

Noncommutative localization in homotopy theory

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1 Introduction

In a sense, noncommutative localization is at the center of homotopy theory, or even more accurately, one form of it *is* homotopy theory. After all, Gabriel and Zisman [9] and later Quillen [18] observed that the homotopy category of CW-complexes can be obtained from the category of topological spaces by formally inverting the maps which are weak homotopy equivalences. More generally, the homotopy category of any Quillen model category [6] [11] can be built by formally inverting maps. In a slightly different direction, the process of *localization with respect to a map* (§2) has recently developed into a powerful tool for making homotopy-theoretic constructions [2, §4]; roughly speaking, localizing with respect to f involves converting an object X into a new one, $L_f(X)$, with the property that, as far as mapping into $L_f(X)$ goes, f looks like an equivalence.

In this paper we will show how the Cohn noncommutative localization described in [19] can be interpreted as an instance of localization with respect to a map (3.2). Actually, we produce a derived form of the Cohn localization, and show that the circumstances in which the Cohn localization is most useful are exactly those in which the higher derived information vanishes (3.3). Finally, we sketch how the derived Cohn localization can sometimes be computed by using a derived form of the categorical localization construction from Gabriel and Zisman (§4).

1.1 The context. It is necessary to choose what to work with: algebraic objects, such as rings, chain complexes, and differential graded algebras (DGAs), or geometric ones, such as ring spectra and module spectra [7] [12]. Since this paper focuses mostly on Cohn localization, we've picked the algebraic option. If R is a ring, the term *R -module* will refer to an

This research was partially supported by NSF grant DMS02-04169

(unbounded) chain complex over R . See [21], [1], or [22, §10] for algebraic accounts of how to work with these complexes, and [13] for a topological approach. The differentials in our complexes always lower degree by one, and all unspecified modules are *left* modules. To maintain at least a little topological standing, we denote the i 'th homology group of an R -module X by $\pi_i X$; this is in fact isomorphic to the i 'th homotopy group of the Eilenberg-MacLane spectrum corresponding to X [7]. A map between R -modules which induces an isomorphism on π_* is called a *quasi-isomorphism* or *equivalence*; the *homotopy category of R -modules* (also known as the *derived category of R*) is obtained from the category of R -modules by formally inverting the equivalences. A *cofibration sequence* of R -modules is one which becomes a distinguished triangle in the derived category. If $f : X \rightarrow Y$ is a map of R -modules, the *cofibre* C of f is the chain complex mapping cone of f [22, 1.2.8], and there is a cofibration sequence $X \rightarrow Y \rightarrow C$. We use Σ for the shift or suspension operator.

An ordinary module M over R gives rise to an R -module in our sense by treating M as a chain complex concentrated in degree 0; we refer to such an M as a *discrete* module over R , and we do not distinguish in notation between M and its associated complex.

We will sometimes work in a context which includes differential graded algebras (DGAs) [22, 4.5.2]. In this setting a ring is identified with the associated DGA concentrated in degree 0.

1.2 Tensor and Hom. The symbol \otimes_R refers to the tensor product of two R -modules, and Hom_R to the complex of homomorphisms between two R -modules (for this last, see [22, 2.7.4], but reindex so that all of the differentials reduce degree by one). For our purposes, both \otimes_R and Hom_R are always taken in the derived sense, so that modules are to be replaced by suitable resolutions before the tensor product or function object is formed. Along the same lines, $\text{End}_R(X)$ denotes the DGA given by the derived endomorphism complex of the R -module X .

These conventions are such that if R is a ring, M a discrete right module over R , and N a discrete left module, then $M \otimes_R N$ is a complex with

$$\pi_i(M \otimes_R N) \cong \begin{cases} \text{Tor}_i^R(M, N) & i \geq 0 \\ 0 & i < 0. \end{cases}$$

Similarly, if M and N are discrete left R -modules, $\text{Hom}_R(M, N)$ is a complex with

$$\pi_i \text{Hom}_R(M, N) \cong \begin{cases} \text{Ext}_R^{-i}(M, N) & i \leq 0 \\ 0 & i > 0. \end{cases}$$

In particular, $\pi_i \text{End}_R(M) \cong \text{Ext}_R^{-i}(M, M)$.

2 Localization with respect to a map

Suppose that R is a ring and that $f : A \rightarrow B$ is a map of R -modules.

2.1 Definition. An R -module Y is said to be *f-local* if f induces an equivalence $\text{Hom}_R(B, Y) \rightarrow \text{Hom}_R(A, Y)$.

In other words, Y is *f-local* if, as far as mapping into Y is concerned, f looks like an equivalence.

