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§ 1. INTRODUCTION

1.1 SUMMARY.

We describe a version of obstruction theory for simplicial sets, which involves *canonical obstruction cocycles* and then use this to obtain a similar theory for diagrams of simplicial sets. An application of the latter (to the problem of realizing diagrams in the homotopy category by means of diagrams of simplicial sets) will be given in [4].

1.2. NOTATION, TERMINOLOGY, ETC.

(i) SIMPLICIAL SETS. We freely use some of the basics of simplicial theory as can be found in [6] and [2, Ch. VIII]. The category of simplicial sets will be denoted by **S**.

(ii) FUNDAMENTAL GROUPOIDS. A *groupoid G* is a small category in which every map is invertible; its nerve NG is a disjoint union of aspherical simplicial sets. For $X \in \mathbf{S}$, its *fundamental groupoid* is the groupoid $\Pi_1 X$ which has the vertices of X as objects and the "homotopy classes of paths" between them as maps. Clearly Π_1 *is a functor* $\Pi_1 : \mathbf{S} \to \mathbf{G}$, where **G** denotes the category of groupoids (and functors between them).

(iii) HOMOTOPY MODULES OVER FUNDAMENTAL GROUPOIDS. A *module over a groupoid G* is a functor $M: \mathbf{G} \to$ (*abelian groups*). For $X \in \mathbf{S}$ and $n \geq 2$, the functor $\Pi_n X$ which sends a vertex $x \in X$ to the n-th homotopy group $\pi_n(X,x)$ is an example of a module over $\Pi_1 X$. Moreover, if **M** denotes the

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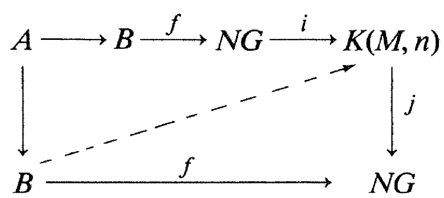
category of modules over groupoids (a map $M_1 \to M_2 \in \mathbf{M}$, where $M_i : G_i \to$ $\to$(*abelian groups*), consists of a functor $g : G_1 \to G_2$ together with a natural transformation $M_1 \to M_2 g$), then Π_n *is clearly a functor* $\Pi_n : \mathbf{S} \to \mathbf{M}$.

(iv) EILENBERG-MACLANE OBJECTS. Given a module M over a groupoid G and an integer $n \geq 0$, the associated *Eilenberg-MacLane object* $K(M, n)$ is the simplicial set which has as n-simplices the pairs (u, v) such that u is an n-simplex $G_0 \to \cdots \to G_n$ of NG and v is an n-simplex of the Eilenberg-MacLane complex $K(MG_0, n)$ [6, § 23]. *The forgetful map* $j : K(M, n) \to NG$ *is a fibration and has an obvious cross section* $i : NG \to K(M, n)$.

(v) COHOMOLOGY WITH LOCAL COEFFICIENTS. Given a map $A \to B \in \mathbf{S}$ and a module M over a groupoid G, *the n-th relative cohomology group with local coefficients induces by a map* $f : B \to NG \in \mathbf{S}$ (or equivalently a map $f' : \Pi_1 B \to$ $\to G \in \mathbf{G}$) is the abelian group given by the formula

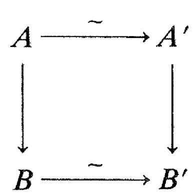
$$H^n(B, A; M) = \pi_0 \operatorname{hom}(B, K(M, n))$$

where hom denotes the function complex in the between category $A \downarrow \mathbf{S} \downarrow NG$ [5, p. 46]. The vertices of $\operatorname{hom}(B, K(M, n))$ thus are the dotted arrows which make the following diagram commute



They form an abelian group $Z^n(B, A; M)$ and are called *cocycles*. Two cocycles are called *cohomologous* if they are in the same component of $\operatorname{hom}(B, K(M, n))$.

