A PARAMETRIZED INDEX THEOREM FOR THE ALGEBRAIC K-THEORY EULER CLASS

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Abstract. Riemann-Roch theorems assert that certain algebraically defined wrong way maps (transfers) in algebraic K–theory agree with topologically defined ones [BaDo]. Bismut and Lott [BiLo] proved such a Riemann–Roch theorem where the wrong way maps are induced by the projection of a smooth fiber bundle, and the topologically defined transfer map is the Becker–Gottlieb transfer. We generalize and refine their theorem, and prove a converse stating that the Riemann–Roch condition is equivalent to the existence of a fiberwise smooth structure. In the process, we prove a family index theorem where the K–theory used is algebraic K–theory, and the fiber bundles have topological (not necessarily smooth) manifolds as fibers.

0. Introduction

This work is inspired by a recent paper of Bismut and Lott [BiLo], especially by their "Riemann–Roch theorem for flat complex vector bundles" . — Suppose that $p : E \to B$ is a smooth fiber bundle with compact fiber F. That is, F is a smooth manifold, possibly with boundary, and the structure group of p is the diffeomorphism group of F, a topological group. Let V be a flat complex vector bundle on E, and let V_i be the complex vector bundle on B whose fiber over $b \in B$ is the homology group with twisted coefficients $H_i(F_b; V)$ where F_b is the fiber over b. Then the Atiyah–Singer index theorem for families implies the following equation

$$\mathrm{tr}^*[V] = \sum (-1)^i [V_i] \quad \in K^0_{\mathrm{top}}(B) \tag{0–1}$$

where $K^0_{\mathrm{top}}(B)$ is the topological complex K–theory group, and where tr is the *Becker–Gottlieb transfer*, a stable map from B to E determined by the bundle p. See [BecSch] .

Notice now that all vector bundles in the equation are flat. Hence the two sides of the equation can be viewed and will be viewed as elements in $[B, K(\mathbb{C})]$, the set of homotopy classes of maps from B to the *algebraic* K–theory space $K(\mathbb{C})$ of the *discrete* field $\mathbb{C}$. It is natural to ask whether equation (0–1) holds in $[B, K(\mathbb{C})]$.

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Recall that the secondary characteristic classes of Kamber and Tondeur yield homomorphisms $c_k : [B, K(\mathbb{C})] \to H^k(B;\mathbb{R})$, for odd k. The Bismut–Lott Riemann–Roch theorem, BLRR in the sequel, states that

$$c_k\,(\mathrm{tr}^*[V]) = \sum(-1)^i c_k[V_i] \quad \in H^k(B;\mathbb{R}) \tag{0–2}$$

To prove this Bismut and Lott use Bismut's local version of the Atiyah–Singer theorem for families. (They also assume that the fibers of p are *closed* manifolds, but we have been told that this was mostly to simplify the presentation.) Lott then issued the challenge to topologists to find a more topological proof.

Our generalization is indeed based on a much more topological approach. Let V be a bundle of finitely generated projective left R–modules on E, where R is *any* ring. This determines in the usual way a homotopy class of maps $[V]$ from E to the algebraic K–theory space $K(R)$. (The map factors through the classifying space of the category of f.g. projective left R–modules and their isomorphisms.) Each fiber $F_b = p^{-1}(b)$ of p has homology groups $H_i(F_b; V)$ with (twisted) coefficients in V, and for fixed i these homology groups form a bundle V_i of left R–modules on B. We assume that the fibers of V_i are again projective, in which case they are also finitely generated. Then each V_i determines a homotopy class of maps $[V_i] : B \to K(R)$. In §9 we prove

$$\mathrm{tr}^*[V] = \sum(-1)^i[V_i] \qquad \in [B, K(R)]\,. \tag{0–3}$$

Our method for proving (0–3) yields a stronger formula, (0–4) below, which we call the *A-theory Riemann Roch Theorem*, ARR in the sequel. In this theorem Quillen's algebraic K-thory of rings is replaced by Waldhausen's K-theory of spaces.

Relaxing the conditions, we now suppose merely that $p : E \to B$ is a fibration where B is a CW space and the fibers of p are homotopy equivalent to *compact* CW spaces. Then both sides of equations (0–1), (0–2), (0–3), and (0–4) are well defined for p. However, the equations might fail. In fact we show that p satisfies the A-theory Riemann-Roch theorem *if and only if* p is fiber homotopy equivalent to a smooth bundle with compact fiber. In another paper we shall show that (0–2) and hence (0–3) can fail for a fiber bundle whose fibers are closed *topological* manifolds.

ARR relates different types of fiberwise Euler characteristics. It seems appropriate to discuss various refinements of the notion *Euler characteristic.* According to Becker and Gottlieb, the refined Euler characteristic of a space F homotopy equivalent to a compact CW–space is an element $\langle F\rangle_{\mathrm{bg}}$ in $Q(F_+) = \Omega^\infty\Sigma^\infty F_+$. When F is connected, the component of $\langle F\rangle_{\mathrm{bg}}$ in $\pi_0(Q(F_+)) \cong \mathbb{Z}$ is the "usual" Euler characteristic of F. The Becker–Gottlieb–Euler characteristic can be applied to *families.* For example, the fibration $p : E \to B$ above, with fiber F_x over $x \in B$, determines another fibration $Q_B(E_+) \to B$ whose fiber over $x \in B$ is $Q((F_x)_+)$, and a continuous section $\langle p\rangle_{\mathrm{bg}}$ of it which selects the point $\langle F_x\rangle_{\mathrm{bg}}$ in the fiber over x, for each x. Composing $\langle p\rangle_{\mathrm{bg}}$ with the inclusion $Q_B(E_+) \hookrightarrow Q(E_+)$, one obtains what is called the *Becker–Gottlieb transfer*, a map from B to $Q(E_+)$.

Waldhausen associates with F a space $A(F)$, which is the universal receptacle for relative Euler characteristics of retractive spaces over F with suitable finiteness conditions. (A retractive space over F is a space X together with maps $r : X \to F$ and $s : F \to X$ such that $rs = \mathrm{id}$.) In particular, the relative Waldhausen–Euler characteristic of the retractive space $\mathbb{S}^0 \times F$ (with $r(x, y) = y$ and $s(y) = (1, y)$) is a point $\langle F \rangle$ in $A(F)$. This construction can also be applied to families: The fibration $p : E \to B$ determines another fibration $A_B(E) \to B$ whose fiber over $x \in B$ is the space $A(F_x)$, and a section $\langle p \rangle$ of it which selects the point $\langle F_x \rangle$ in the fiber over x, for each x.

There is a well known natural splitting [Wald2], consisting of $\imath : Q(F_+) \to A(F)$ and $t : A(F) \to Q(F_+)$ with $t\imath \simeq \mathrm{id}$. It is a folk theorem (which we learned from John Klein, and which we shall neither use nor prove in this paper) that t takes Waldhausen–Euler characteristics to Becker–Gottlieb–Euler characteristics:

$$t\langle F \rangle \text{ “ = ” } \langle F \rangle_{\mathrm{bg}} .$$

(This formula is not really useful unless it is applied to families, in which case it means that two sections of a certain fibration are homotopic.) We show that if F is a smooth compact manifold with boundary, then also

$$\imath\langle F \rangle_{\mathrm{bg}} \text{ “ = ” } \langle F \rangle .$$

(Again this is not really useful unless it is applied to families, but here *family* means: fiber bundle with smooth compact manifold fibers.) Hence if a fibration $p : E \to B$ is fiber homotopy equivalent to a smooth bundle with compact fibers, then we must have a vertical homotopy

$$\imath \cdot \langle p \rangle_{\mathrm{bg}} \quad \simeq \quad \langle p \rangle . \tag{0–4}$$

Our main theorem is that this condition (ARR) is not only necessary, but also sufficient.