2.2 Definition. A map $X \rightarrow X'$ of R -modules is said to be an *f-local equivalence* if it induces an equivalence $\text{Hom}_R(X', Y) \rightarrow \text{Hom}_R(X, Y)$ for every *f-local* R -module Y . An *f-localization* of X is a map $\epsilon : X \rightarrow L_f(X)$, such that $L_f(X)$ is *f-local* and ϵ is an *f-local equivalence*.

2.3 Remark. It is not hard to see that any two *f-localizations* of X are equivalent, so that we can speak loosely of *the f-localization* of X . For any map f and R -module X , the *f-localization* $L_f(X)$ of X exists, and the construction of $L_f(X)$ can be made functorial in X (see [10], or 2.12 below). The functor L_f preserves equivalences, is idempotent up to equivalence, and preserves cofibration sequences up to equivalence. An R -module X is *f-local* if and only if $X \rightarrow L_f(X)$ is an equivalence. A map g of R -modules is an *f-local equivalence* if and only if $L_f(g)$ is an equivalence.

2.4 Remark. Let C be the cofibre of f . For any R -module Y there is a cofibration sequence

$$\text{Hom}_R(C, Y) \rightarrow \text{Hom}_R(B, Y) \rightarrow \text{Hom}_R(A, Y).$$

This shows that Y is *f-local* if and only if $\text{Hom}_R(C, Y)$ is contractible, i.e, if and only if Y is local with respect to $0 \rightarrow C$. This last condition is sometimes expressed by saying that Y is *C-null* [8, 1.A.4]. The *f-localization* functor L_f can also be interpreted as a *C-nullification* functor.

2.5 Proposition. *Up to equivalence, the R -module $L_f(R)$ is a DGA, in such a way that the localization map $R \rightarrow L_f(R)$ is a morphism of DGAs.*

Proof. Let $Y = L_f(R)$ and \mathcal{E} be the endomorphism DGA $\text{End}_R(Y)$. Since Y is *f-local*, the map $R \rightarrow Y$ induces an equivalence

$$\mathcal{E} = \text{End}_R(Y) \xrightarrow{\sim} \text{Hom}_R(R, Y) = Y.$$

The action of R on Y then gives a double commutator map

$$R \rightarrow \text{End}_{\mathcal{E}}(Y) \sim \text{End}_{\mathcal{E}}(\mathcal{E}) \sim \mathcal{E} \sim Y.$$

It is easy to see that this is essentially the localization map $R \rightarrow Y$. Identifying Y with $\text{End}_{\mathcal{E}}(Y)$ gives the required DGA structure. \square

From now on we will treat $L_f(R)$ as a DGA and $R \rightarrow L_f(R)$ as a homomorphism of DGAs.

2.6 Definition. The localization functor L_f is *smashing* if for every R -module X the map $X \sim R \otimes_R X \rightarrow L_f(R) \otimes_R X$ is an f -localization map.

2.7 Remark. For any R -module X , the natural map $X \rightarrow L_f(R) \otimes_R X$ is an f -local equivalence; one way to see this is to pick an f -local Y and consider the chain of equivalences

$$\text{Hom}_R(L_f(R) \otimes_R X, Y) \sim \text{Hom}_R(X, \text{Hom}_R(L_f(R), Y)) \sim \text{Hom}_R(X, Y).$$

The question of whether L_f is smashing, then, is the question of whether for every X the R -module $L_f(R) \otimes_R X$ is f -local.

2.8 Remark. If L_f is smashing then the category of f -local R -modules is equivalent, from a homotopy point of view, to the category of $L_f(R)$ -modules. In particular, the homotopy category of f -local R -modules is equivalent to the homotopy category of $L_f(R)$ -modules.

2.9 Examples. Let $R = \mathbb{Z}$, pick a prime p , and let f be the map $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$. Then L_f is smashing, and $L_f(X) \sim \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} X$.

On the other hand, if f is the map $\mathbb{Z}[1/p] \rightarrow 0$, then L_f is the Ext- p -completion functor [2, 2.5], which is the total left derived functor of the p -completion functor. In particular, $L_f(\mathbb{Z}) \sim \mathbb{Z}_p$ and $L_f(\mathbb{Z}/p^\infty) \sim \Sigma\mathbb{Z}_p$. Since $\Sigma\mathbb{Z}_p$ is not equivalent to $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}/p^\infty$, L_f is *not* smashing in this case.

The main positive result about smashing localizations is due to Miller. Recall that an R -module A is said to be *small* if $\text{Hom}_R(A, -)$ commutes up to equivalence with arbitrary coproducts. This is the same as saying that A is finitely built from R , or that A is equivalent to a chain complex of finite length made up of finitely generated projective R -modules.

