $L_{\Sigma f} E \to L_{\Sigma f} X$ are also equivalent to principal fibrations. The appearance of the suspension in $L_{\Sigma f}$ is essential (§4).

Let G be the homotopy fibre of $E \to X$, or in other words the group of the principal fibration. It turns out that the fibre of $\mathrm{Cell}_A E \to \mathrm{Cell}_A X$ is *not* in general equivalent to $\mathrm{Cell}_A G$ (but see 2.2). For a simple example of this, let $A = S^{n+1}$, so that Cell_A is the n-connected Postnikov cover functor (for $n = 1$, the universal cover functor). If the map $\pi_{n+1}E \to \pi_{n+1}X$ is not surjective, the homotopy fibre of $\mathrm{Cell}_A E \to \mathrm{Cell}_A X$ has nontrivial homotopy in dimension n, and so this homotopy fibre is not even A-cellular, much less equivalent to $\mathrm{Cell}_A G$.

Along the same lines, if f is the map $S^n \to *$, then $L_{\Sigma f}$ is the n'th Postnikov section functor. Again, if $\pi_{n+1}E \to \pi_{n+1}X$ is not surjective the homotopy fibre of $L_{\Sigma f} E \to L_{\Sigma f} X$ is not equivalent to $L_{\Sigma f} G$ (but see 3.2).

Relationship to previous work. The behavior of fibration sequences under localization functors and cellularization functors is considered in [**1**], [**3**], and [**5**]. The general conclusion is that localization and cellularization functors preserve neither fibration sequences nor principal fibration sequences, although under

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additional assumptions the failure is measured by a finite product of Eilenberg-Mac Lane spaces [**5**] [**1**]. Here we show that the functors Cell_A and $L_{\Sigma f}$ at least respect the principal nature of fibrations. In §4 we describe a case in which a localization functor L_f (no suspension on the f) takes a principal fibration to a map which is not equivalent to a principal fibration.

Recall that a space is said to be a polyGEM if it is built up from generalized Eilenberg-Mac Lane spaces (GEMs) by a finite number of principal fibrations. It is known that any cellularization or localization of a GEM is again a GEM. This paper grew out of an effort to answer the long-standing question of whether any cellularization or localization of a polyGEM is again a polyGEM.

1.1. Notation, terminology, and background. We use $\mathcal{S}$ for the category of spaces and $\mathcal{S}_*$ for the category of pointed spaces; the word "equivalence" in either of these categories stands for weak homotopy equivalence; "connected" refers to path connectivity and "component" to path component. A map $E \to X$ is *equivalent to a principal fibration* if there exists a classifying map $X \to BG$ such that $E \to X \to BG$ is a homotopy fibre sequence, or more generally if such a classifying map exists after taking the homotopy pullback of $E \to X$ over a CW-approximation to X.

The functor Cell_A takes $\mathcal{S}_*$ to $\mathcal{S}_*$, and in discussing the functor Cell_A we always assume that A is a pointed, connected CW-complex. Let Map_* denote the space of basepoint-preserving maps. Call a map $f \in \mathcal{S}_*$ an *A-cellular equivalence* if $\mathrm{Map}_*(A, f)$ is an equivalence. Given $X \in \mathcal{S}_*$, the map $\mathrm{Cell}_A(X) \to X$ is characterized up to homotopy by three properties [**5**]: $\mathrm{Cell}_A(X)$ is an ordinary cell complex, $\mathrm{Cell}_A(X) \to X$ is an A-cellular equivalence, and $\mathrm{Map}_*(\mathrm{Cell}_A(X), f)$ is an equivalence whenever f is an A-cellular equivalence. These properties imply that the functor Cell_A is idempotent up to homotopy and transforms A-cellular equivalences into homotopy equivalences.