Returning to $p : E \to B$, note that we may narrow the search for smooth bundles with compact fiber on B, fiber homotopy equivalent to p, by prescribing the vertical tangent bundle of the manifold fiber bundles that we are searching for. We prescribe it by specifying an n–dimensional vector bundle γ on E. An n–dimensional *smooth structure* on p and γ consists of

(1) a fiber bundle $p^\sharp : E^\sharp \to B$ whose fibers are smooth compact n–manifolds with boundary ;
(2) a fiber homotopy equivalence $\phi : E^\sharp \to E$ (over B) ;
(3) a vector bundle isomorphism of $\phi^*\gamma$ with the vertical tangent bundle of $E^\sharp$.

We can make a space or simplicial set $\mathcal{S}^D_n(p, \gamma)$ out of these structures (D for differentiable). A k–simplex would be a "continuous family" of n–dimensional smooth structures on p and γ, parametrized by the simplex Δ^k. For our theorem, we want a large n, in fact $n = \infty$. To this end we have stabilization maps $\mathcal{S}^D_n(p, \gamma) \hookrightarrow \mathcal{S}^D_{n+1}(p, \gamma \oplus \varepsilon)$, given by taking product with the unit interval and rounding off corners. Let $\mathcal{S}^D(p, \gamma)$ be the union (direct limit) of the $\mathcal{S}^D_{n+k}(p, \gamma \oplus \varepsilon^k)$ for $k \geq 0$.

0.1. Theorem. *$\mathcal{S}^D(p,\gamma)$ is naturally homotopy equivalent to the homotopy fiber of*

$$\Gamma\big(Q_B(E)\to B\big) \quad \xrightarrow{\imath\cdot} \quad \Gamma\big(A_B(E)\to B\big)$$

over the point $\langle p\rangle$. (Here Γ denotes section spaces, and $\imath\cdot$ is composition with $\imath$.)

0.2. Remark (on parametrized Reidemeister torsion). For connected pointed F, the space $A(F)$ can also be described as the algebraic K–theory of the "ring space" $Q(\Omega(F)_+)$ (whose ring of components is the group ring $\mathbb{Z}[\pi_1(F)]$). In this sense $\langle F\rangle$ contains algebraic K–theory information about F, when defined.

Returning to $p : E \to B$ in Theorem 0.1 again, suppose that V is a bundle of finitely generated projective left R-module for some (discrete) ring R. This gives rise to a map $\lambda : A(E) \to K(R)$ which, in non–rigorous language, takes the Waldhausen Euler characteristic of a retractive space $X \leftrightarrows E$ to the Euler characteristic of the relative singular chain complex (coefficients in V) of the pair (X,E). Suppose further that for each $x \in B$, the cellular chain complex of the fiber F_x with coefficients in V is contractible. Then the image of $\langle F_x\rangle$ under the composition

$$A(F_x) \xrightarrow{\subset} A(E) \xrightarrow{\lambda} K(R)$$

is "trivialized", in other words: we have lifted $\langle F_x\rangle$ to

$$\text{hofiber}\,[A(F_x)\to\cdots\to K(R)] \tag{0–5}$$

where *hofiber* means homotopy fiber. We call this lift the *homotopy Reidemeister torsion* of F_x. Varying x, we have the parametrized homotopy Reidemeister torsion, a refinement of $\langle p\rangle$. It is a section of a certain fibration on B whose fiber over $x\in B$ is the space (0–5), for each x. If in addition $p : E \to B$ is a smooth fiber bundle with compact fibers, then by theorem 0.1 or condition (0–4) we have a more subtle refinement of $\langle p\rangle$: a section of a fibration on B whose fiber over $x \in B$ is

$$\text{hofiber}\,[Q(F_+)\xrightarrow{\imath}\cdots\to K(R)]$$

for each x. This would be called the parametrized *smooth Reidemeister torsion* of the smooth bundle p. Note that it depends on V.

Earlier, Igusa and Klein [IgK] used parametrized generalized Morse functions to define the parametrized Reidemeister torsion of a smooth fiber bundle. Theirs belongs to exactly the same section space as ours. Our philosophy is much closer to that of [BiLo], in fact identical to it. However, it is not clear to us at this stage whether the parametrized Reidemeister torsions produced by [BiLo] are in agreement with ours. Put differently, Bismut and Lott show that a weaker form of our condition (0–4) is satisfied for smooth fiber bundles with compact fibers, but we don't know whether their homotopy (given by the differential form $\mathcal{T}_*$ in [BiLo, Thm.0.2]) is always in agreement with the one we produce. It certainly is in agreement if the base space of the fiber bundle is a point ; this is the theorem of Cheeger and Müller, [Che], [Mü]. Put differently again, our theorem 0.1 and [BiLo] together make it possible to *state* a family version of the Cheeger–Müller theorem, and we would like to know whether this is in fact true.— As for the "weaker form", note that our theorem 0.1 is an integral result, whereas [BiLo] is about rational topology. Even rationally, our theorem 0.1 is stronger. For example, [BiLo, Thm 0.1] is an equality between *certain homotopy invariants* of two maps from B to the algebraic K–theory space $K(\mathbb{R})$, both defined in terms of $p : E \to B$ and a representation of $\pi_1(E)$. Our result implies that the two maps are in fact homotopic. See §10. (At the time of writing, the rational homotopy type of $K(\mathbb{R})$ is unknown, but it is known that $K_i(\mathbb{R}) \otimes \mathbb{Q}$ is infinite dimensional for some values of i.)

Our proof of theorem 0.1 is mainly by reduction to two other theorems. One of these is an analogue of theorem 0.1 for fiber bundles with compact *topological* manifolds as fibers. The other is an A–theory index theorem for families.

To state the topological analogue of theorem 0.1, we recall that $A(F)$ is the infinite loop space associated to a spectrum $\boldsymbol{A}(F)$:

$$A(F) = \Omega^\infty \boldsymbol{A}(F)\,.$$

The functor $F \mapsto \boldsymbol{A}(F)$ takes homotopy equivalences to homotopy equivalences. In this situation one has a natural *assembly map*:

$$\alpha : F_+ \wedge \boldsymbol{A}(*) \longrightarrow \boldsymbol{A}(F) \tag{0–6}$$

which is the obvious isomorphism when F is a point. (Strictly speaking, the domain of α is not $F_+ \wedge \boldsymbol{A}(*)$ but something homotopy equivalent to it.) The assembly map is perhaps better known to analysts under the name *Kasparov map*. We abbreviate

$$\boldsymbol{A}^{\%}(F) = F_+ \wedge \boldsymbol{A}(*)\,, \qquad A^{\%}(F) = \Omega^\infty(\boldsymbol{A}^{\%}(F))$$

and we still write $\alpha : A^{\%}(F) \to A(F)$ for the map of infinite loop spaces induced by (0–6). Given $p : E \to B$ as in theorem 0.1, we write $A^{\%}_B(E)$ for the total space of the fibration on B with fiber $A^{\%}(F_x)$ over $x \in B$, and we write $\mathcal{S}^T(p,\gamma)$ for the topological manifold analogue of $\mathcal{S}^D(p,\gamma)$. Here γ can be any microbundle [Mil1], [Kis] on E.

