Maps Between Classifying Spaces

by

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§1. Introduction.

Suppose that π is a finite group and that G is a compact Lie group. If $\rho\colon \pi \longrightarrow G$ is a homomorphism, let $Z_G(\rho)$ denote the centralizer in G of the image of ρ. The group $Z_G(\rho)$ is a closed subgroup of G, and the obvious group homomorphism

$$Z_G(\rho)\times\pi \longrightarrow G$$

passes to a classifying space map

$$BZ_G(\rho)\times B\pi \longrightarrow BG$$

which has as adjoint a map

$$BZ_G(\rho) \longrightarrow \mathrm{Hom}(B\pi,BG).$$

(Here $\mathrm{Hom}(B\pi,BG)$ is the space of unpointed maps from $B\pi$ to BG.) Taking a disjoint union over G-conjugacy classes $\langle\rho\rangle$ of homomorphisms $\rho\colon \pi \longrightarrow G$ gives a map

$$\coprod_{\langle\rho\rangle} BZ_G(\rho) \longrightarrow \mathrm{Hom}(B\pi,BG).$$

Fix some prime number p. The purpose of this paper is to prove the following theorem.

<u>1.1</u> <u>Theorem</u>. If π is a finite p-group and G is a compact Lie group, then the above map

$$\coprod_{\langle\rho\rangle} BZ_G(\rho) \longrightarrow \mathrm{Hom}(B\pi, BG).$$

is a strong mod p equivalence.

Remark. A map $f\colon X \longrightarrow Y$ of spaces is a strong mod p equivalence if it satisfies the following three conditions:

(i) f induces an isomorphism $\pi_0 X \longrightarrow \pi_0 Y$.

(ii) f induces an isomorphism $\pi_1(X,x) \longrightarrow \pi_1(Y,f(x))$ for each basepoint x in X.

(iii) f induces an isomorphism

$$H_*(\tilde{X}_x, \mathbb{Z}/p) \longrightarrow H_*(\tilde{Y}_{f(x)}, \mathbb{Z}/p)$$

for each basepoint x in X.

Here $\tilde{X}_x$ denotes the universal cover of the component of $\tilde{X}$ containing x and $\tilde{Y}_{f(x)}$ the universal cover of the component of $\tilde{Y}$ containing $f(x)$. Although there is no unique map $\tilde{f}\colon \tilde{X}_x \longrightarrow \tilde{Y}_{f(x)}$ induced by f, any two of the obvious candidates differ by a covering transformation of $\tilde{Y}_{f(x)}$, so that condition (iii) has an invariant meaning.

Remark. Theorem 1.1 is based upon a remarkable result of H. Miller (see §2) Lashof, May and Segal [LMS] have proved a statement like 1.1 under the assumption that G is abelian.

A groupoid reformulation. A groupoid is a small category in which every morphism is invertible. Any groupoid V has a classifying space or nerve BV [BK, XI, §2] [S, §4]. If V and W are two groupoids, Hom(V,W) will denote the groupoid in which objects are functors $V \longrightarrow W$ and morphisms are natural transformations. There is a natural map (cf. [S, Prop. 2.1])

$$BHom(V,W) \longrightarrow Hom(BV,BW).$$

A <u>compact Lie groupoid</u> is a groupoid V in which the morphism sets are compact differentiable manifolds in such a way that the composition and inverse maps are smooth; this extra topology is taken into account in forming the classifying space BV [S, §5]. Just as a group can be treated as a groupoid with one object, so a compact Lie group can be treated as a compact Lie groupoid with one object; in this interpretation the classifying space construction for (compact Lie) groupoids specializes to the usual one for (compact Lie) groups. If π is a finite group and V is a compact Lie group, then $Hom(\pi,V)$ is naturally a compact Lie groupoid and Theorem 1.1 can be restated in the following form.

<u>1.1′ Theorem</u>. If π is a finite p-group and G is a compact Lie group, then the natural map

$$BHom(\pi,G) \longrightarrow Hom(B\pi,BG)$$

is a strong mod p equivalence.

This is the statement that we will work with through most of the rest of the paper. Note that the map of 1.1′ is actually a homotopy equivalence if π and G are discrete groups.

<u>Notation and terminology</u>. Throughout the paper, p will denote a fixed prime number. The symbol R will denote the ring $\mathbb{Z}/p$, π a finite p-group, and σ the cyclic group of order p. All homology and cohomology is taken with coefficients in R. Most groupoids are compact Lie groupoids, and any functor

between two such groupoids is understood to give smooth maps between the appropriate morphism spaces.