2.10 Proposition. [15] *Let $f : A \rightarrow B$ be a map of R -modules. If A and B are small, or more generally if the cofibre C of f is equivalent to a coproduct of small R -modules, then L_f is smashing.*

2.11 Lemma. *If the cofibre C of $f : A \rightarrow B$ is equivalent to a coproduct of small R -modules, then the class of f -local R -modules is closed under arbitrary coproducts.*

Proof. Write $C \sim \coprod_{\alpha} C_{\alpha}$, where each C_{α} is small. Then Y is f -local if and only if Y is C -null (2.4), which is the case if and only if Y is C_{α} -null for each α . The lemma now follows from the fact that $\mathrm{Hom}_R(C_{\alpha}, -)$ commutes up to equivalence with coproducts. \square

Proof of 2.10. Consider the class of R -modules X for which $L_f(R) \otimes_R X$ is f -local. We have to show that this is the class of all R -modules (2.7). However, the class contains R itself, is closed under cofibration sequences (2.3), is closed under arbitrary coproducts (2.11), and is closed under equivalences. The usual method for constructing resolutions shows that this is enough to give the desired result. \square

2.12 Construction of $L_f(X)$. We will sketch an explicit description of $L_f(X)$, at least up to equivalence, in the case in which the cofibre C of f is equivalent to a coproduct of small R -modules. Actually, we will assume that C itself is small, since the adjustments to handle the general case are mostly notational.

Recall that the *homotopy colimit* of a sequence $X_0 \xrightarrow{\sigma_0} X_1 \xrightarrow{\sigma_1} \dots$ of R -modules is the cofibre of the map $\sigma : \coprod X_i \rightarrow \coprod X_i$ given by $\sigma(x_i) = \sigma_i(x_i) - x_i$. The description of $L_f(X)$ depends on two observations.

1. $\mathrm{Hom}_R(C, -)$ commutes up to equivalence with sequential homotopy colimits.
2. If U is a coproduct of copies of suspensions of C , $g : U \rightarrow X$ is a map of R -modules, and X' is the cofibre of g , then $X \rightarrow X'$ is an f -local equivalence.

Item (1) is clear from the description above of sequential homotopy colimits. For (2), pick an f -local Y , consider the cofibre sequence

$$\mathrm{Hom}_R(X', Y) \rightarrow \mathrm{Hom}_R(X, Y) \rightarrow \mathrm{Hom}_R(U, Y),$$

and observe that the term on the right is trivial (2.4).

Consider a set of representatives $g_{\alpha} : \Sigma^{n_{\alpha}} C \rightarrow X$ for all nontrivial homotopy classes of maps from suspensions of C to X . Let $U = \coprod_{\alpha} \Sigma^{n_{\alpha}} C$, let $g : U \rightarrow X$ be the sum of the maps $\{g_{\alpha}\}$, and let $\Phi(X)$ denote the cofibre

of g . There is a natural map $X \rightarrow \Phi(X)$. Iterate the process to construct a sequential diagram

$$X \rightarrow \Phi(X) \rightarrow \Phi^2(X) \rightarrow \cdots \rightarrow \Phi^n(X) \rightarrow \cdots, \quad (2.13)$$

and let $\Phi^\infty(X) = \text{hocolim}_n \Phi^n(X)$. We claim that $X \rightarrow \Phi^\infty(X)$ is an f -localization map, so that $\Phi^\infty(X) \sim L_f(X)$. The fact that $\Phi^\infty(X)$ is f -local follows from 2.4 and (1) above, since every map from a suspension of C into $\Phi^\infty(X)$ factors up to homotopy through $\Phi^n(X)$ for some n , and so is null homotopic, since it becomes null homotopic by construction in $\Phi^{n+1}(X)$. To see that $X \rightarrow \Phi^\infty(X)$ is an f -local equivalence, observe that by (2) above and induction the map $X \rightarrow \Phi^n(X)$ is an f -local equivalence for each $n \geq 1$. For an f -local Y it is now possible to compute

$$\begin{aligned} \text{Hom}_R(\Phi^\infty(X), Y) &\sim \text{holim}_n \text{Hom}_R(\Phi^n(X), Y) \\ &\sim \text{holim}_n \text{Hom}_R(X, Y) \\ &\sim \text{Hom}_R(X, Y). \end{aligned}$$

2.14 Remark. The standard construction of $L_f(X)$ is similar to the above, but slightly more complicated [10, 4.3]. To make the construction functorial, and not just functorial up to equivalence, it is necessary to build $\Phi(X)$ by using *all* maps from suspensions of C to X , not just a set of representatives of nontrivial homotopy classes. We neglected to mention above that C should have been replaced up to equivalence by a projective complex (cofibrant model); in the general setting there's also a slight adjustment [10, 4.2.2] to deal with the fact that X might not be fibrant, in other words, to deal with the fact that not every map $\Sigma^n C \rightarrow X$ in the homotopy category is necessarily represented by an actual map $\Sigma^n C \rightarrow X$. Finally, if C is not small the countable homotopy colimit in 2.13 has to be replaced by a parallel transfinite construction [10, 10.5].