0.3. Theorem. *$\mathcal{S}^T(p,\gamma)$ is naturally homotopy equivalent to the homotopy fiber of*

$$\Gamma\big(A^{\%}_B(E) \to B\big) \quad \xrightarrow{\alpha\cdot} \quad \Gamma\big(A_B(E) \to B\big)$$

over the point $\langle p \rangle$.

We find this much easier to prove than theorem 0.1, and the reasons are not all that hard to understand. Namely, it turns out that for a compact euclidean neighbourhood retract F the Waldhausen–Euler characteristic $\langle F \rangle$ has a canonical lift to $A^{\%}(F)$: the *microcharacteristic*

$$\langle\!\langle F \rangle\!\rangle \in A^{\%}(F)\,.$$

By saying that $\langle\!\langle F \rangle\!\rangle$ *lifts* $\langle F \rangle$, we mean that it comes with a sufficiently canonical path in $A(F)$ from $\alpha\langle\!\langle F \rangle\!\rangle$ to $\langle F \rangle$. If $p : E \to B$ is a fibration with compact ENR fibers, otherwise as before, then the microcharacteristic can be taken fiberwise, which gives rise to a section

$$\langle\!\langle p \rangle\!\rangle : B \longrightarrow A^{\%}_B(E)\,.$$

This explains the left–hand column in a homotopy commutative square (notation of theorem 0.3)

$$(0\text{–}7)\qquad \begin{array}{ccc} \mathcal{S}^T(p,\gamma) & \longrightarrow & * \\ \Big\downarrow & & \Big\downarrow{\scriptstyle *\mapsto\langle p\rangle} \\ \Gamma\big(A^{\%}_B(E)\to B\big) & \xrightarrow{\ \alpha\cdot\ } & \Gamma\big(A_B(E)\to B\big)\,. \end{array}$$

Using results of Waldhausen, we can show that (0–7) is a homotopy pullback square—which proves theorem 0.3.

In our index theorem, we use a very general type of *Euler class* for euclidean bundles. For simplicity, suppose first that ξ is an n–dimensional vector bundle, with fibers ξ_x, on some space F which may be an ENR or a CW–space. We can assume that the structure group of ξ is $\mathrm{O}(\mathbb{R}^n) = \mathrm{O}(n)$, and we denote by $\mathrm{O}(\xi)$ the bundle of groups on F whose fiber over $x \in F$ is $\mathrm{O}(\xi_x)$.

Choose $k \gg 0$, form $\xi \oplus \varepsilon^k$ where ε^k is trivial k–dimensional. There is a bundle inclusion

$$(0\text{–}8)\qquad \mathrm{O}(\varepsilon^k)/\,\mathrm{O}(\varepsilon^{k-1}) \quad\hookrightarrow\quad \mathrm{O}(\xi\oplus\varepsilon^k)/\,\mathrm{O}(\xi\oplus\varepsilon^{k-1})$$

whose domain is a trivial bundle with fiber $\mathrm{O}(k)/\,\mathrm{O}(k-1) \cong \mathbb{S}^{k-1}$. Taking adjoints, we obtain an "interesting" section $e^D(\xi)$ (not the trivial section) of the bundle

$$(0\text{–}9)\qquad \Omega^{k-1}_F(\mathrm{O}(\xi\oplus\varepsilon^k)/\,\mathrm{O}(\xi\oplus\varepsilon^{k-1}))$$

on F, where Ω^{k-1}_F denotes fiberwise $(k-1)$–fold loops. Call it the *Euler section.* This was first constructed by Becker in [Be]. The fibers of (0–9) are homeomorphic to $\Omega^{k-1}(\mathrm{O}(k+n)/\,\mathrm{O}(k+n-1)) \cong \Omega^{k-1}\mathbb{S}^{k+n-1}$. Letting $k\to\infty$, we may say that $e^D(\xi)$ represents a class in the n–th cohomology of F with twisted coefficients in the sphere spectrum $\boldsymbol{S}^0$. The twist depends on ξ. In fact, the total space of the bundle (0–9), with $k=\infty$, is fiber homotopy equivalent over F to $Q_F(\mathrm{fthom}(\xi))$, where $\mathrm{fthom}(\xi)$ is the fiberwise one–point compactification of the total space of ξ. Use the point at infinity as base point in each fiber of $\mathrm{fthom}(\xi)$.

Next, suppose that ξ is a bundle with fibers homeomorphic to $\mathbb{R}^n$ and with structure group $\mathrm{TOP}(n)$. Keeping the orthogonal groups in the left–hand side of (0–8), and replacing orthogonal groups by homeomorphism groups in the right–hand side of (0–8), define a bundle

$$(0\text{–}10)\qquad \mathrm{EU}(\xi) := \Omega^{k-1}_F(\mathrm{TOP}(\xi\oplus\varepsilon^k)/\,\mathrm{TOP}(\xi\oplus\varepsilon^{k-1}))$$

on F, and an Euler section $e^T(\xi)$ of it. The fibers of the bundle (0–10) are homeomorphic to $\Omega^{k-1}(\mathrm{TOP}(n+k)/\,\mathrm{TOP}(n+k-1)$, so that $e^T(\xi)$ represents a class in the n–th cohomology of F with twisted coefficients in a spectrum

$$\{\mathrm{TOP}(i+1)/\,\mathrm{TOP}(i) \;\mid\; i \geq 0\}\,.$$

Thanks to Waldhausen [Wald2] this spectrum is homotopy equivalent to $\boldsymbol{A}(*)$. Using this remarkable result, one finds that the total space of (0–10), with $k=\infty$, is fiber homotopy equivalent over F to $A^{\%}_F(\text{fthom}(\xi))$.

The Hopf index theorem, perhaps the mother of all index theorems, states that the Poincaré dual of the Euler class of the tangent bundle of a compact manifold is the Euler characteristic of the manifold. Following Heinz Hopf, we now assume that F is a compact smooth or topological manifold and ask what happens to the Euler *section(s)* under Poincaré duality when ξ is the tangent bundle τ^F. Note first that the Euler sections can be *trivialized* over the boundary— they are not exactly trivial over the boundary, just vertically nullhomotopic by a canonical nullhomotopy. This uses the existence and essential uniqueness of collars, [Bro]. Then it is admissible to write

$$\text{(0–11)}\qquad \begin{aligned} e^D(\tau) &\in \Gamma_{\text{in}}\big(Q_F(\text{fthom}(\tau)) \to F\big) &&\text{(smooth case)}\\ e^T(\tau) &\in \Gamma_{\text{in}}\big(A^{\%}_F(\text{fthom}(\tau)) \to F\big) &&\text{(topological case)} \end{aligned}$$

where Γ_{in} denotes spaces of sections with support in the interior of F. By Poincaré duality, the section spaces in (0–11) are homotopy equivalent to

$$\text{(0–12)}\qquad \begin{aligned} &Q(F_+) &&\text{(smooth case)}\\ &A^{\%}(F) &&\text{(topological case)}\,. \end{aligned}$$

(Idea: (0–11) is relative cohomology twisted by the tangent bundle, and (0–12) is absolute untwisted homology. The coefficient spectra are $\boldsymbol{S}^0$ and $\boldsymbol{A}(*)$, respectively. Compare [WW1, §3].) Still inspired by Hopf, one may guess that $e^D(\tau)$ maps to the Becker–Gottlieb–Euler characteristic $\langle F\rangle_{\text{bg}} \in Q(F_+)$ under Poincaré duality (smooth case), and that $e^T(\tau)$ maps to the microcharacteristic $\langle\!\langle F\rangle\!\rangle \in A^{\%}(F)$ under Poincaré duality (topological case). The smooth case is implicit in [BeGo], also for families. See also Appendix A. The topological case, in the family version, is our index theorem. Notation: $p: E \to B$ is a fiber bundle with compact topological n–manifolds F_x as fibers, and τ is the vertical tangent bundle [Mil1], [Kis] on E, a fiber bundle with fibers $\cong \mathbb{R}^n$.