When it comes to constructing function spaces and completions, the reader is expected to replace the spaces involved by their singular complexes and to work with simplicial techniques [BK, Part 2] [Ma].

§2. Homotopy fixed point sets.

Suppose that X is a space with an action of the p-group π. Let X^{π} and $X^{h\pi}$ denote respectively the fixed-point set and the homotopy fixed-point set of this action. By definition [BK, XI, §3], $X^{h\pi}$ is $\mathrm{Hom}_{\pi}(E\pi,X)$, where $\mathrm{Hom}_{\pi}(--,--)$ denotes the space of π-maps and $E\pi$ is the universal cover of the standard simplicial $B\pi$. The unique π-map $E\pi \longrightarrow *$, where $*$ is the one-point space with a trivial π action, induces a map

$$X^{\pi} = \mathrm{Hom}_{\pi}(*,X) \longrightarrow \mathrm{Hom}_{\pi}(E\pi,X) = X^{h\pi}.$$

Let R denote the ring $\mathbb{Z}/p$, and R_{∞} the R-completion functor of [BK, Part I]. Functoriality gives a composite map

$$R_{\infty}(X^{\pi}) \longrightarrow R_{\infty}(X)^{\pi} \longrightarrow (R_{\infty}X)^{h\pi}$$

which fits into a commutative diagram

$$\begin{array}{ccc} X^{\pi} & \longrightarrow & X^{h\pi} \\ \downarrow & & \downarrow \\ R_{\infty}(X^{\pi}) & \longrightarrow & (R_{\infty}X)^{h\pi} \end{array}$$

The following theorem is at the foundation of the rest of the paper.

2.1 Theorem. [Mi] If π is a finite p-group and X is a finite π-complex, the above map

$$R_{\infty}(X^{\pi}) \longrightarrow (R_{\infty}X)^{h\pi}$$

is a homotopy equivalence.

We will need the following consequence of 2.1.

2.2 Proposition. If π is a finite p-group, G is a compact Lie group, and π

acts on G via group automorphisms, then the natural map

$$G^{\pi} \longrightarrow G^{h\pi}$$

induces an isomorphism on mod p homology.

The proof of 2.2 rests on two lemmas. Recall that a nilpotent space X is mod p acyclic (ie. $\tilde{H}_*(X) = 0$) iff X is connected and the homotopy groups of X are uniquely p-divisible [BK, V, 2.7, 3.3].

<u>2.3</u> <u>Lemma</u>. If the π-space X is nilpotent and mod p acyclic, then $X^{h\pi}$ is also nilpotent and mod p acyclic.

<u>Proof</u>. The space $X^{h\pi}$ can be identified with the space of sections of the fibration over $B\pi$ associated to the action of π on X. The group $H^i(\pi,A)$ vanishes whenever A is a uniquely p-divisible abelian group with a π-action, so a direct obstruction-theory argument in the above fibration shows that $X^{h\pi}$ is non-empty and connected. (This obstruction-theory argument involves in the very lowest dimension filtering $\pi_1 X$ by its lower central series subgroups and observing that the associated graded is uniquely p-divisible.) It follows that there is an invariantly defined action of π on $\pi_* X$; some further obstruction theory shows that the homotopy groups of $X^{h\pi}$ are exactly the fixed subgroups of this action. It is now not hard to see that $\pi_i X^{h\pi}$ is uniquely p-divisible -- this is trivial for $i>1$ and follows again by induction on the lower central series filtration of $\pi_1 X$ if $i=1$. The lemma is immediate.

<u>2.4</u> <u>Lemma</u>. Suppose that X is a connected nilpotent space upon which the

group π acts. If either

(i) X is 1-connected, or

(ii) $X = K(A,1)$ where A is a finitely-generated abelian group

then the natural map $X^{h\pi} \longrightarrow (R_\infty X)^{h\pi}$ induces an isomorphism on mod p homology.

<u>Remark</u>. Condition (ii) can be weakened to

(ii)′ X has only a finite number of non-zero homotopy groups.

Some condition on X beyond nilpotency is needed in 2.4.