2.15 Other structure. There is more that can be said if the cofibre C of f is small. Let $C^\# = \text{Hom}_R(C, R)$. There is a ‘‘homology theory’’ on the category of R -modules determined by the functor $X \mapsto \pi_*(C^\# \otimes_R X)$; let $X \rightarrow \hat{X}$ denote Bousfield localization with respect to this theory [8, 1.E.4] [10, xi]. Then for any X there is a homotopy fibre square

$$\begin{array}{ccc} X & \longrightarrow & \hat{X} \\ \downarrow & & \downarrow \\ L_f(X) & \longrightarrow & L_f(\hat{X}). \end{array}$$

In the case in which $R = \mathbb{Z}$ and f is the map $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$, this is the arithmetic square

$$\begin{array}{ccc} X & \longrightarrow & X_p^\wedge \\ \downarrow & & \downarrow \\ \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} X & \longrightarrow & \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} (X_p^\wedge). \end{array}$$

See [3] for other results along these lines and for some (mostly commutative) examples.

3 The Cohn localization

In this section we construct the Cohn localization from the point of view of §2. Let R be a ring and let $\{f_\alpha : P_\alpha \rightarrow Q_\alpha\}$ be some set of maps between discrete (1.1) finitely generated projective R -modules. Let f denote $\coprod_\alpha f_\alpha$. The main results are as follows.

3.1 Proposition. *The DGA $L_f(R)$ is (-1) -connected, i.e., $\pi_i L_f(R)$ vanishes for $i < 0$.*

If P is a discrete R -module, let $P^\# = \text{Ext}_R^0(P, R)$ denote its usual dual, and note that $P^\#$ is a discrete right R -module. For each $f_\alpha : P_\alpha \rightarrow Q_\alpha$, let $f_\alpha^\# : Q_\alpha^\# \rightarrow P_\alpha^\#$ be the dual of f_α , and let S denote the set $\{f_\alpha^\#\}$. Recall that a ring homomorphism $R \rightarrow R'$ is said to be *S -inverting* if for each α the map $\text{Tor}_0^R(f_\alpha^\#, R')$ is an isomorphism. A *Cohn localization* of R with respect to S is an initial object $R \rightarrow S^{-1}R$ in the category of S -inverting ring homomorphisms $R \rightarrow R'$ [19, Part 1].

The map $R \rightarrow L_f(R)$ of DGAs (2.5) induces a ring homomorphism $R = \pi_0 R \rightarrow \pi_0 L_f(R)$.

3.2 Proposition. *The map $R \rightarrow \pi_0 L_f(R)$ is a Cohn localization of R with respect to S .*

From now on we will denote $\pi_0 L_f(R)$ by \mathcal{L} . In light of 3.2, we think of the DGA $L_f(R)$ as a *derived Cohn localization* of R with respect to S .

Recall [17] that the ring homomorphism $R \rightarrow \mathcal{L}$ is said to be *stably flat* if $\text{Tor}_i^R(\mathcal{L}, \mathcal{L}) = 0$ for $i > 0$. It is in the stably flat case that Cohn localization leads to K -theory localization sequences.

3.3 Proposition. *The map $R \rightarrow \mathcal{L}$ is stably flat if and only if the groups $\pi_i L_f(R)$ vanish for $i > 0$.*

In other words, $R \rightarrow \mathcal{L}$ is stably flat if and only if $L_f(R)$ is equivalent as a DGA to \mathcal{L} , or if and only if the “higher derived functors” of Cohn localization, given by $\pi_i L_f(R)$, $i > 0$, vanish.

The rest of this section is taken up with proofs. Observe to begin with that the objects P_α and Q_α are small (§2) as R -modules, and so the cofibre of f is a coproduct of small objects. It follows that L_f is smashing (2.10) and that up to equivalence there is a relatively simple construction for $L_f(X)$ (2.12).

3.4 Proposition. *An R -module X is f -local if and only if each group $\pi_i X$ is f -local.*

3.5 Remark. It might be useful to spell out the meaning of this. The object R is an ordinary ring and the object X is a chain complex over R . Each group $\pi_i X$ is a discrete R -module, which can be treated as a chain complex over R concentrated in degree 0. The proposition states that X is f -local if and only if, for each $i \in \mathbb{Z}$, the R -module obtained from $\pi_i X$ is f -local.