0.4. Theorem. *The image of $e^T(\tau) \in \Gamma_{\text{in}}\big(A^{\%}_E(\text{fthom}(\tau)) \to E\big)$ under fiberwise Poincaré duality*

$$\Gamma_{\text{in}}\big(A^{\%}_E(\text{fthom}(\tau)) \to E\big) \quad\xrightarrow{\simeq}\quad \Gamma\big(A^{\%}_B(E) \to B\big)$$

is vertically homotopic to the fiberwise microcharacteristic $\langle\!\langle p\rangle\!\rangle$, by a preferred vertical homotopy.

Back to the notation and assumptions of theorem 0.1: using Euler *sections* (not Euler characteristics) and Poincaré duality, we construct the vertical arrows in a commutative diagram

$$\text{(0–13)}\qquad \begin{array}{ccc} \mathcal{S}^D(p,\gamma) & \xrightarrow{\ \subset\ } & \mathcal{S}^T(p,\gamma)\\ \big\downarrow & & \big\downarrow\\ \Gamma\big(Q_B(E_+) \to B\big) & \xrightarrow{\ \imath\cdot\ } & \Gamma\big(A^{\%}_B(E) \to B\big)\,. \end{array}$$

By smoothing theory (not just [HiMa], but also [Mor1], [Mor2], [Mor3], [BuLa], [KiSi]), (0–13) is a homotopy pullback square. By the index theorem 0.4, the left–hand column in the homotopy pullback square (0–7) agrees with the right–hand column in (0–13). Joining the squares we have

$$\begin{array}{ccccc}
\mathcal{S}^D(p,\gamma) & \xrightarrow{\ \subset\ } & \mathcal{S}^T(p,\gamma) & \longrightarrow & * \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle *\mapsto\langle p\rangle} \\
\Gamma\big(Q_B(E_+)\to B\big) & \xrightarrow{\ \imath\cdot\ } & \Gamma\big(A_B^{\%}(E)\to B\big) & \xrightarrow{\ \alpha\cdot\ } & \Gamma\big(A_B(E)\to B\big)
\end{array} ,$$

and deletion of the middle column completes the proof of theorem 0.1.

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Section Headings

1. Fibrations and Fiber Bundles

1.1. Notation. We introduce several topological categories. Each of these has a discrete class of objects. The morphism spaces should be regarded as simplicial sets. The categories are quite large, so that their classifying spaces are *simplicial classes*, to which we nevertheless refer informally as *spaces*.

(1) $\mathfrak{homeo}_n$: Objects are compact n–manifolds M. The space of morphisms from M to M' in $\mathfrak{homeo}_n$ is the space of homeomorphisms $M \to M'$.
(2) $\mathfrak{invemb}_n$: Objects are the same as for $\mathfrak{homeo}_n$, but the space of morphisms from M to M' in $\mathfrak{invemb}_n$ is the space of *invertible* embeddings from M to M'. An embedding (=injective continuous map) $e : M \to M'$ is *invertible* if there exists another embedding $f : M' \to M$ such that fe and ef are isotopic to the identity embeddings of M and M', respectively.
(3) $\mathfrak{G}_n$: Objects are compact ENR's equipped with an n–microbundle γ. The space of morphisms from (F,γ) to (F',γ') is the space of homotopy equivalences $F \to F'$ covered by microbundle maps $\gamma \to \gamma'$.

1.2. More notation and some motivation. Note that

$$\mathfrak{homeo}_n \subset \mathfrak{invemb}_n \subset \mathfrak{G}_n$$

where the second "inclusion" is defined by $M \mapsto (M, \tau^M)$. Product with the identity from I to I gives compatible "stabilization" functors from $\mathfrak{homeo}_n$ to $\mathfrak{homeo}_{n+1}$, from $\mathfrak{invemb}_n$ to $\mathfrak{invemb}_{n+1}$, and from $\mathfrak{G}_n$ to $\mathfrak{G}_{n+1}$. Let

$$\mathfrak{homeo}\,, \qquad \mathfrak{invemb}\,, \qquad \mathfrak{G}$$

be the resulting stabilized categories. (For example, an object in $\mathfrak{G}$ is a triple (m, F, γ) where $m \geq 0$ and (F, γ) is an object in $\mathfrak{G}_m$. A morphism in $\mathfrak{G}$ from (m, F, γ) to (n, F', γ') is a morphism in $\mathfrak{G}_n$ from $(F \times I^{n-m}, \gamma \times \varepsilon^{n-m})$ to (F', γ') provided $n \geq m$, and if $n < m$ there is no morphism.) Then

$$B\mathfrak{homeo} \simeq \operatorname*{hocolim}_n B\mathfrak{homeo}_n\,,$$
$$B\mathfrak{invemb} \simeq \operatorname*{hocolim}_n B\mathfrak{invemb}_n\,,$$
$$B\mathfrak{G} \simeq \operatorname*{hocolim}_n B\mathfrak{G}_n$$

by inspection or by [Tho1].

Note that $B\mathfrak{homeo}_n$ classifies (up to isomorphism) fiber bundles whose fibers are compact n–manifolds. $B\mathfrak{G}_n$ classifies (up to fiber homotopy equivalence) fibrations $p : E \to B$ where the fibers are homotopy equivalent to compact CW–spaces and where E is equipped with an n–microbundle γ. We obtain similar but more stable interpretations for $B\mathfrak{homeo}$ and $B\mathfrak{G}$. Loosely speaking, $B\mathfrak{homeo}$ classifies fiber bundles with compact manifold fibers, and $B\mathfrak{G}$ classifies fibrations $p : E \to B$ where the fibers are homotopy equivalent to compact CW–spaces and where E is equipped with a *stable* microbundle γ. If $p : E \to B$ is such a fibration, classified by $c : B \to B\mathfrak{G}$, then the homotopy fiber of

$$\operatorname{map}(B, B\mathfrak{homeo}) \hookrightarrow \operatorname{map}(B, B\mathfrak{G}) \tag{1–1}$$

over the point c is homotopy equivalent to $\mathcal{S}^T(p, \gamma)$.

By contrast, the spaces $B\mathfrak{invemb}_n$ and $B\mathfrak{invemb}$ are not meant to be "interpreted". We use them to bridge a gap.