<u>Proof of 2.4</u>. If X is one-connected, then the homotopy fiber F of the map $X \longrightarrow R_\infty X$ is a nilpotent space which is mod p acyclic. For each point x in $(R_\infty X)^{h\pi}$, the homotopy fiber F_x over x of the map $X^{h\pi} \longrightarrow (R_\infty X)^{h\pi}$ is essentially the homotopy fixed point set of an action of π on F determined by x. By lemma 2.3, F_x is mod p acyclic; this finishes case (i). For (ii), note that $R_\infty X = K(\hat{A},1)$, where $\hat{A}$ is the tensor product of A with the ring of p-adic integers. Let A' be the quotient of A by its subgroup of elements of finite order prime to p. The exact sequence

$$0 \longrightarrow A' \longrightarrow \hat{A} \longrightarrow \hat{A}/A' \longrightarrow 0$$

where $\hat{A}/A'$ is uniquely p-divisible, easily gives that $H^i(\pi,A)$ is isomorphic to $H^i(\pi,\hat{A})$ for $i>0$ and that the map $H^0(\pi,A) \longrightarrow H^0(\pi,\hat{A})$ has kernel and cokernel which are uniquely p-divisible. By interpreting $X^{h\pi}$ as the space of sections of the fibration over $B\pi$ associated to the π-action on X, one sees that $X^{h\pi}$ is non-empty iff a certain extension class in $H^2(\pi,A)$ vanishes. This extension class vanishes iff its image in the isomorphic group $H^2(\pi,\hat{A})$ vanishes, so that $X^{h\pi}$ is non-empty iff $(R_\infty X)^{h\pi}$ is non-empty. If $X^{h\pi}$ is non-empty then, given a choice of basepoint, its components are in natural bijective correspondence with $H^1(\pi,A)$, and each component has the homotopy type of $K(B,1)$ where $B = H^0(\pi,A)$. Since the same is true of $(R_\infty X)^{h\pi}$ with A replaced with $\hat{A}$, the lemma follows in a straightforward way.

Proof of 2.2. Suppose first of all that G is not connected. Write

$$G = \coprod_{i=1}^{n} G_i$$

where each G_i is a union of components of G, G_0 is the identity component of G, and the subsets $\pi_0 G_i$ of $\pi_0 G$ run through the orbits of π on $\pi_0 G$. There are parallel decompositions

$$G^\pi = \coprod_i (G_i)^\pi$$

$$G^{h\pi} = \coprod_i (G_i)^{h\pi}.$$

If $(G_i)^\pi = \phi$, then $(G_i)^{h\pi} = \phi$ by [J] [DFZ] so that the map $(G_i)^\pi \longrightarrow (G_i)^{h\pi}$ is vacuously a homology isomorphism. If $(G_i)^\pi$ is non-empty, it is clear that G_i consists of a single connected component of G and that left multiplication by any element x of $(G_i)^\pi$ produces a π-homeomorphism $G_0 \longrightarrow G_i$. This implies that the map $(G_i)^\pi \longrightarrow (G_i)^{h\pi}$ is a homology equivalence iff the map $(G_0)^\pi \longrightarrow (G_0)^{h\pi}$ is a homology equivalence. This fact justifies restricting attention to G_0 or, equivalently, assuming that G is connected.

By [BtD, p. 229, 233], it is possible to find a finite cover $\tilde{G}$ of G such that $\tilde{G}$ is isomorphic as a group to $K\times T$ where K is 1-connected semisimple and T is a torus. By passing to a further cover, if necessary, it is easy to arrange in addition that $\pi_1\tilde{G}$ be a π-invariant subgroup of $\pi_1 G$. By covering-space theory there is a unique action of π on $\tilde{G}$ which lifts the given action of π on G and fixes the identity element of $\tilde{G}$; this "canonical" action of π on $\tilde{G}$ is an action through group automorphisms. Let Q denote the deck transformation group $\pi_1 G/\pi_1\tilde{G}$. If the canonical action of π on $\tilde{G}$ is chosen as a basepoint, then up to conjugacy by elements of Q the actions of π on $\tilde{G}$ which lift the given action on G are in bijective correspondence with elements of the cohomology group $H^1(\pi,Q)$. For each $\alpha \in H^1(\pi,Q)$, choose an action of π on $\tilde{G}$ corresponding to α and let $\tilde{G}_\alpha$ denote the resulting π-space. Let $q\colon \tilde{G} \longrightarrow G$ be the projection map. It is easy to see that if α and β are distinct elements of $H^1(\pi,Q)$ the intersection $q(\tilde{G}_\alpha^\pi) \cap q(\tilde{G}_\beta^\pi)$ is empty and that consequently there is a disjoint union decomposition