The functor L_f takes $\mathcal{S}$ to $\mathcal{S}$, but in discussing L_f we often assume that f is a map between pointed CW-complexes. The basepoint is for convenience; it plays no role in the construction of $L_f X$. Let Map denote the ordinary mapping space and $\mathrm{Map}^{\mathrm{h}}$ the derived mapping space obtained by replacing the domain space by an equivalent cell complex. Say that a space $Y \in \mathcal{S}$ is *L_f-local* if $\mathrm{Map}(f, Y)$ is an equivalence, and that a map $g \in \mathcal{S}$ is an *L_f-equivalence* if $\mathrm{Map}^{\mathrm{h}}(g, Y)$ is an equivalence for every L_f-local space Y. The map $X \to L_f X$ is determined up to equivalence by two properties: $X \to L_f X$ is an L_f-equivalence and $L_f X$ is L_f-local. These properties imply that L_f is idempotent up to equivalence and transforms L_f-equivalences into equivalences.

We say that a functor F between appropriate categories of spaces is *continuous* if it preserves equivalences; this property guarantees that up to equivalence F can be applied fibrewise in a fibration [**5**, I.F]. The functor F is said to be *augmented* if there are natural maps $FX \to X$, and *co-augmented* if there are natural maps $X \to FX$.

2. The cellularization of a principal fibration

In this section we prove the following theorem.

2.1. Theorem. *Suppose that $E \to X$ in $\mathcal{S}_*$ is equivalent to a principal fibration and that A is a pointed, connected CW-complex. Then the natural map* $\mathrm{Cell}_A E \to \mathrm{Cell}_A X$ *is equivalent to a principal fibration.*

2.2. Remark. In the setting of 2.1, let G be the structure group of $E \to X$ and G' the structure group of $\mathrm{Cell}_A E \to \mathrm{Cell}_A X$. It follows from the fact that Map_* preserves fibration sequences that $G \to G'$ is an A-cellular equivalence, and so induces an equivalence $\mathrm{Cell}_A G \to \mathrm{Cell}_A G'$.

The proof needs some terminology and a few lemmas.

2.3. Definition. Suppose that $C : \mathcal{S}_* \to \mathcal{S}_*$ is an augmented continuous functor (1.1). The functor C is said to have a *natural presentation as a homotopy fibre* if there exists a co-augmented continuous functor $Q : \mathcal{S} \to \mathcal{S}$ such that for any pointed space X the augmentation $CX \to X$ and the co-augmentation $X \to QX$ combine to give a homotopy fibre sequence

$$CX \to X \to QX\,.$$

2.4. Remark. In forming QX, X is to be treated as an unpointed space by forgetting the basepoint. The above condition amounts to the statement that CX is naturally weakly homotopy equivalent, as a pointed space, to the homotopy fibre of $X \to QX$ over the image in QX of the basepoint in X. (Note that the basepoint of X also gives a natural basepoint for this homotopy fibre.)

For the rest of this section A is a pointed connected CW-complex.

2.5. Lemma. *Suppose that $C : \mathcal{S}_* \to \mathcal{S}_*$ is an augmented continuous functor which has a natural presentation as a homotopy fibre. Then if $E \to X$ in $\mathcal{S}_*$ is equivalent to a principal fibration, so is the composite map $CE \to E \to X$.*

Proof. Let Q be the functor associated as in 2.3 to the presentation of C as a homotopy fibre. Let $X \to BG$ be the classifying map for $E \to X$, so that there is a homotopy fibre sequence $E \to X \to BG$. Consider the following diagram, in which all of the rows and columns are homotopy fibre sequences.

$$\begin{array}{ccccc}
CE & \longrightarrow & E & \longrightarrow & QE \\
\big\downarrow{\scriptstyle =} & & \big\downarrow & & \big\downarrow \\
CE & \longrightarrow & X & \longrightarrow & Q'X \\
\big\downarrow & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p'} \\
* & \longrightarrow & BG & \xrightarrow{=} & BG
\end{array}$$

Here p' is obtained by applying the functor Q fibrewise to p (1.1); since p does not necessarily have a section (i.e., the fibres of p are not necessarily supplied with compatible basepoints) it is necessary in forming $Q'X$ that Q be given as a functor of unpointed spaces. The middle row exhibits $X \to Q'X$ as a classifying map for $CE \to X$. □