1.3. Proposition. *The inclusion $B\mathfrak{invemb} \to B\mathfrak{G}$ is a homotopy equivalence.*

1.4. Proposition. *The homotopy fiber of $B\mathfrak{homeo}_n \hookrightarrow B\mathfrak{invemb}_n$ over an object M^n is homotopy equivalent to the h–cobordism space $\mathfrak{H}(\partial M)$.*

1.5. Corollary. *The homotopy fiber of $B\mathfrak{homeo} \hookrightarrow B\mathfrak{invemb}$ over an object M^n is homotopy equivalent to the stabilized h–cobordism space*

$$\operatorname*{hocolim}_k \mathfrak{H}(M \times I^k) = \Omega\,\mathrm{Wh}^{\mathrm{TOP}}(M)\,.$$

1.6. Remarks. (1) Strictly speaking we define $\mathfrak{H}(\partial M)$ in 1.4 as the space of *invertible* cobordisms on ∂M. By definition, a cobordism $(W; \partial M, L)$ is invertible if there exist cobordisms $(V_1; L, \partial M)$ and $(V_2; L, \partial M)$ such that $W \amalg_L V_1$ and $V_2 \amalg_{\partial M} W$ are product cobordisms. Invertible cobordisms are h–cobordisms [Stal, Thm.2]. We do not know whether the converse is true, but we do know that any counterexample to the three–dimensional Poincaré conjecture gives rise to a non–invertible h–cobordism with base $\mathbb{S}^2$ (delete two small open disks from the fake 3–sphere).

(2) Invertible cobordisms arise in the proof of 1.4 as follows: Let $e : M \to M'$ be an invertible embedding, and suppose that $e(M)$ is contained in the interior of M (both M and M' are compact). Then the closure of $M' \smallsetminus e(M)$ is an invertible cobordism from ∂M to $\partial M'$. The proof uses some isotopy extension theory as in [Ce] or [EdKi]. We are indebted to Bill Browder for pointing this out to us.

(3) Let $(W; K, L)$ be an invertible cobordism (K and L closed). The space of embeddings of W in $K \times [0,1)$ extending the inclusion of $K \cong K \times \{0\}$ in $K \times [0,1)$ is contractible. (Use invertibility and the uniqueness of collars, [Bro]).

1.8. Remark. The topological categories $\mathfrak{homeo}_n$ and $\mathfrak{homeo}$ are clearly topological groupoids. Also, $\mathfrak{invemb}_n$, $\mathfrak{invemb}$, $\mathfrak{G}_n$ and $\mathfrak{G}$ are *groupoid–like*, that is, they become groupoids when all morphism spaces in sight are replaced by their π_0.

Proof of 1.3. Let (F, γ) be an object of $\mathfrak{G}$. Let M, M' be objects of $\mathfrak{invemb}_n$, for some large n, such that both M and M' are in the same component of $\mathfrak{G}$ as (F, γ). Then $M \times I^k$ and $M' \times I^k$ are tangentially homotopy equivalent for large k, and it follows easily that there exists an invertible embedding of $M \times I^k$ in $M' \times I^k$. (Compare [Maz] ; use topological immersion theory [Gau] and a general position argument.) Hence the inclusion $B\mathfrak{invemb} \hookrightarrow B\mathfrak{G}$ is a bijection on components, and we may restrict attention to the component of $B\mathfrak{invemb}$ containing a particular object M^n, say, and to the image component in $B\mathfrak{G}$. Since $\mathfrak{invemb}$ and $\mathfrak{G}$ are *groupoid–like*, these components are homotopy equivalent to the classifying spaces $B\,\mathrm{end}(M)$ and $B\,\mathrm{END}(M)$, where $\mathrm{end}(M)$ is the topological monoid of endomorphisms of M in $\mathfrak{invemb}$ and $\mathrm{END}(M)$ is the larger topological monoid of endomorphisms of (M, τ^M) in $\mathfrak{G}$. Again, topological immersion theory and a general position argument show that the inclusion $\mathrm{end}(M) \hookrightarrow \mathrm{END}(M)$ is a homotopy equivalence. Hence the inclusion of classifying spaces is also a homotopy equivalence. □

Remark. The classifying space $B\,\mathrm{end}(M)$ is homotopy equivalent to the classifying space of the group of stabilized homeomorphisms $\mathrm{int}(M) \to \mathrm{int}(M)$. Taking this into account, we can say that 1.3 is a variant on the *open fiber smoothing theorem* of Casson and Gottlieb [CaGo].

Proof of 1.4. We shall use Quillen's Theorem B [Qui], or rather Waldhausen's simplicial version of it [Wald2, §4], with the "addendum". We need the following special case: Let $\mathcal{Z} : \mathcal{C} \to \mathcal{D}$ be a simplicial functor between simplicial categories. Suppose that the simplicial sets $\mathrm{ob}(\mathcal{C})$ and $\mathrm{ob}(\mathcal{D})$ are discrete. For an object D in $\mathcal{D}[0]$, let $\mathcal{Z}/D$ be the simplicial category whose objects in degree n are pairs (C, f) where C is an object of $\mathcal{C}[0]$ and $f : \mathcal{Z}(C) \to D$ is a morphism in $\mathcal{D}[n]$ (that is, a morphism from the n–fold degeneracy of $\mathcal{Z}(C)$ to the n–fold degeneracy of D). The morphisms in $\mathcal{Z}/D$ are commutative triangles. *Suppose* that, for any morphism $e : D \to D'$ in $\mathcal{D}[0]$, the transition functor

$$e_* : \mathcal{Z}/D \to \mathcal{Z}/D' \quad ; \quad (C, f) \mapsto (C, ef)$$

induces a homotopy equivalence of the classifying spaces, $B(\mathcal{Z}/D) \simeq B(\mathcal{Z}/D')$.

Then the square

$$\begin{array}{ccc} \mathcal{Z}/D & \xrightarrow{\text{forget}} & \mathcal{C} \\ \Big\downarrow \mathcal{Z}\cdot & & \Big\downarrow \mathcal{Z} \\ \mathrm{id}_{\mathcal{D}}/D & \xrightarrow{\text{forget}} & \mathcal{D} \end{array}$$

is a homotopy pullback square (and, of course, the lower left–hand term is contractible).

We apply all this with $\mathcal{C} = \mathfrak{homeo}_n$, $\mathcal{D} = \mathfrak{invemb}_n$, and $D = M^n$, and $\mathcal{Z}$ equal to the inclusion functor. Let $\mathcal{F}_1$ be the homotopy functor represented by the space $B(\mathcal{Z}/M)$. Then for any finite CW complex X, an element in $\mathcal{F}_1(X)$ is represented by a pair $(q : E \to X, \beta)$ where q is a fiber bundle over X with compact n–manifold fibers, and β is a map from E to M such that β restricted to any fiber of p is an invertible embedding. Two such pairs (q_0, β_0) and (q_1, β_1) are *equivalent* if there exists a bundle isomorphism $\iota : E_0 \to E_1$ such that $\beta_1\iota$ is isotopic to β_0. In particular, every element in $\mathcal{F}_1(X)$ has a representative (q, β) such that $\mathrm{im}(\beta) \cap \partial M = \emptyset$. We call such a representative regular.

Let $\mathcal{F}_2$ be the homotopy functor represented by $\mathfrak{H}(\partial M)$. For any finite CW complex X, an element in $\mathcal{F}_1(X)$ is represented by an *h-cobordism bundle*, i.e. a fiber bundle $\eta : H \to X$ such that η contains as a subbundle the product bundle $X\times\partial M \to X$ and such that each fiber of η is an (invertible) h-cobordism with base the copy of ∂M in that fiber. Another such bundle η_1 represents the same element if there exists a bundle isomorphism $\eta \to \eta_1$ extending the identity on the common subbundles $X\times\partial M \to X$.