$$G^\pi = \coprod_{\alpha\in H^1(\pi,Q)} q(\tilde{G}_\alpha^\pi).$$

For each α, the projection $\tilde{G}_\alpha^\pi \longrightarrow q(\tilde{G}_\alpha^\pi)$ is a principal covering map with group $Q^\pi = H^0(\pi,Q)$. This analysis can also be applied to the covering $q'\colon \mathrm{Hom}(E\pi,\tilde{G}) \longrightarrow \mathrm{Hom}(E\pi,G)$; in this case it gives a parallel disjoint union decomposition

$$G^{h\pi} = \coprod_{\alpha\in H^1(\pi,Q)} q'(\tilde{G}_\alpha^{h\pi})$$

such that for each α the projection $\tilde{G}_\alpha^{h\pi} \longrightarrow q'(\tilde{G}_\alpha^{h\pi})$ is a principal covering map with group Q^π. To finish the proof, it is enough to show that each map $\tilde{G}_\alpha^\pi \longrightarrow \tilde{G}_\alpha^{h\pi}$ is a homology isomorphism. If $\tilde{G}_\alpha^\pi = \phi$, then $\tilde{G}_\alpha^{h\pi} = \phi$ by [J] [DFZ], so that the desired result is vacuously true. If $\tilde{G}_\alpha^\pi$ is non-empty, then left multiplication by any element of $\tilde{G}_\alpha^\pi$ gives a π-homeomorphism $\tilde{G}_0 \longrightarrow \tilde{G}_\alpha$; this fact justifies restricting attention to the canonical action $\tilde{G}_0$ or, what comes to the same thing, assuming that $G = K \times T$ where K is 1-connected semisimple and T is a torus.

In this case, the action of π on $G = K \times T$ must preserve the sequence

$$1 \longrightarrow T \longrightarrow G \longrightarrow K \longrightarrow 1$$

since T is invariantly determined as the connected component of the center of G. There are no non-trivial group homomorphisms from K to T, so the above sequence is uniquely split and π must consequently preserve the given product decomposition of G. This means that there is an action of π on K and an action of π on T such that $G^{\pi} = K^{\pi} \times T^{\pi}$ and $G^{h\pi} = K^{h\pi} \times T^{h\pi}$. Let X be either K or T. The space X^{π} is nilpotent since it is a group; by [BK II], the map $X^{\pi} \longrightarrow R_{\infty}(X^{\pi})$ is a mod p homology isomorphism. By 2.1 the map $R_{\infty}(X^{\pi}) \longrightarrow (R_{\infty}X)^{h\pi}$ is a mod p homology isomorphism. By 2.4, the map $X^{h\pi} \longrightarrow (R_{\infty}X)^{h\pi}$ is a mod p homology isomorphism. These statements immediately imply that the map $X^{\pi} \longrightarrow X^{h\pi}$ is a mod p homology isomorphism, which completes the proof of the proposition.

§3. Homotopy classes of maps.

As usual, let σ denote the cyclic group of order p. The following proposition, which is the goal of this section, is a very special case of 1.1

3.1 Proposition. If G is a compact Lie group such that $\pi_0 G$ is a p-group, then the map of §1 induces an isomorphism

$$\pi_0 B\mathrm{Hom}(\sigma, G) \longrightarrow \pi_0 \mathrm{Hom}(B\sigma, BG).$$

Two basic results go into the proof of 3.1. Recall that R is the ring $\mathbb{Z}/p$ and that R_∞ is the R-completion functor of [BK, part I]. Let A-alg denote the category of unstable algebras over the mod p Steenrod algebra.

3.2 Theorem. [L] If X is a connected space of finite type, then the natural map

$$\pi_0 \mathrm{Hom}(B\sigma, R_\infty X) \longrightarrow \mathrm{Hom}_{A\text{-}alg}(H^*X,\ H^*B\sigma)$$

is an isomorphism.

Remark. The natural map of 3.2 is the edge homomorphism in the unstable Adams spectral sequence for the homotopy groups of $\mathrm{Hom}(B\sigma, X)$.

3.3 Theorem. [A] [MW] If G is a compact Lie group, then the natural map

$$\pi_0 B\mathrm{Hom}(\sigma, G) \longrightarrow \mathrm{Hom}_{A\text{-}alg}(H^*BG, H^*B\sigma)$$

is an isomorphism.