Proof of 3.4. If P is a discrete projective module over R , it is not hard to see that there are natural isomorphisms

$$\pi_i \operatorname{Hom}_R(P, X) \cong \operatorname{Ext}_R^0(P, \pi_i X).$$

This is clearly true if P is free, and follows in general from a retract argument. The proposition then follows from definition 2.1. \square

3.6 Lemma. *Suppose that $h : P \rightarrow Q$ is a map of discrete finitely generated projective R -modules, W is the cofibre of h , X is an R -module which is (-1) -connected, $g : \Sigma^n W \rightarrow X$ is a map which is not null homotopic, and X' is the cofibre of g . Then X' is (-1) -connected.*

Proof. By a retract argument, we can assume that P and Q are free, so that $P \cong R^i$ and $Q \cong R^j$. In view of the definition of W , there is a cofibration sequence

$$\operatorname{Hom}_R(W, X) \rightarrow X^j \rightarrow X^i.$$

The corresponding long exact homotopy sequence shows that $\pi_k \operatorname{Hom}_R(W, X)$ vanishes for $k < -1$. Since $g : \Sigma^n W \rightarrow X$ is essential, it follows that $n \geq -1$. This gives a cofibration sequence

$$X \rightarrow X' \rightarrow \Sigma^{n+1} W \tag{3.7}$$

with $n + 1 \geq 0$. It is clear that there are isomorphisms

$$\pi_k W = \begin{cases} \text{coker}(h) & k = 0 \\ \text{ker}(h) & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

and so in particular that $\pi_k W = 0$ for $k < 0$. The proof is completed by looking at the long exact homotopy sequence of 3.7. \square

Proof of 3.1. This follows from 3.6 and the construction of $L_f(R)$ sketched in 2.12. \square

Proof of 3.2. If P is a discrete finitely generated projective R -module, then for any discrete R -module M , there is a natural isomorphism

$$\text{Tor}_0^R(P^\#, M) \cong \text{Ext}_R^0(P, M).$$

In particular, as in the proof of 3.4, the map $\text{Tor}_0^R(f_\alpha^\#, M)$ is an isomorphism for all α if and only if M is f -local. By 3.4, \mathcal{L} is f -local, and it follows that $R \rightarrow \mathcal{L}$ is S -inverting.

Now, suppose that $R \rightarrow R'$ is an arbitrary S -inverting ring homomorphism. As above, the ring R' is f -local as an R -module, and this implies that the map $R \rightarrow L_f(R)$ induces an equivalence

$$\text{Hom}_R(L_f(R), R') \rightarrow \text{Hom}_R(R, R') \sim R'. \quad (3.8)$$

In conjunction with 3.1, the universal coefficient spectral sequence

$$\text{Ext}_R^i(\pi_j L_f(R), R') \Rightarrow \pi_{-i-j} \text{Hom}_R(L_f(R), R')$$

shows that $\pi_0 \text{Hom}_R(L_f(R), R')$ is isomorphic to $\text{Ext}_R^0(\mathcal{L}, R')$. Applying π_0 to the equivalence 3.8 thus shows that every homomorphism $R \rightarrow R'$ of discrete modules over R extends uniquely to a homomorphism $\mathcal{L} \rightarrow R'$. In particular, the given ring homomorphism $u : R \rightarrow R'$ extends uniquely to $v : \mathcal{L} \rightarrow R'$. To show that v is a ring homomorphism, it is enough to show that for each element λ of \mathcal{L} , the two maps $a, b : \mathcal{L} \rightarrow R'$ given by $a(x) = v(x\lambda)$ and $b(x) = v(x)v(\lambda)$ are the same. Both a and b are maps of discrete R -modules, and so it is in fact enough to show that a and b agree when composed with the map $R \rightarrow \mathcal{L}$. But this is just the statement that v is a map of discrete R -modules. \square

3.9 Lemma. *Suppose that X and Y are respectively right and left R -modules such that $\pi_i X$ and $\pi_i Y$ vanish for $i < 0$. Then there are natural isomorphisms*

$$\pi_i(X \otimes_R Y) \cong \begin{cases} \mathrm{Tor}_0^R(\pi_0 X, \pi_0 Y) & i = 0 \\ 0 & i < 0. \end{cases}$$

Proof. This follows from the Künneth spectral sequences

$$\begin{aligned} \pi_i((\pi_j X) \otimes_R Y) &\Rightarrow \pi_{i+j}(X \otimes_R Y) \\ \mathrm{Tor}_i^R(M, \pi_j Y) &\Rightarrow \pi_{i+j}(M \otimes_R Y). \end{aligned}$$

In the second spectral sequence, M is a discrete right R -module (e.g., $\pi_k X$ for some $k \geq 0$). \square

3.10 Lemma. *The natural map*

$$L_f(R) \sim R \otimes_R L_f(R) \rightarrow L_f(R) \otimes_R L_f(R)$$

is an equivalence. The natural map $\mathcal{L} \cong \mathrm{Tor}_0^R(R, \mathcal{L}) \rightarrow \mathrm{Tor}_0^R(\mathcal{L}, \mathcal{L})$ is an isomorphism.