2.6. Lemma. *The functor* Cell_A *has a natural presentation as a homotopy fibre.*

Proof. This lemma is due in a slightly different form to Chacholski [**4**], and we will use terminology from his paper in translating his result into the above one. (This translation can also be achieved by an appeal to [**4**, 20.3].) Let P denote the functor given by localization (1.1) with respect to the map $f : \Sigma A \to *$; this is also called the nullification functor with respect to ΣA. Given a pointed space X, let MX be obtained from X by attaching copies of $\mathrm{Cone}(A)$ along maps $h : A \to X$ running through a set of representatives for the homotopy classes of pointed maps $A \to X$. (What Chacholski uses in [**4**] is the homotopy cofibre of $\bigvee_h A \to X$, but this is the same up to homotopy.) Chacholski proves that $\mathrm{Cell}_A(X)$ is equivalent to the homotopy fibre of the composite map $X \to MX \to P(MX)$ over the basepoint in MX given by the image of the basepoint of X. This presents two problems for us: the formation of MX is not functorial (choices of representatives for homotopy classes are involved) and the formation of MX seems to depend on the basepoint in X (the choices involve representatives of homotopy classes of pointed maps). But there is a simple remedy: let $\hat{M}X$ be obtained from X be attaching a copy of $\mathrm{Cone}(A)$ for *every* map $A \to X$. Certainly $\hat{M}X$ is not usually equivalent to MX, but we leave it to the reader to check that the inclusion $MX \to \hat{M}X$ induces an equivalence between the basepoint component $P(MX)_0$ of $P(MX)$ and the basepoint component $P(\hat{M}X)_0$ of $P(\hat{M}X)$. The point is that $(\hat{M}X)_0$ is obtained from $(MX)_0$ by attaching copies of $\mathrm{Cone}(A)$ via maps $A \to (MX)_0$ which are null homotopic. These attachments essentially wedge on copies of ΣA which, by the basic properties of a localization functor (1.1), can be coned off without affecting the space that results from subsequently applying P. It follows that $\mathrm{Cell}_A(X)$ is equivalent to the homotopy fibre of $X \to P(\hat{M}X)$, and so $\mathrm{Cell}_A(X) \to X \to P(\hat{M}X)$ is the desired presentation of $\mathrm{Cell}_A(X)$ as a natural homotopy fibre.

The fact that the functor $X \mapsto P(\hat{M}X)$ is continuous follows from the fact that $X \mapsto P(MX)$ is. Each component of $P(\hat{M}X)$ is equivalent to $P(MY)_0$, where Y is X with a strategically chosen basepoint. □

2.7. Remark. If $A = S^{n+1}$, so that Cell_A is the n-connected Postnikov cover functor, the space $P(\hat{M}X)$ above is the n'th Postnikov stage of X.

2.8. Lemma. *Let $E \to X$ be as in 2.1, and let E' be defined by the homotopy fibre square*

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ \mathrm{Cell}_A(X) & \longrightarrow & X \end{array} .$$

Then the natural map $\mathrm{Cell}_A(E') \to \mathrm{Cell}_A(E)$ is an equivalence.

Proof. By the basic properties of Cell_A (1.1), it is enough to check that the map $\mathrm{Map}_*(A, E') \to \mathrm{Map}_*(A, E)$ is an equivalence. This follows from the fact that $\mathrm{Map}_*(A, –)$ preserves homotopy fibre squares and the fact that $\mathrm{Cell}_A(X) \to X$ is an A-cellular equivalence (1.1). □

Proof of 2.1. Let $E' \to \mathrm{Cell}_A(X)$ be as in 2.8. By construction this map is equivalent to a principal fibration, and since $\mathrm{Cell}_A(E')$ is equivalent to $\mathrm{Cell}_A(E)$ (2.8) it is enough to prove that the composite $\mathrm{Cell}_A(E') \to E' \to \mathrm{Cell}_A(X)$ is equivalent to a principal fibration. This follows from 2.5 and 2.6. □

3. The localization $L_{\Sigma f}$ of a principal fibration

In this section we prove the following theorem.