Proof of 3.1. In view of 3.2 and 3.3, it is enough to show that if X (eg. BG) is a connected space of finite type such that π_1X is a p-group, then the natural map $X \longrightarrow R_\infty X$ induces an isomorphism

$$\pi_0 \mathrm{Hom}(B\sigma, X) \longrightarrow \pi_0 \mathrm{Hom}(B\sigma, R_\infty X).$$

For such an X, $\pi_1 R_\infty X$ is isomorphic to $\pi_1 X$ [BK, IV, 5.1] and the universal cover of $R_\infty X$ is the p-completion of the universal cover of X [BK, II, 5.1]. It follows that the homotopy fiber F of the map $X \longrightarrow R_\infty X$ is a nilpotent space which is mod p acyclic; in particular, the homotopy groups of F are uniquely p-divisible and in this case the fundamental group of F is even abelian. The proof is completed by applying obstruction theory to the lifting diagram

$$\begin{tikzcd} & X \arrow[d] \\ B\sigma \arrow[r] \arrow[ur, dashed] & R_\infty X \end{tikzcd}$$

It is not hard to go a bit further (cf. proof of 2.4) and show that the map $\mathrm{Hom}(B\sigma, X) \longrightarrow \mathrm{Hom}(B\sigma, R_\infty X)$ induces an isomorphism on mod p homology.

§4. The cyclic group of order p.

The purpose of this section is to prove a few basic results which will be used in §5 to give a proof of 1.1 by induction on the order of the group π involved. Recall that σ denotes the cyclic group of order p.

4.1 Proposition. If G is a compact Lie group such that $\pi_0 G$ is a p-group, then the natural map

$$BHom(\sigma,G) \longrightarrow Hom(B\sigma,BG)$$

is a strong mod p equivalence.

The proof of 4.1 depends upon a well-known fact which perhaps was first pointed out by D. Sullivan.

4.2 Lemma. If G is a topological group, then the free loop space fibration $Hom(S^1,BG) \longrightarrow BG$ is fiber-homotopy equivalent to the fibration over BG associated to the action of G on itself by conjugation.

Proof. It is easy to see that the free loop space fibration extends to a homotopy fiber square

$$\begin{array}{ccc} Hom(S^1,BG) & \longrightarrow & BG \\ \downarrow & & \downarrow \Delta \\ BG & \xrightarrow{\Delta} & BG\times BG \end{array}$$

in which Δ is the diagonal map. Looping down shows that the right-hand vertical map is equivalent to the fibration over $BG\times BG$ associated to the natural action of $G\times G$ on the coset space $G\times G/\Delta(G)$. The lemma follows from the fact that for any G the restriction to the diagonal copy of G in $G\times G$ of the action of $G\times G$ on $G\times G/\Delta(G)$ is equivalent to the action of G on itself by conjugation.

<u>Proof of 4.1</u>. By 3.1 the map

$$\pi_0 B\mathrm{Hom}(\sigma,G) \longrightarrow \pi_0 \mathrm{Hom}(B\sigma,BG)$$

is as equivalence, so, given a homomorphism $\rho\colon \sigma \longrightarrow G$, it is enough to show that the loop space map

$$L(\rho)\colon \Omega(B\mathrm{Hom}(\sigma,G),\rho) \longrightarrow \Omega(\mathrm{Hom}(B\sigma,BG),\rho)$$

is an isomorphism on mod p homology. The domain of $L(\rho)$ is essentially $Z_G(\rho)$. The range of $L(\rho)$ is the space of dotted arrows in the diagram

$$\begin{tikzcd} & \mathrm{Hom}(S^1,BG) \arrow[d] \\ B\sigma \arrow[ur, dashed] \arrow[r, "B\rho"] & BG \end{tikzcd}$$

By 4.2, this space is equivalent to the space of sections of the bundle over $B\sigma$ with fiber G associated to the action of σ on G obtained by composing $\rho\colon \sigma \longrightarrow G$ with the conjugation action of G on itself. Equivalently, the range of $L(\rho)$ is the homotopy fixed point set of this action of σ on G. Since $Z_G(\rho)$ is the actual fixed point set of the action, the desired result follows easily from 2.1.

If $1 \longrightarrow K \longrightarrow G \longrightarrow \gamma \longrightarrow 1$ is a short exact sequence of compact Lie groups in which γ is finite, we will let $\Gamma(G\longrightarrow\gamma)$ denote the compact Lie groupoid in which the objects are sections $s\colon \gamma \longrightarrow G$ of the projection $G \longrightarrow \gamma$ and morphisms $s_1 \longrightarrow s_2$ are elements k of K such that $ks_1k^{-1} = s_2$. There is a natural map

$$B\Gamma(G\longrightarrow\gamma) \longrightarrow \Gamma(BG\longrightarrow B\gamma)$$

where $\Gamma(BG\longrightarrow B\gamma)$ denotes the space of sections of the fibration $BG \longrightarrow B\gamma$.