Note that *this notion of cohomology has "homotopy meaning" if the map* $A \to B \in \mathbf{S}$ *is a cofibration,* i.e. if



is a commutative diagram in which the horizontal maps are weak (homotopy) equivalences and the vertical maps are cofibrations, then the induced map $H^n(B', A'; M) \to H^n(B, A; M)$ is an isomorphism. In this case the above definition of cohomology with local coefficients is equivalent to the usual one.

(vi) COSKELETONS. For every $X \in \mathbf{S}$ and integer $n \geq 0$, the *n-skeleton* of X is the smallest subcomplex $sk_n X \subset X$ containing all the n-simplices of X and [1] the *n-coskeleton* of X is the simplicial set $cosk_n X$ which has as its j-simplices the maps $sk_n \Delta[j] \to X \in \mathbf{S}$. The obvious map $X \to cosk_n X$ is $1-1$ and onto in dimensions $\leq n$ and hence induces isomorphisms of the homotopy groups in dimensions $<n$. If X is fibrant (i.e. satisfies the extension condition), then so is $cosk_n X$ and its homotopy groups are trivial in dimensions $\geq n$. Consequently the sequence

$$X = \lim_{\leftarrow} cosk_i X \to \cdots \to cosk_{n+1} X \to cosk_n X \to \cdots$$

is, up to homotopy, a Postnikov decomposition of X. To obtain the corresponding *k-invariants* one notes that, for every integer $n \geq 1$, there is an obvious cocycle

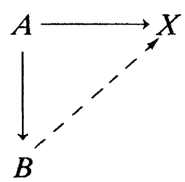
$$k^{n+1} X \in Z^{n+1}(cosk_n X, cosk_{n+1} X; \Pi_n X)$$

which assigns to every $(n+1)$-simplex $w: sk_n \Delta[n+1] \to X \in \mathbf{S}$ of $cosk_n X$ the corresponding element of $\pi_n(X, w(0))$.

§ 2. AN OBSTRUCTION THEORY FOR SIMPLICIAL SETS

In this section we describe a version of obstruction theory for simplicial sets which involves *canonical* obstruction cocycles. We first deal with

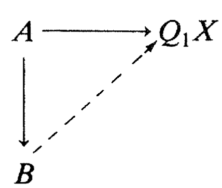
2.1 OBSTRUCTIONS TO EXTENSIONS OF MAPS. Given a solid arrow diagram of simplicial sets



in which the maps $A \to B$ is a cofibration and X is fibrant (i.e. satisfies the extension condition), the aim of obstruction theory is to find inductive algebraic conditions for the existence of a dotted arrow which makes the resulting triangle commutative. To do this one considers the sequence

$$X = \lim_{\leftarrow} Q_i X \to \cdots \to Q_n X \to Q_{n-1} X \to \cdots \to Q_1 X$$

in which $Q_n X = cosk_{n+1} X$ (1.2(vi)) for all $n \geq 1$ and notes that, given a solid arrow diagram



the existence of a dotted arrow which makes the resulting triangle commutative is equivalent to the corresponding (already algebraic) existence problem for the associated map of the fundamental groupoids. It thus remains, given a commutative solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & Q_nX \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & Q_{n-1}X \end{array} \qquad n>1$$

to find an algebraic condition for the existence of a dotted arrow which makes the upper triangle commutative and the lower triangle homotopy-commutative rel A (i.e. commutative up to weak equivalences in the under category $A\downarrow\mathbf{S}$ [5, p. 46]) and this is done in

2.2 THEOREM. *Given a commutative solid arrow diagram*

$$\begin{array}{ccc} A & \longrightarrow & Q_nX \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & Q_{n-1}X \end{array} \qquad n>1$$

in which $A\to B$ is a cofibration and X is fibrant, there exists a dotted arrow which makes the upper triangle commutative and the lower triangle homotopy-commutative rel. A iff the cocycle $h^{n+1}\in Z^{n+1}(B,A;\Pi_nX)$, obtained by pulling back the canonical cocycles (1.2(vi)) $k^{n+1}X\in Z^{n+1}(Q_{n-1}X, Q_nX;\Pi_nX)$, *is cohomologous to* 0.