Define a natural tranformation $\psi : \mathcal{F}_1 \cong \mathcal{F}_2$ as follows. Send an element in $\mathcal{F}_1(X)$ with regular representative $(q : E \to X, \beta)$ to the class of the h–cobordism bundle on X whose total space is the closure of the complement of the image of

$$(q, \beta) : E \longrightarrow X\times M\,.$$

The fibers of this so–called h–cobordism bundle are indeed invertible h–cobordisms, by remark 1.6, item (2). It follows from 1.6, item (3), that ψ is a natural bijection, so that

$$B(\mathcal{Z}/M) \simeq \mathfrak{H}(\partial M)\,.$$

Now we can easily verify that the hypothesis of Quillen's Theorem B is satisfied. Let $e : M \to M'$ be a morphism in $\mathfrak{invemb}_n$. The homotopy class of the transition map

$$e_* : B(\mathcal{Z}/M) \simeq \mathfrak{H}(\partial M) \quad\longrightarrow\quad B(\mathcal{Z}/M') \simeq \mathfrak{H}(\partial M')$$

will only depend on the connected component of e, so we may assume that e is regular. The closure of the complement of $\mathrm{im}(e)$ is then an h–cobordism W from ∂M to $\partial M'$, and e_* is simply concatenation with this h–cobordism, as a map from $\mathfrak{H}(\partial M)$ to $\mathfrak{H}(\partial M')$. Since W is invertible, e_* is a homotopy equivalence. Applying Theorem B now, we find that the homotopy fiber in 1.4 is homotopy equialent to the classifying space of the categorical fiber, which is $B(\mathcal{Z}/M) \simeq \mathfrak{H}(\partial M)$. □

Remark. By the remark following the proof of 1.3, the component of $B\mathfrak{invemb}_n$ containing the object M^n classifies fiber bundles with fibers homeomorphic to $\mathrm{int}(M)$.

The corresponding component of $B\mathfrak{homeo}_n$ classifies fiber bundles with fibers homeomorphic to M. This shows that 1.4 becomes the Kuiper–Lashof theorem [KuiLa1], [KuiLa2] when $M = \mathbb{D}^n$. See also [Cm].

Proof and explanation of 1.5. The space $\Omega\,\mathrm{Wh}^{\mathrm{TOP}}(M)$ may be defined as the homotopy colimit of the spaces $\mathfrak{H}(M\times I^k)$, for $k\to\infty$, under the *upper stabilization maps*

$$\mathfrak{H}(M\times I^k) \longrightarrow \mathfrak{H}(M\times I^{k+1})$$

described in and [Wald2]. It turns out that $\Omega\,\mathrm{Wh}^{\mathrm{TOP}}(M)$ is a homotopy invariant functor in the variable M. For more precision and details, see [BuLa], [Wald2] and [Wald3]. From 1.4, we see that the homotopy fiber Φ in 1.5 can be described as the homotopy colimit (here telescope) of a diagram

$$\mathfrak{H}(\partial M) \to \mathfrak{H}(\partial(M\times I)) \to \mathfrak{H}(\partial(M\times I^2)) \to \mathfrak{H}(\partial(M\times I^3)) \to \cdots$$

where the maps are given as follows. Go from $\mathfrak{H}(\partial(M\times I^k))$ to $\mathfrak{H}(\partial(M\times I^k)\times I)$ by upper stabilization ; then go from there to $\mathfrak{H}(\partial(M\times I^{k+1}))$ using the map induced by the inclusion

$$\partial(M\times I^k)\times I \subset \partial(M\times I^{k+1}))\,.$$

Since upper stabilization commutes with inclusion–induced maps, we conclude that

$$\Phi \simeq \operatorname*{hocolim}_k \operatorname*{hocolim}_j \mathfrak{H}(\partial(M\times I^k)\times I^j) = \operatorname*{hocolim}_k \Omega\,\mathrm{Wh}^{\mathrm{TOP}}(\partial(M\times I^k))\,.$$

Since $\Omega\,\mathrm{Wh}^{\mathrm{TOP}}$ is a homotopy functor, and commutes up to homotopy equivalence with the special homotopy colimits that we are using (telescopes), we find

$$\Phi \simeq \Omega\,\mathrm{Wh}^{\mathrm{TOP}}(\operatorname*{hocolim}_k(\partial(M\times I^k))) \simeq \Omega\,\mathrm{Wh}^{\mathrm{TOP}}(M)\,. \quad \square$$

2. Euler Characteristics and Microcharacteristics

Our goal in this section is to set up diagram (0–7) and to prove Theorem 0.3. Subsections 2.1 and 2.4 are essentially taken from [WW3], and subsection 2.3 is from [WWAs]. The reader should have an intuitive understanding of homotopy direct and homotopy inverse limits [BK], [HoVo]. See also [WWAs]. Our notation is: holim for homotopy inverse limits alias homotopy projective limits, and hocolim for homotopy direct limits. Another fundamental concept that we use freely is the K–theory of a category with cofibrations and weak equivalences, [Wald3].

Terminology: In this section, *space* means a space homotopy equivalent to a CW–space. A space is *homotopy finite* if it is homotopy equivalent to a compact CW–space. A retractive space over a space X consists of a space Y and maps $s: X\to Y$, $r: Y\to X$ such that $rs = \mathrm{id}_X$ and s is a closed embedding having the homotopy extension property. A map over X and relative to X between retractive spaces over X is a *cofibration* if the underlying map of spaces is a closed embedding having the homotopy extension property : it is a *weak equivalence* if the underlying map of spaces is a homotopy equivalence. With these notions of cofibration and weak equivalence, the category of retractive spaces over X is indeed a category with cofibrations and weak equivalences, in the sense of [Wald 3]. This follows from [Str]. A retractive space over X is *homotopy finite* if it is the codomain of a weak equivalence from another retractive space over X which is a CW–space relative to X, with finitely many cells. Define $A(X)$ as the K–theory of the category of homotopy finite retractive spaces over X. This must be regarded as a simplicial *class*, since we have made no effort to downsize the category of homotopy finite retractive spaces over X. Also, a small effort is required to make $A(X)$ into a functor of the variable X. We leave this to the reader.

2.1. Euler characteristics

Let Y be a homotopy finite space. Any homotopy finite retractive space over Y determines a point in $A(Y)$; specifically, the retractive space

$$\mathbb{S}^0 \times Y \underset{s}{\overset{r}{\rightleftarrows}} Y$$

where r is the projection and s identifies Y with $\{1\} \times Y$, determines a point

$$\langle Y \rangle \in A(Y).$$

2.1.1. Definition. We call $\langle Y \rangle$ the *Euler characteristic* of Y. (Note that we are interested in the point $\langle Y \rangle$, not just in its connected component.)

A homotopy equivalence $f : X \longrightarrow Y$ between homotopy finite spaces induces another homotopy equivalence $f_* : A(X) \longrightarrow A(Y)$. It also determines a path $\langle f \rangle$ in $A(Y)$ from $f_*\langle X \rangle$ to $\langle Y \rangle$. Namely, $f_*\langle X \rangle$ is the point in $A(Y)$ corresponding to the retractive space

$$\{-1\} \times X \cup \{1\} \times Y \rightleftarrows Y$$

where the retraction is equal to f on $\{-1\} \times X$. Now f gives a weak equivalence from this retractive space to

$$\mathbb{S}^0 \times Y \rightleftarrows Y.$$

The weak equivalence determines a path in $A(Y)$.