4.3 Proposition. If $1 \longrightarrow K \longrightarrow G \longrightarrow \sigma \longrightarrow 1$ is a short exact sequence of compact Lie groups in which $\pi_0 K$ is a p-group, then the map

$$B\Gamma(G\longrightarrow\sigma) \longrightarrow \Gamma(BG\longrightarrow B\sigma)$$

is a strong mod p equivalence.

Proof. There is a commutative diagram

$$\begin{array}{ccc} B\mathrm{Hom}(\sigma,G) & \longrightarrow & \mathrm{Hom}(B\sigma,BG) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ B\mathrm{Hom}(\sigma,\sigma) & \longrightarrow & \mathrm{Hom}(B\sigma,B\sigma) \end{array}$$

in which the lower horizontal map is a homotopy equivalence and the upper one is a strong mod p equivalence. The homotopy fiber of the map a over the identity map of σ is $B\Gamma(G\longrightarrow\sigma)$ (Quillen, Theorem B) while the corresponding homotopy fiber of b is $\Gamma(BG\longrightarrow B\sigma)$. Since $\pi_i B\mathrm{Hom}(\sigma,\sigma) = 0$ for $i>1$, the lemma follows easily from a comparison of long exact homotopy sequences.

Given 4.3, it is possible to remove the π_0 restriction in 4.1.

4.4 Proposition. For any compact Lie group G the natural map

$$B\mathrm{Hom}(\sigma,G) \longrightarrow \mathrm{Hom}(B\sigma,BG)$$

is a strong mod p equivalence.

Proof. Let K be the connected component of G, so that there is a short exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow \pi_0 G \longrightarrow 1.$$

There is a commutative diagram

$$\begin{array}{ccc} B\mathrm{Hom}(\sigma, G) & \longrightarrow & \mathrm{Hom}(B\sigma, BG) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ B\mathrm{Hom}(\sigma, \pi_0 G) & \longrightarrow & \mathrm{Hom}(B\sigma, B\pi_0 G) \end{array}$$

in which the lower horizontal map is a homotopy equivalence. Pick $\rho\colon \sigma \longrightarrow \pi_0 G$. The homotopy fiber of the map a over ρ is then $B\Gamma(G_\rho \longrightarrow \sigma)$, where G_ρ is the pullback over $\rho\colon \sigma \longrightarrow \pi_0 G$ of $G \longrightarrow \pi_0 G$; the corresponding homotopy fiber of b is $\Gamma(BG_\rho \longrightarrow B\sigma)$. By 4.3, the map

$$B\Gamma(G_\rho \longrightarrow B\sigma) \longrightarrow \Gamma(BG_\rho \longrightarrow B\sigma)$$

is a strong mod p equivalence. Since $\pi_i B\mathrm{Hom}(\sigma, \pi_0 G)$ vanishes for $i > 1$, the proposition follows easily from a comparison of long exact homotopy sequences.

4.5 Proposition. If $1 \longrightarrow K \longrightarrow G \longrightarrow \sigma \longrightarrow 1$ is a short exact sequence of compact Lie groups, then the natural map

$$B\Gamma(G \longrightarrow \sigma) \longrightarrow \Gamma(BG \longrightarrow B\sigma)$$

is a strong mod p equivalence.

Proof. This follows from 4.4 in exactly the same way as 4.3 follows from 4.1.

An action of σ on a compact Lie groupoid G is, as usual, a way of assigning to each element x of σ a functor $F_x\colon G \longrightarrow G$ such that $F_{xy} = F_x F_y$. A groupoid G with such an action is called a σ-groupoid. For example, if $\mathcal{E}\sigma$ is the canonical "contractible" groupoid which has the elements of σ as objects and exactly one map from any given object to any other, then there is a unique action of σ on $\mathcal{E}\sigma$ which induces left translation on the object set; this naturally makes $\mathcal{E}\sigma$ into a σ-groupoid. If G is a σ-groupoid, let $G^{h\sigma}$ stand for the compact Lie groupoid $\mathrm{Hom}_\sigma(\mathcal{E}\sigma, G)$ of σ-equivariant functors $\mathcal{E}\sigma \longrightarrow G$. The objects of $\mathrm{Hom}_\sigma(\mathcal{E}\sigma, G)$ are in fact exactly the functors $H: \mathcal{E}\sigma \longrightarrow G$ which commute with the respective actions of σ on the two categories; a morphism $H_0 \longrightarrow H_1$ is then a natural transformation between the two functors which is itself equivariant in the obvious sense.