3.1. THEOREM. *Suppose that $E \to X$ in $\mathcal{S}$ is equivalent to a principal fibration. Assume that E and X are connected, and let f be a map between pointed CW-complexes. Then the natural map $L_{\Sigma f}E \to L_{\Sigma f}X$ is equivalent to a principal fibration.*

3.2. REMARK. In the setting of 3.1, let G be the structure group of $E \to X$ and G' the structure group of $L_{\Sigma f}E \to L_{\Sigma f}X$. The proof of 3.1 below shows that the natural map $G \to G'$ induces an equivalence $L_f G \to L_f G'$.

3.3. REMARK. We do not know whether or not the connectivity assumption on E can be removed from 3.1 (we would like to thank the referee for questioning a claim to the contrary in an earlier draft) but the connectivity assumption on X is necessary. If $X = X_0 \sqcup X_1$ is the disjoint union of two connected spaces, and

$$p = p_0 \sqcup p_1 : E_0 \sqcup E_1 \to X_0 \sqcup X_1$$

is a principal fibration over X with E_0 and E_1 connected, then the maps $L_{\Sigma f}(p_0)$ and $L_{\Sigma f}(p_1)$ are equivalent to principal fibrations (3.1) but $L_{\Sigma f}(p) \sim L_{\Sigma f}(p_0) \sqcup L_{\Sigma f}(p_1)$ is not necessarily principal. The problem is that the group associated to $L_{\Sigma f}(p_0)$ might not be the same as the group associated to $L_{\Sigma f}(p_1)$.

The following proposition describes the main construction we use. In the statement of this proposition and in what follows we repeatedly refer to “the” (homotopy) fibre of a map to a connected space B. By this we mean the (homotopy) fibre over some chosen basepoint in B.

3.4. PROPOSITION. *Suppose that $p : Y \to B$ is a fibration between connected spaces. Assume that the fibre of p is both connected and $L_{\Sigma f}$-local (1.1). Then there exists a space B', depending functorially on p, which lies in a homotopy fibre square*

$$\begin{array}{ccc} Y & \longrightarrow & L_{\Sigma f}Y \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' . \end{array}$$

Moreover, the map $B \to B'$ induces an equivalence $L_{\Sigma f}B \sim L_{\Sigma f}B'$.

The assumption that the fibre of p is connected is needed in the proof of 3.4 to show that the homotopy fibre powers of Y over B are connected. The proof of 3.4 depends on a few lemmas.

3.5. LEMMA. *Suppose that $p : Y \to B$ is a fibration between connected spaces. Assume that p has a section $s : B \to Y$, and let F be the fibre of p. Then the homotopy fibre of s is naturally weakly homotopy equivalent to ΩF.*

PROOF. Let U be the homotopy fibre of $B \to Y$. The space U can be identified by examining the following 3×3 fibration square (each row and each column of which is a fibration sequence).

$$\begin{array}{ccccc}
\Omega F & \longrightarrow & * & \longrightarrow & F \\
\Big\downarrow{\scriptstyle\sim} & & \Big\downarrow & & \Big\downarrow \\
U & \longrightarrow & B & \xrightarrow{s} & Y \\
\Big\downarrow & & \Big\downarrow{\scriptstyle=} & & \Big\downarrow{\scriptstyle p} \\
* & \longrightarrow & B & \xrightarrow{=} & B
\end{array}$$

The diagram shows that to obtain the desired naturality, the basepoint in F used to form ΩF should be the image under s of the basepoint in B that specifies F. □

Suppose that B is connected. Given a map $p : Y \to B$ with homotopy fibre F, we say that L_f *strongly preserves* p if the natural map from F to the homotopy fibre F' of $L_f Y \to L_f B$ is an equivalence.