We can continue in this manner, looking e.g. at composable sequences of homotopy equivalences. Perhaps the best way to express the naturality properties of Euler characteristics is to use homotopy inverse limits. Let $\mathfrak{p}$ be a covariant functor from a small category $\mathcal{C}$ to homotopy finite spaces. Suppose $\mathfrak{p}$ takes all morphisms in $\mathcal{C}$ to homotopy equivalences. For each c in $\mathcal{C}$, we have a functor from the "over" category $\mathcal{C}\downarrow c$ to the category of retractive spaces over $\mathfrak{p}(c)$ which takes $b \to c$ to the retractive space $\mathfrak{p}(b) \amalg \mathfrak{p}(c)$. Since this functor takes all morphisms in $\mathcal{C}\downarrow c$ to weak equivalences, it induces a map

$$B(\mathcal{C}\downarrow c) \longrightarrow A(\mathfrak{p}(c))\,. \tag{2–1}$$

Note that $c \mapsto A(\mathfrak{p}(c))$ is another functor taking all morphisms in $\mathcal{C}$ to homotopy equivalences. We can regard (2–1) as a natural transformation between functors in the variable c. By definition, such a natural transformation is a point in the homotopy inverse limit

$$\operatorname*{holim}_{c \text{ in } \mathcal{C}} A(\mathfrak{p}(c)) .$$

By inspection, our point projects to $\langle \mathfrak{p}(c) \rangle \in A(\mathfrak{p}(c))$, for each c in $\mathcal{C}$. We refer to this type of naturality as *lax naturality.* The Euler characteristic $\langle Y \rangle$ is lax natural with respect to homotopy equivalences.

We like to think of points in the homotopy inverse limit above as "sections" of a certain "fibration". This is explained in the next construction.

2.1.2. Construction. Let $\mathcal{C}$ be a small category. Let $\mathfrak{v}(c) = B(\mathcal{C}\downarrow c)$ for each c in $\mathcal{C}$. Suppose that $\mathfrak{q}$ is a covariant functor from $\mathcal{C}$ to spaces taking all morphisms to homotopy equivalences. Form the commutative square

$$\begin{array}{ccc} \operatorname{hocolim} \mathfrak{q} \times \mathfrak{v} & \longrightarrow & \operatorname{hocolim} \mathfrak{q} \\ \big\downarrow v^*q & & \big\downarrow q \\ \operatorname{hocolim} \mathfrak{v} & \xrightarrow{v} & \operatorname{hocolim} * \end{array}$$

(all homotopy colimits over $\mathcal{C}$, and v and q are the obvious projections). This is a pullback square where the horizontal arrows are homotopy equivalences and the vertical arrows are *quasifibrations* in the sense of [DoTho]. A point in the homotopy *inverse* limit $\operatorname{holim} \mathfrak{q}$ is a natural transformation $t : \mathfrak{v} \to \mathfrak{q}$, giving rise to another natural transformation $(t, \mathrm{id}) : \mathfrak{v} \to \mathfrak{q} \times \mathfrak{v}$, and this induces a *section* t_* of the quasifibration on the left. Thus we have maps

$$\operatorname{holim} \mathfrak{q} \longrightarrow \Gamma(v^*q) \xleftarrow{\simeq} \Gamma(q)$$

where Γ denotes the section spaces of the associated fibrations. *Exercise:* The left–hand map, from $\operatorname{holim} \mathfrak{q}$ to $\Gamma(v^*q)$, is also a homotopy equivalence (spectral sequence argument ; use the skeleton filtration of $B\mathcal{C}$). In this way, $\operatorname{holim} \mathfrak{q}$ is an accceptable model or substitute for $\Gamma(q)$.

2.1.3. Remark. Again let $\mathfrak{p}$ from $\mathcal{C}$ to spaces be a covariant functor taking all morphisms to homotopy equivalences. Since the projection $p : \operatorname{hocolim} \mathfrak{p} \to B\mathcal{C}$ is a quasifibration in the sense of Dold–Thom, we can convert it into a fibration using the Serre construction, without changing the homotopy types of the fibers. To increase the usefulness of this observation we add another: essentially *every* fibration arises in this way. Indeed, suppose that $p : E \to B$ is a fibration (where E and B are homotopy equivalent to CW–spaces, according to our conventions). Let $\operatorname{simp}(B)$ be the category of singular simplices of B. An object of $\operatorname{simp}(B)$ is a map $f : \Delta^j \to B$ for some $j \geq 0$, and a morphism from $f : \Delta^j \to B$ to $g : \Delta^k \to B$ is a monotone map v from $\{1, \dots, j\}$ to $\{1, \dots, k\}$ such that $gv_* = f$. Define $\mathfrak{p}$ and $\mathfrak{w}$ from $\operatorname{simp}(B)$ to spaces by $\mathfrak{p}(f) = f^*E$ for $f : \Delta^j \to B$, and $\mathfrak{w}(f) = \Delta^j$. There is a commutative diagram

$$\begin{array}{ccccc} \operatorname{hocolim} \mathfrak{p} & \xleftarrow{=} & \operatorname{hocolim} \mathfrak{p} & \longrightarrow & E \\ \big\downarrow \text{proj.} & & \big\downarrow & & \big\downarrow p \\ \operatorname{hocolim} * & \xleftarrow{\text{proj.}} & \operatorname{hocolim} \mathfrak{w} & \xrightarrow{\text{eval.}} & B \end{array}$$

where middle arrow is induced by the projections $f^*E \to \Delta^j$ for $f : \Delta^j \to B$, and "eval." is induced by the evaluation transformation from $\mathfrak{w}$ to the constant functor with value B. (All homotopy colimits are over simp(B).) Both squares in the diagram are homotopy pullback squares, and the maps in the lower row are homotopy equivalences [Se2].

2.1.4. Example. Let $\mathfrak{p}$ be a functor from $\mathcal{C}$ to spaces taking all morphisms to homotopy equivalences, and such that each $\mathfrak{p}(c)$ is homotopy finite. Then $c \mapsto A(\mathfrak{p}(c))$ is another functor taking all morphisms to homotopy equivalences. Now write

$$B = B\mathcal{C}, \qquad E := \operatorname{hocolim} \mathfrak{p}, \qquad A_B(E) := \operatorname{hocolim} A \cdot \mathfrak{p}\,.$$

Let $p : E \to B$ be the projection and write $\langle p \rangle$ for the distinguished element in $\operatorname{holim} A \cdot \mathfrak{p}$ that we get from (2–1). We call $\langle p \rangle$ the *Euler section* of p. Informally, it is a section $B \to A_B(E)$ of the projection $A_B(E) \to B$.

We can also write $E = \operatorname{colim} \mathfrak{p} \times \mathfrak{v}$ (notation of 2.1.2), which explains the second arrow in

$$\text{(2–2)} \qquad \operatorname{hocolim} A \cdot \mathfrak{p} \xleftarrow{\simeq} \operatorname{hocolim} A \cdot \mathfrak{p} \longrightarrow A(E)\,.$$

Informally, (2–2) is a map from $A_B(E)$ to $A(E)$. We mention it here because the composition

$$B \xrightarrow{\langle p \rangle} A_B(E) \longrightarrow A(E)$$

appears implicitly on the right–hand side of [BiLo, Thm. 0.1]. More details are given below in subsection 2.2 and in section 10.