Proof. Since $L_f(L_f(R)) \sim L_f(R)$, the first statement follows from the fact that L_f is smashing (2.10). The second then follows from 3.1 and 3.9. \square

Proof of 3.3. Suppose that $\pi_i L_f(R) = 0$ for $i > 0$, or in other words (3.1), that $L_f(R) \sim \mathcal{L}$. It follows from 3.10 that $\mathcal{L} \otimes_R \mathcal{L} \sim \mathcal{L}$; applying π_* then gives isomorphisms

$$\mathrm{Tor}_i^R(\mathcal{L}, \mathcal{L}) \cong \pi_i(\mathcal{L} \otimes_R \mathcal{L}) \cong \begin{cases} \mathcal{L} & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose on the other hand that $\pi_i(\mathcal{L} \otimes_R \mathcal{L}) \cong \mathrm{Tor}_i^R(\mathcal{L}, \mathcal{L})$ vanishes for $i > 0$. Consider the class of all \mathcal{L} -modules X with the property that the natural map

$$X \sim R \otimes_R X \rightarrow \mathcal{L} \otimes_R X$$

is an equivalence. This class includes \mathcal{L} (3.9, 3.10), is closed under equivalences, is closed under cofibration sequences, and is closed under arbitrary coproducts. As in the proof of 2.10, this is enough to show that the class contains all \mathcal{L} -modules. In particular, for any discrete \mathcal{L} -module M there are isomorphisms

$$\mathrm{Tor}_i^R(\mathcal{L}, M) \cong \begin{cases} M & i = 0 \\ 0 & i > 0. \end{cases}$$

Each group $\pi_j L_f(R)$ is a module over $\mathcal{L} = \pi_0 L_f(R)$, and so it follows from the Künneth spectral sequence

$$\mathrm{Tor}_i^R(\mathcal{L}, \pi_j L_f(R)) \Rightarrow \pi_{i+j}(\mathcal{L} \otimes_R L_f(R))$$

that the natural map $L_f(R) \rightarrow \mathcal{L} \otimes_R L_f(R)$ is an equivalence, and in particular that the R -module structure on $L_f(R)$ extends to an \mathcal{L} -module structure. This structure can be used to factor the natural map $R \rightarrow L_f(R)$ as a composite $R \rightarrow \mathcal{L} \rightarrow L_f(R)$. Applying L_f to this composite gives a diagram

$$L_f(R) \rightarrow \mathcal{L} \rightarrow L_f^2(R)$$

in which we have used 3.4 to identify $L_f(\mathcal{L}) \sim \mathcal{L}$. The composite map $L_f(R) \rightarrow L_f^2(R)$ is an equivalence, since $R \rightarrow L_f(R)$ is an f -local equivalence. Applying π_* shows that $\pi_i L_f(R) \cong 0$ for $i > 0$. \square

4 Localization of categories

In this section we sketch without proof a connection between the Cohn localization of a ring and the process of forming the derived localization of a category. The connecting link between the two is the notion of *ring with several objects*.

4.1 Derived localization of categories. Suppose that \mathbf{C} is a small category and \mathbf{W} a subcategory which contains all the objects of \mathbf{C} . The *localization of \mathbf{C} with respect to \mathbf{W}* is a functor $\mathbf{C} \rightarrow \mathbf{W}^{-1}\mathbf{C}$ which is initial in the category of all functors with domain \mathbf{C} which take the arrows in \mathbf{W} into isomorphisms. A derived form of this localization can be constructed by forming a free simplicial resolution $(F\mathbf{C}, F\mathbf{W})$ of the pair (\mathbf{C}, \mathbf{W}) and taking the dimensionwise localization $(F\mathbf{W})^{-1}F\mathbf{C}$ [5]. This results in a category $L(\mathbf{C}, \mathbf{W})$ with the same objects as \mathbf{C} , but enriched over simplicial sets. Up to an enriched analog of categorical equivalence, $L(\mathbf{C}, \mathbf{W})$ is the same as the hammock localization of [4], and from this point of view there is a natural functor $\mathbf{C} \rightarrow L(\mathbf{C}, \mathbf{W})$. This functor is universal, in an appropriate sense, among functors from \mathbf{C} to categories enriched over simplicial sets which send the arrows of \mathbf{W} into maps which are invertible up to homotopy.

4.2 Examples. The following examples do not involve small categories, but it is still possible to make sense of them. Let \mathbf{C} be the category of topological spaces and \mathbf{W} the subcategory of weak homotopy equivalences. Let X and Y be spaces with CW -approximations X' and Y' . Then the set of maps $X \rightarrow Y$ in $\mathbf{W}^{-1}\mathbf{C}$ is isomorphic to the set of homotopy classes of maps

$X' \rightarrow Y'$; the simplicial set of maps $X \rightarrow Y$ in $L(\mathbf{C}, \mathbf{W})$ is equivalent to the singular complex of the mapping space $\text{Map}(X', Y')$.