3.6. Lemma. *Suppose that $p : Y \to B$ is a fibration between connected spaces. Assume that p has a section $s : B \to Y$, and that the fibre F of p is $L_{\Sigma f}$-local. Then $L_{\Sigma f}$ strongly preserves both p and s.*

Proof. Assume without loss of generality that F, Y, and B have basepoints that are compatible under the maps $F \to Y$, $Y \to B$ and $B \to Y$; note that F is connected (because p has a section). Recall the following properties of $L_{\Sigma f}$:

(1) If X is a pointed space, there is a natural weak homotopy equivalence between $\Omega L_{\Sigma f} X$ and $L_f \Omega X$ [**5**, 3.A.1].
(2) The functor L_f preserves products, i.e., for any U and V the natural map $L_f(U \times V) \to L_f(U) \times L_f(V)$ is an equivalence [**5**, 1.A.8].

The fibration Ωp is a fibration of loop spaces with a section Ωs, and so it is equivalent to a product fibration. By (1) ΩF is L_f-local, and so by (2) L_f strongly preserves Ωp. Another application of (1) shows that $L_{\Sigma f}$ strongly preserves p. It follows from 3.5 that $L_{\Sigma f}$ strongly preserves s as well. □

3.7. Simplicial spaces. Suppose that U_* is a *simplicial space*, in other words, a functor $\Delta^{\mathrm{op}} \to \mathcal{S}$ [**6**, I.1]. By the *realization* $|U_*|$ of U_* we mean the geometric realization of the diagonal [**6**, IV.1] of the bisimplicial set obtained by applying the singular complex functor [**6**, I.1] levelwise to U. The space $|U_*|$ is one model for the homotopy colimit of U_* (cf. [**3**, XII.4.3]); it is weakly equivalent to a less elaborate construction if the degeneracy maps in U_* are closed cofibrations [**9**, A.1(iv)].

The following proposition is a restatement of a result of Bousfield and Friedlander [**2**] [**6**, IV.4]. A *basepoint* for a simplicial space U_* is a basepoint in U_0; the images of this basepoint under iterated degeneracy maps give compatible basepoints for U_n, $n \geq 0$. If $U_* \to V_*$ is a map of simplicial spaces and V_* is pointed, then there is an associated fibre simplicial space Z_*, where Z_n is the inverse image in U_n of the basepoint in V_n.

3.8. Proposition. *Suppose that $U_* \to V_*$ is a map of simplicial spaces. Assume that V_* is a simplicial object in the category of pointed connected spaces, and that each map $U_n \to V_n$ is a fibration of spaces; let Z_* be the fibre of $U_* \to V_*$. Then there is a natural fibration sequence of spaces*

$$|Z_*| \to |U_*| \to |V_*| \,.$$

If Y is a space, let $\operatorname{cosk}_0 Y$ denote the simplicial space which in degree n contains $Y^{n+1} \cong \operatorname{Map}(\Delta[n]_0, Y)$; face maps are given by deletions and degeneracy maps by repetition [**6**, VII.1]. For example, if G is a discrete group, $\operatorname{cosk}_0 G$ is the usual simplicial model for EG. If $Y \to B$ is a fibration, let $\operatorname{cosk}_0^B Y$ denote the analogous simplicial space which in degree n contains the $(n+1)$-fold fibre power $\times_B^{n+1} Y$ of Y over B.

Let cB be the constant simplicial space with B at each level and with all of the face and degeneracy maps given by identities. Note that there is a natural map $\operatorname{cosk}_0^B Y \to \operatorname{cosk}_0^B B \cong cB$ of simplicial spaces.

3.9. Lemma. *For any nonempty space Y, $|cY|$ is weakly equivalent to Y and $|\operatorname{cosk}_0 Y|$ is contractible. For any surjective fibration $Y \to B$ of spaces, the natural map $|\operatorname{cosk}_0^B Y| \to |cB|$ is an equivalence.*

Proof. The first two statements are elementary. When it comes to the third, at the cost of working component by component in B we can assume that B is connected and pointed. Let F be the fibre of $Y \to B$. The proof consists in observing that the fibre of $\operatorname{cosk}_0^B Y \to cB$ is $\operatorname{cosk}_0 F$, and then applying 3.8 and the second statement of the lemma. □