This is an immediate consequence of the following lemma, which itself is readily verified

2.3 LEMMA. *Let $X\in\mathbf{S}$ be fibrant. Then the commutative diagram*

$$\begin{array}{ccc} Q_nX & \longrightarrow & N\Pi_1X \\ \downarrow & & \downarrow \\ Q_{n+1}X & \longrightarrow & K(\Pi_nX, n+1) \end{array} \qquad n>1$$

which (1.2) *represents the canonical cocycle $k^{n+1}X$, is, up to homotopy, a fibre square* (i.e. the map from Q_nX to the homotopy inverse limit [2, Ch. XI] of the remaining diagram is a weak equivalence).

We end with indicating the changes that have to be made to generalize these results to

2.4 OBSTRUCTIONS TO LIFTINGS OF MAPS. The problem here is, given a commutative solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & Y \end{array} \qquad \text{in } \mathbf{S}$$

in which the map $A \to B$ is a cofibration and the map $X \to Y$ is a fibration, to find inductive algebraic conditions for the existence of a dotted arrow which makes both triangles commutative. To do this one considers the factorization of the map $X \to Y$

$$X = \lim_{\leftarrow} Q_i'X \to \cdots \to Q_n'X \to Q_{n-1}'X \to \cdots \to Q_0'X \to Y$$

in which, for $n > 0$, $Q_n'X$ is the "$(n+1)$-coskeleton of X over Y," i.e. the simplicial set which has as its j-simplices the commutative diagrams

$$\begin{array}{ccc} sk_{n+1}\Delta[j] & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta[j] & \longrightarrow & Y \end{array}$$

and $Q_0'X$ is the "restricted 1-coskeleton of X over Y," i.e. simplicial set which has as its j-simplices the commutative diagrams

$$\begin{array}{ccc} sk_1\Delta[j] & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta[j] & \longrightarrow & Y \end{array}$$

in which the top map is homotopic to a constant. The map $Q_0'X \to Y$ then is readily verified to be a fibration which induces isomorphisms of the higher homotopy groups and hence, given a commutative solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & Q_0'X \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & Y \end{array}$$

the existence of a dotted arrow which makes both triangles commutative is equivalent to the corresponding (already algebraic) existence problem for the fundamental groupoids. It thus remains, given a commutative solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & Q'_n X \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & Q'_{n+1} X \end{array} \qquad\qquad n \geq 1$$

to find an algebraic condition for the existence of a dotted arrow which makes the upper triangle commutative and the lower triangle homotopy-commutative rel. A, which can be done as above. The details are straightforward and will be left to the reader.

§ 3. AN OBSTRUCTION THEORY FOR DIAGRAMS OF SIMPLICIAL SETS

We now describe an obstruction theory for diagrams of simplicial sets which completely parallels the obstruction theory for simplicial sets of § 2.

We start with some comments on

3.1 DIAGRAMS OF SIMPLICIAL SETS. Let $\mathbf{D}$ be a small category and let $\mathbf{S}^{\mathbf{D}}$ denote the category of $\mathbf{D}$-diagrams of simplicial sets, i.e. the category which has as objects the functors $\mathbf{D} \to \mathbf{S}$ and as maps the natural transformations between them. Then [3] $\mathbf{S}^{\mathbf{D}}$ admits a *closed simplicial model category* structure in which a map $f: X \to Y \in \mathbf{S}^{\mathbf{D}}$ is a weak equivalence (resp. a fibration) iff, for every object $D \in \mathbf{D}$, the map $fD: XD \to YD \in \mathbf{S}$ is a weak equivalence (resp. a fibration) and in which the simplicial structure is the obvious one, i.e. the one induced by the usual simplicial structure on $\mathbf{S}$.