Since $B\mathcal{E}\sigma$ is the usual free contractible σ-space $E\sigma$, there is a natural map $B(G^{h\sigma}) \longrightarrow (BG)^{h\sigma}$.

<u>4.6</u> <u>Proposition</u>. If G is a compact Lie groupoid with an action of σ, then the natural map

$$B(G^{h\sigma}) \longrightarrow (BG)^{h\sigma}$$

is a strong mod p equivalence.

<u>Proof</u>. We can clearly assume that σ acts transitively on $\pi_0 BG$. In this case, if BG is not connected then $B(G^{h\sigma})$ and $(BG)^{h\sigma}$ are both empty, so assume that BG is connected. If G_1 and G_2 are σ-groupoids then any equivariant functor $F: G_1 \longrightarrow G_2$ which is an (abstract) equivalence of topological groupoids will induce an equivalence of categories $G_1^{h\sigma} \longrightarrow G_2^{h\sigma}$ as well as a homotopy equivalence $(BG_1)^{h\sigma} \longrightarrow (BG_2)^{h\sigma}$. Consequently, by taking the cartesian product of G with $\mathcal{E}\sigma$ (if necessary) and passing to a suitable full subcategory we can assume that σ acts simply transitively on the objects of G. Under this assumption it is possible to form the quotient category G/σ; this quotient category is a compact Lie <u>group</u> G which is furnished with a

natural map $G \longrightarrow \sigma$. The proposition now follows from 4.5, since $G^{h\sigma}$ can be identified with $\Gamma(G \longrightarrow \sigma)$ and $(BG)^{h\sigma}$ with $\Gamma(BG \longrightarrow B\sigma)$.

§5. Proof of the main theorem.

The proof of 1.1 begins with setting up some inductive machinery. As usual, π will denote a p-group and σ the cyclic group of order p.

Suppose that

$$1 \longrightarrow \kappa \longrightarrow \pi \xrightarrow{f} \sigma \longrightarrow 1$$

is a short exact sequence of groups. Let $\tilde{\kappa}$ be the groupoid in which an object is an element y of σ and a morphism $y_1 \longrightarrow y_2$ is an element x of π such that $f(x)y_1 = y_2$. Composition of morphisms in $\tilde{\kappa}$ corresponds to multiplication of elements of π. It is possible to give $\tilde{\kappa}$ the structure of a σ-groupoid (§4) by declaring that an element $y \in \sigma$ act by sending the morphism $x\colon y_1 \longrightarrow y_2$ to $x\colon y_1y^{-1} \longrightarrow y_2y^{-1}$; the quotient category $\tilde{\kappa}/\sigma$ is then exactly the group π.

If G is a compact Lie group, let $\mathrm{Hom}(\tilde{\kappa},G)$ denote the evident compact Lie groupoid in which the objects are functors $\tilde{\kappa} \longrightarrow G$ and the morphisms are natural transformations. The group σ acts on $\mathrm{Hom}(\tilde{\kappa},G)$ as well as on the space $\mathrm{Hom}(B\tilde{\kappa},BG)$; for naturality reasons there is a σ-equivariant map

$$B\mathrm{Hom}(\tilde{\kappa},G) \longrightarrow \mathrm{Hom}(B\tilde{\kappa},BG).$$

<u>5.1</u> <u>Proposition</u>. If

$$1 \longrightarrow \kappa \longrightarrow \pi \longrightarrow \sigma \longrightarrow 1$$

is a short exact sequence of groups, G is a compact Lie group, and $\tilde{\kappa}$ is as above, then there is a commutative diagram

$$\begin{tikzcd} B\mathrm{Hom}(\pi,G) \arrow[r] \arrow[d, "a"'] & \mathrm{Hom}(B\pi,BG) \arrow[d, "b"] \\ B(\mathrm{Hom}(\tilde{\kappa},G)^{h\sigma}) \arrow[r] & \mathrm{Hom}(B\tilde{\kappa},BG)^{h\sigma} \end{tikzcd}$$

in which the vertical arrows are homotopy equivalences.

Remark. For G a σ-groupoid the "homotopy fixed-point groupoid" $G^{h\sigma}$ is defined in §4.