Proof of 3.4. Let F be the fibre of $Y \to B$ and $\times_B^n Y$ the n-fold fibre power of Y over B. Any one of the projection maps $\times_B^n Y \to Y$ has a connected $L_{\Sigma f}$-local fibre F^{n-1}, and also a section given by the diagonal map $d : Y \to \times_B^n Y$. It follows from 3.6 that $L_{\Sigma f}$ strongly preserves both q and d. Let $U_* = cY$ be a constant simplicial space and let $V_* = \operatorname{cosk}_0^B Y$; diagonal maps as above give a map $U_* \to V_*$. The realization $|U_*|$ is equivalent to Y and $|V_*|$ is equivalent to B (3.9); under these equivalences the map $|U_*| \to |V_*|$ corresponds to $Y \to B$. Let Z_* be the fibre of $U_* \to V_*$. Since each V_n is connected, it follows from 3.8 that there is a natural fibration sequence

$$|Z_*| \to |U_*| \to |V_*|$$

so that in particular $|Z_*| \sim F$. (It's interesting to relate Z_* to the bar construction on ΩF.) As indicated above, the functor $L_{\Sigma f}$ strongly preserves the fibration sequences $Z_n \to U_n \to V_n$, and so again by 3.8 there is a fibration sequence

$$F \sim |Z_*| \to |L_{\Sigma f} U_*| \to |L_{\Sigma f} V_*| \,.$$

Since $|L_{\Sigma f} U_*| \sim L_{\Sigma f} Y_*$ (recall that $U_* = cY$ and use 3.9), this gives a fibration sequence $F \to L_{\Sigma f} Y \to B'$. It is easy to check that $L_{\Sigma f} Y \to B'$ fits into the desired homotopy fibre square. The final statement is a consequence of [**5**, 1.D.3] and the homotopy colimit formula for B'. □

Proof of 3.1. Let $X \to B$ be a classifying map for $E \to X$; by adjusting the spaces involved up to equivalence, we can assume that $X \to B$ is a fibration with fibre E. Let $Y \to B$ be obtained by applying $L_{\Sigma f}$ fibrewise to $X \to B$, so that the homotopy fibres of $Y \to B$ are equivalent to $L_{\Sigma f} E$ and there is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B & \xrightarrow{=} & B \end{array}$$

which induces $E \to L_{\Sigma f}E$ on homotopy fibres and induces an equivalence $L_{\Sigma f}X \to L_{\Sigma f}Y$. The map $L_{\Sigma f}X \sim L_{\Sigma f}Y \to B'$ provided by 3.4 is the desired classifying map for $L_{\Sigma f}E \to L_{\Sigma f}X$; the same lemma gives the statement in 3.2.

4. The localization L_f of a principal fibration

We give an example of a principal fibration $E \to X$ and a localization functor L_f such that the map $L_f E \to L_f X$ is not equivalent to a principal fibration. This shows that the appearance of Σf in 3.1 is essential.

Let π be a nonabelian finite subgroup of the group S^3 of unit quaternions, for instance, the subgroup generated by the quaternions $\{i, j, k\}$. The left translation action of π on S^3 gives principal fibration sequences

$$\pi \to S^3 \to S^3/\pi \qquad \text{and} \qquad S^3 \to S^3/\pi \to B\pi\,.$$

We will concentrate on the right-hand sequence. Let L_f be localization with respect to the map $f : B\pi \to *$, or equivalently nullification with respect to $B\pi$ (see the proof of 2.6). According to H. Miller's solution of the Sullivan Conjecture [**8**], $L_f(S^3/\pi)$ is equivalent to S^3/π, because any finite complex such as S^3/π is L_f-local (1.1). On the other hand, for trivial reasons $L_f(B\pi)$ is contractible. Applying L_f to the principal fibration $S^3/\pi \to B\pi$ thus gives the map $S^3/\pi \to *$. This map is certainly not equivalent to a principal fibration, because, for instance, S^3/π has a nonabelian fundamental group (namely, π) and so cannot be equivalent to a topological group.

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