Next we define, as in 1.2(v), a suitable notion of

3.2 COHOMOLOGY OF DIAGRAMS. Given a small category $\mathbf{D}$, a map $A \to B \in \mathbf{S}^{\mathbf{D}}$ and (1.2) a $\mathbf{D}$-diagram $G \in \mathbf{G}^{\mathbf{D}}$ of groupoids and a $\mathbf{D}$-diagram $M \in \mathbf{M}^{\mathbf{D}}$ which restricts to G, *the n-th relative cohomology group with local coefficients induced by a map* $f: B \to NG \in \mathbf{S}^{\mathbf{D}}$ (or equivalently a map $f': \Pi_1 B \to$ $\to G \in \mathbf{G}^{\mathbf{D}}$) is the abelian group given by the formula

$$H^n(B, A; M) = \pi_0 \hom(B, K(M, n))$$

where hom denotes the function complex in the between category $A \downarrow \mathbf{S}^{\mathbf{D}} \downarrow NG$. The vertices of $\hom(B, K(M, n))$ form an abelian group $Z^n(B, A; M)$ and are called *cocycles*. Two cocycles are called *cohomologous* if they are in the same components of $\hom(B, K(M, n))$. And, as in 1.2(v), *this notion has "homotopy meaning" if the map* $A \to B \in \mathbf{S}^{\mathbf{D}}$ *is* (3.1) *a cofibration.*

3.3 REMARKS

(i) If the map $A \to B \in \mathbf{S}^{\mathbf{D}}$ is a cofibration (3.1), then [3, 3.3] readily implies that $\hom(B, K(M, n))$ has the homotopy type of the homotopy inverse limit of a diagram which, for every map $D_0 \to D_1 \in \mathbf{D}$, contains the function complex (1.2(v)) $\hom(BD_0, K(MD_1, n))$. It follows [2, Ch. XI, § 7], that there is a spectral sequence which converges to $H^*(B, A; M)$ and has as E_2-term the higher derived functors of the inverse limit functor applied to the diagram of groups $H^*(BD_0, AD_0; MD_1)$.

(ii) A considerable simplification occurs if A is empty, G is trivial and $B = N(\mathbf{D}{\downarrow} -)$. In this case $H^*(B, A; M)$ consists of the higher derived functors of the inverse limit functor applied to the $\mathbf{D}$-diagram M.

(iii) Another simplification occurs if M is a constant $\mathbf{D}$-diagram with value $M_0 \in \mathbf{M}$, in which case $H^*(B, A; M) \approx H^*(\varinjlim B, \varinjlim A; M_0)$.

Now we are ready to discuss:

3.4 OBSTRUCTIONS TO EXTENSIONS OF MAPS. Given a solid arrow diagram in $\mathbf{S}^{\mathbf{D}}$

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ B & & \end{array}$$

in which the map $A \to B$ is a cofibration and X is fibrant (3.1), our aim is to find inductive algebraic conditions for the existence of a dotted arrow which makes the resulting triangle commutative. To do this one proceeds as in 2.1 and one thus has to prove

3.5 THEOREM. *Given a small category* $\mathbf{D}$ *and a commutative solid arrow diagram in* $\mathbf{S}^{\mathbf{D}}$

$$\begin{array}{ccc} A & \longrightarrow & Q_n X \\ \big\downarrow & \nearrow & \big\downarrow \\ B & \longrightarrow & Q_{n-1} X \end{array} \qquad\qquad n > 1$$

in which $A \to B$ *is a cofibration and* X *is fibrant, there exists a dotted arrow which makes the upper triangle commutative and the lower triangle homotopy-commutative rel.* A *iff the cocycle* $h^{n+1} \in Z^{n+1}(B, A; \Pi_n X)$, *obtained by pulling back the canonical cocycle* (1.2(v) and 3.2)

$$k^{n+1}X \in Z^{n+1}(Q_{n-1}X, Q_n X; \Pi_n X),$$

is cohomologous to 0.

This is an immediate consequence of the following lemma, which itself follows readily from 2.3 and 3.1.

3.6 LEMMA. *Let $X \in \mathbf{S}^{\mathbf{D}}$ be fibrant. Then the commutative diagram*

$$\begin{array}{ccc} Q_nX & \longrightarrow & N\Pi_1X \\ \big\downarrow & & \big\downarrow \\ Q_{n-1}X & \longrightarrow & K(\Pi_nX, n+1) \end{array} \qquad n>1$$

which represents the canonical cocycle $k^{n+1}X$ is, up to homotopy, a fibre square.

It remains to deal with

3.7 OBSTRUCTIONS TO LIFTINGS OF MAPS. This one does as in 2.4. The details are again left to the reader.

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