Proof. The map a is the composite

$$\mathrm{Hom}(\pi,G) = \mathrm{Hom}(\tilde{\kappa}/\sigma,G) = \mathrm{Hom}(\tilde{\kappa},G)^{\sigma} \longrightarrow \mathrm{Hom}(\tilde{\kappa},G)^{h\sigma}$$

while the map b is the composite

$$\mathrm{Hom}(B\pi,BG) = \mathrm{Hom}((B\tilde{\kappa})/\sigma,BG) = \mathrm{Hom}(B\tilde{\kappa},BG)^{\sigma} \longrightarrow \mathrm{Hom}(B\tilde{\kappa},BG)^{h\sigma}.$$

It is clear that the diagram commutes. Since $\mathrm{Hom}(\tilde{\kappa},G)^{h\sigma}$ is $\mathrm{Hom}_{\sigma}(\tilde{\kappa}\times\mathcal{E}\sigma,G)$ and $\mathrm{Hom}(B\tilde{\kappa},BG)^{h\sigma}$ is $\mathrm{Hom}_{\sigma}(B\tilde{\kappa}\times E\sigma,BG)$, the proposition reduces to showing that the map

$$\mathrm{Hom}_{\sigma}(\tilde{\kappa},G) \longrightarrow \mathrm{Hom}_{\sigma}(\tilde{\kappa}\times\mathcal{E}\sigma,G)$$

is an equivalence of topological groupoids and that the map

$$\mathrm{Hom}_{\sigma}(B\tilde{\kappa},BG) \longrightarrow \mathrm{Hom}(B\tilde{\kappa}\times E\sigma,BG)$$

is a homotopy equivalence. Both statements follow easily from the fact that σ acts freely on $\tilde{\kappa}$ and therefore also freely on $B\tilde{\kappa}$.

Proof of 1.1. The theorem is certainly true if π is the trivial group. We can assume by solvability and induction on the order of π that there is a short exact sequence

$$1 \longrightarrow \kappa \longrightarrow \pi \longrightarrow \sigma \longrightarrow 1$$

such that the theorem is known to be true for the group κ. By the argument in the proof of 4.1 we will in fact be done if we can show that the map

$$\pi_0 B\mathrm{Hom}(\pi,G) \longrightarrow \pi_0\mathrm{Hom}(B\pi,BG)$$

is an isomorphism. Let $\tilde{\kappa}$ be as above. By 5.1 it is enough to show that the map

$$\pi_0 B(\mathrm{Hom}(\tilde{\kappa},G)^{h\sigma}) \longrightarrow \pi_0 \mathrm{Hom}(B\tilde{\kappa},BG)^{h\sigma}$$

is an isomorphism. By 4.5, it is in turn enough to show that the map

$$\pi_0 (B\mathrm{Hom}(\tilde{\kappa},G))^{h\sigma} \longrightarrow \pi_0 \mathrm{Hom}(B\tilde{\kappa},BG)^{h\sigma}$$

is an isomorphism. The category $\tilde{\kappa}$ is equivalent to the category of the group κ, so by induction the map

$$B\mathrm{Hom}(\tilde{\kappa},G) \longrightarrow \mathrm{Hom}(B\tilde{\kappa},BG)$$

is a strong mod p equivalence. It will be sufficient to prove, therefore, that if $X \longrightarrow Y$ is an equivariant map between σ-spaces which is a strong mod p equivalence, then the induced map $X^{h\sigma} \longrightarrow Y^{h\sigma}$ gives an isomorphism on π_0. This last statement is in fact a straightforward exercise in obstruction theory, since each homotopy fiber of the map $X \longrightarrow Y$ is a connected simple space with homotopy groups which are uniquely p-divisible (cf. proof of 2.3).

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References.

[A]

J. F. Adams, private communication, Fall, 1985.

[BtD]

T. Bröcker and T. tom Dieck, Representations of Compact Lie Groups, Springer(1985).

[BK]

A. K. Bousfield and D. M. Kan, Homotopy Limits, Completions and Localizations, Lecture Notes in Math. 304, Springer(1972).

[DFZ]

E. Dror-Farjoun and A. Zabrodsky, to appear.

[J]

S. Jackowski, to appear.

[L]

J. Lannes, Sur la cohomologie modulo p des p-groupes abeliens elementaires to appear.

[LMS]

R. Lashof, J. P. May and G. Segal, Equivariant bundles with abelian structural group, Contemporary Math. 19(1983), pp. 167-176.

[Ma]

J. P. May, Simplicial Objects in Algebraic Topology, Van Nostrand, Princeton, 1967.

[Mi]

H. Miller, The fixed-point conjecture, to appear.

[MW]

H. Miller and C. Wilkerson, to appear.

[Q]
D. G. Quillen, Higher algebraic K-theory: I, Lecture Notes in Math. 341, Springer (1973), pp 85-147.

[S]
G. Segal, Classifying spaces and spectral sequences, Publ. Math. IHES 34(1968), pp. 105-112.