Let R be a ring, \mathbf{C} the category of unbounded chain complexes over R , i.e., the category of R -modules, and $\mathbf{W} \subset \mathbf{C}$ the subcategory of quasi-isomorphisms. Then $\mathbf{W}^{-1}\mathbf{C}$ is the derived category of R . If X and Y are R -modules, then the homotopy groups of the simplicial set of maps $X \rightarrow Y$ in $L(\mathbf{C}, \mathbf{W})$ are $\pi_i \text{Hom}_R(X, Y)$, $i \geq 0$.

4.3 Rings with several objects. A *ring T with several objects* is a small additive category [16]; a discrete T -module is an additive functor from T to abelian groups. There is a category of discrete T -modules in which the morphisms are natural transformations between functors. Define a T -module to be a chain complex of discrete T -modules, i.e., an additive functor from T to the category of chain complexes over \mathbb{Z} . One can build a homotopy theory of T -modules in which the weak equivalences are natural transformations which are objectwise quasi-isomorphisms (see [20] for geometric versions of this). We use the notation $\text{Hom}_T(X, Y)$ for the derived chain complex of maps between two T -modules X, Y .

Every object $x \in T$ gives rise to a discrete small projective T -module P_x , where P_x assigns to y the group of maps $x \rightarrow y$ in T . Suppose that $\{f_\alpha : P_{x_\alpha} \rightarrow P_{y_\alpha}\}$ is a set of maps between such projectives, and let $f = \coprod_\alpha f_\alpha$. The ideas in §2 give for any T -module X an f -local module $L_f(X)$ and an f -local equivalence $X \rightarrow L_f(X)$. There is an associated category $L_f(T)$ enriched over chain complexes (in other words, $L_f(T)$ is “a DGA with several objects”); this has the same objects as T , and the function complex of maps x to y in $L_f(T)$ is given by $\text{Hom}_T(L_f(y), L_f(x))$. There is a functor $i : T \rightarrow L_f(T)$ and by the same smallness argument used in the proof of 2.10, the functor L_f can be identified as (derived) left Kan extension along i .

If \mathbf{C} is a simplicial category, let $\mathbb{Z}\mathbf{C}$ denote the simplicial additive category obtained by applying the free abelian group functor dimensionwise to the morphism sets of \mathbf{C} . There is an associated category $N\mathbb{Z}\mathbf{C}$ enriched over chain complexes, formed by normalizing the simplicial abelian groups which appear as morphism objects in $\mathbb{Z}\mathbf{C}$ and using the Eilenberg-Zilber formula [22, 6.5.11] to define composition. If \mathbf{C} is an ordinary category treated as a simplicial category with discrete morphism sets, then $N\mathbb{Z}\mathbf{C}$ is the additive category obtained by taking free abelian groups on the morphism sets of \mathbf{C} , so at what we hope is minimal risk of confusion we will just denote it $\mathbb{Z}\mathbf{C}$.

If (\mathbf{C}, \mathbf{W}) is a pair of categories as above (4.1), then for each morphism $w : x \rightarrow y$ in \mathbf{W} let $f_w : P_y \rightarrow P_x$ be the corresponding map between projective $\mathbb{Z}\mathbf{C}$ -modules and let $f_{\mathbf{W}} = \coprod f_w$.

4.4 Proposition. *Let (\mathbf{C}, \mathbf{W}) and $f = f_{\mathbf{W}}$ be as above. Then in an appropriate enriched sense the two categories $NZL(\mathbf{C}, \mathbf{W})$ and $L_f(\mathbb{Z}\mathbf{C})$ are equivalent.*

“Equivalence” here means that the two categories are related by a zigzag of morphisms between enriched categories with the property that these morphisms give the identity map on object sets and induce quasi-isomorphisms on function complexes.

4.5 Rings. Suppose that T is a ring with a *finite* number of objects, in other words, a small additive category with a finite number of objects. Let $P = \coprod_x P_x$, where the coproduct runs through all of the objects in T and P_x is the projective from 4.3. Let \mathcal{E} be the endomorphism ring of P in the category of discrete T -modules, and $\mathcal{P}(T)$ the ring \mathcal{E}^{op} . The notation $\mathcal{P}(T)$ is meant to suggest that this is a kind of path algebra of T . As an abelian group, $\mathcal{P}(T)$ is isomorphic to the sum $\coprod_{x,y} T(x, y)$ of all of the morphism groups of T ; products are defined by using the composition in T to the extent possible and otherwise setting the products equal to 0. Since P is a small projective generator for the category of discrete T -modules, ordinary Morita theory shows that $\text{Ext}_T^0(P, -)$ gives an equivalence between the category of discrete T -modules and the category of discrete $\mathcal{P}(T)$ -modules. Not surprisingly, this extends to a homotopy-theoretic equivalence between the category of T -modules and the category of $\mathcal{P}(T)$ -modules.

The construction $\mathcal{P}(-)$ can be extended to categories enriched over chain complexes; if T' is such a category, then $\mathcal{P}(T')$ is a DGA.

4.6 Proposition. *Suppose that T is a ring with a finite number of objects, $f : P \rightarrow Q$ is a map between discrete small projective T -modules, and $g : P' \rightarrow Q'$ the corresponding map between discrete finitely-generated $\mathcal{P}(T)$ -modules. Then the DGA $\mathcal{P}(L_f(T))$ is, in an appropriate sense, equivalent to the DGA $L_g(\mathcal{P}(T))$.*

4.7 Remark. The word “equivalent” in the proposition signifies that the DGA $L_g(\mathcal{P}(T))$ is related to $\mathcal{P}(L_f(T))$ by a zigzag of quasi-isomorphisms between DGAs.

4.8 An example. This is along the lines of [19, 2.4]. Let $H \leftarrow G \rightarrow K$ be a two-source of groups. Form a category \mathbf{C} with three objects, x , y , and z and the following pattern of morphisms

$$\begin{array}{ccccc} y & \xleftarrow{H} & x & \xrightarrow{K} & z \\ \downarrow & & \downarrow & & \downarrow \\ H & & G & & K \end{array}$$

This signifies, for instance, that H is the set of maps $x \rightarrow y$, and G is the monoid of endomorphisms of x . The action of G on H by composition is then the translation action determined by the given homomorphism $G \rightarrow H$.

Let ξ denote the pushout of the diagram $H \leftarrow G \rightarrow K$ of groups and X the homotopy pushout of the diagram $BH \leftarrow BG \rightarrow BK$ of spaces. By the van Kampen theorem, $\pi_1 X \cong \xi$.

4.9 Lemma. *The nerve of \mathbf{C} is equivalent to X .*

Let $\mathbf{W} \subset \mathbf{C}$ be the subcategory whose nonidentity morphisms are the maps $x \rightarrow y$ and $x \rightarrow z$ corresponding to the identity elements of H and K , respectively. Note that all of the morphisms in $\mathbf{W}^{-1}\mathbf{C}$ are invertible; in fact, $\mathbf{W}^{-1}\mathbf{C}$ is isomorphic to a connected groupoid with three objects x , y , and z and vertex groups isomorphic to ξ . Let ΩX denote the simplicial loop group of X .

4.10 Proposition. [5] *The simplicial category $L(\mathbf{C}, \mathbf{W})$ is weakly equivalent to a connected simplicial groupoid with three objects x , y , and z and vertex group ΩX .*

Let $f = f_{\mathbf{W}}$ be map between projective $\mathbb{Z}\mathbf{C}$ modules determined as above by \mathbf{W} , and g the corresponding map between projective $\mathcal{P}(\mathbb{Z}\mathbf{C})$ -modules. Note that $\mathcal{P}(\mathbb{Z}\mathbf{C})$ is the matrix ring associated in [19, 2.4] to the amalgamated product $\mathbb{Z}H *_{\mathbb{Z}G} \mathbb{Z}K$ and that g is the sum of the two maps σ_1 and σ_2 described there. Concatenating 4.4 with 4.6 gives the following result.

4.11 Proposition. *The DGA $L_g\mathcal{P}(\mathbb{Z}\mathbf{C})$ is equivalent in an appropriate sense to the 3×3 -matrix algebra on the chain algebra $C_*(\Omega X; \mathbb{Z})$. In particular, for $i \geq 0$ there are natural isomorphisms*

$$\pi_i L_g\mathcal{P}(\mathbb{Z}\mathbf{C}) \cong H_i(\Omega X; \mathbb{Z}) \oplus \cdots \oplus H_i(\Omega X; \mathbb{Z}) \quad (9 \text{ times}).$$

This Cohn localization is stably flat (3.3) if and only if the universal cover of X is acyclic and X itself is equivalent to $B\xi$; this occurs, for instance, if the maps $G \rightarrow H$ and $G \rightarrow K$ are injective. It does not occur if $G = \mathbb{Z}$ and H and K are the trivial group.

Here's another example, which can be treated along the same lines. Let X be a connected space and M the monoid constructed by McDuff [14] with BM weakly equivalent to X . Let R be the monoid ring $\mathbb{Z}M$, and for each $m \in M$ let $f_m : R \rightarrow R$ be given by right multiplication by m . Denote the sum $\coprod_m f_m$ by f . Then for $i \geq 0$ there are natural isomorphisms

$$\pi_i L_f(R) \cong H_i(\Omega X; \mathbb{Z}).$$

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