2.2. Linearized Euler characteristics

Keep the notation of 2.1.4. Suppose in addition that E is equipped with a bundle V of finitely generated projective left R–modules, where R is some (discrete) ring. We want to take a closer look at the composition

$$\langle p \rangle_V : \quad B \xrightarrow{\langle p \rangle} A_B(E) \longrightarrow A(E) \xrightarrow{\lambda} K(R)$$

where the last map, from $A(E)$ to $K(R)$, is induced by an exact functor λ between categories with cofibrations and weak equivalences. For a homotopy finite retractive space

$$X \underset{s}{\overset{r}{\rightleftarrows}} E$$

let $\lambda(X \leftrightarrows E)$ be the singular chain complex of the pair (X, E) with (twisted) coefficients in the bundle of modules $r^*(V)$. Note that $\lambda(X \leftrightarrows E)$ is a chain complex of projective R–modules, homotopy equivalent to a finitely generated one. Such chain complexes form a category with cofibrations and weak equivalences, and its K–theory is $K(R)$. More details can be found below.

Now for the alternative description of the composite map $\langle p \rangle_V$. For each vertex c in B (=object c in $\mathcal{C}$), we have $\mathfrak{p}(c) \subset E$ as the fiber over c. Let $\mathfrak{p}_V(c)$ be the singular chain complex of $\mathfrak{p}(c) \subset E$ with (twisted) coefficients in V. Now $\mathfrak{p}_V$ is a functor from $\mathcal{C}$ to certain chain complexes taking all morphisms to homotopy equivalences. Hence it leads to a map $\langle p_V \rangle$ from $B\mathcal{C} = B$ to $K(R)$. The following is true by construction.

2.2.1. Proposition. $\langle p\rangle_V \simeq \langle p_V\rangle$.

Assuming that each $H_i(\mathfrak{p}_V(c))$ has a finite length resolution by f.g. projective left R–modules, we shall give a homology theoretic description of $\langle p_V\rangle$. Here is some K–theory background.

Recall that an exact category is determined by an embedding of an additive category as a full subcategory of an abelian category $\mathcal{A}$ where $\mathcal{M}$ is closed under extensions in $\mathcal{A}$. See [Qui, p.16] and [Tho2, App.A]. A map f in $\mathcal{M}$ is an admissible monomorphism if it is a monomorphism in $\mathcal{A}$ and the cokernel is isomorphic to an object in $\mathcal{M}$. Dually, a map f in $\mathcal{M}$ is an admissible epimorphism if it is an epimorphism in $\mathcal{A}$ and the kernel is isomorphic to an object in $\mathcal{M}$. The two main examples for us are

$$\mathcal{M} = \mathcal{P}\mathcal{A}\,,$$

the category of projective objects in the abelian category $\mathcal{A}$; and

$$\mathcal{M} = \mathcal{N}\mathcal{P}\mathcal{A}\,,$$

consisting of those objects in the abelian category $\mathcal{A}$ which have finite length resolutions by objects in $\mathcal{P}\mathcal{A}$. The letter $\mathcal{N}$ can be read as *nearly.* If $\mathcal{A}$ is the category of finitely generated left modules over a ring R, then we write $\mathcal{P}_R$ and $\mathcal{N}\mathcal{P}_R$ instead of $\mathcal{P}\mathcal{A}$ and $\mathcal{N}\mathcal{P}\mathcal{A}$.

Notice that any morphism in $\mathcal{P}\mathcal{A}$ which is epi in $\mathcal{A}$ is admissible. A morphism in $\mathcal{P}\mathcal{A}$ which is mono in $\mathcal{A}$ is admissible iff it splits. However, all morphisms in $\mathcal{N}\mathcal{P}\mathcal{A}$ which are epi/mono in $\mathcal{A}$ are admissible. See [Bass, I.6.2]. Quillen's Q–construction associates to an exact category $\mathcal{M}$ an infinite loop space $K(\mathcal{M})$. See [Qui]. Quillen's resolution theorem [Qui] implies that the inclusion $K(\mathcal{P}\mathcal{A}) \longrightarrow K(\mathcal{N}\mathcal{P}\mathcal{A})$ is a homotopy equivalence.

For any category $\mathcal{D}$ with cofibrations $\operatorname{cof}\mathcal{D}$ and weak equivalences $w\mathcal{D}$, Waldhausen [Wald3] has constructed an infinite loop space $\Omega|wS_\bullet(\mathcal{D})|$ which we shall denote by $K(\mathcal{D})$. If $\mathcal{M}$ is an exact category, we can make $\mathcal{M}$ into a category with cofibrations and weak equivalences, by letting $\operatorname{cof}\mathcal{M}$ be the admissible monomorphisms in $\mathcal{M}$ and by letting $w\mathcal{M}$ be the isomorphisms in $\mathcal{M}$. Then there is natural equivalence from $K(\mathcal{M})$ in the sense of Quillen to $K(\mathcal{M})$ in the sense of Waldhausen [Wald3, 1.9], [G, 9.3].

For geometric applications we want to have a chain complex theoretic description of $K(\mathcal{M})$. Let $\operatorname{ch}(\mathcal{M})$ be the category of chain complexes in $\mathcal{M}$ which are graded over $\mathbb{Z}$ and bounded above and below. Homology is defined by first mapping the chain complex into the abelian category $\mathcal{A}$. We make $\operatorname{ch}(\mathcal{M})$ into a category with cofibrations and weak equivalences by letting $\operatorname{cof}\operatorname{ch}(\mathcal{M})$ be the chain maps which are degreewise admissible monomorphisms, and by letting $w\operatorname{ch}(\mathcal{M})$ be the quasi-isomorphisms, i.e. chain maps which induce isomorphisms in homology. The chain complexes concentrated in degre zero form a full subcategory which we identify with $\mathcal{M}$. In many cases the inclusion of Waldhausen K–theories, $K(\mathcal{M}) \to K(\operatorname{ch}(\mathcal{M}))$, is a homotopy equivalence.

2.2.2. Example. *The commutative square of inclusion maps*

$$\begin{array}{ccc} K(\mathcal{PA}) & \longrightarrow & K(\mathrm{ch}(\mathcal{PA})) \\ \downarrow & & \downarrow \\ K(\mathcal{NPA}) & \longrightarrow & K(\mathrm{ch}(\mathcal{NPA})) \end{array}$$

consists entirely of homotopy equivalences.

Proof. The upper horizontal arrow is a homotopy equivalence by [Wald3, 1.7.1] and the approximation theorem [Wald3, 1.6.7]. See [Tho2]. The right–hand vertical arrow is a homotopy equivalence by the approximation theorem, and the left–hand vertical arrow is a homotopy equivalence by Quillen's resolution theorem, mentioned earlier. □

Remark. Let $\mathcal{A}$ be the category of finitely generated left R–modules. Replacing $\mathrm{ch}(\mathcal{PA})$ by the larger category of chain compleces of left projective R–modules which are *homotopy equivalent* to objects in $\mathrm{ch}(\mathcal{PA})$ does not change the homotopy type of the K–theory space. This follows directly from the approximation theorem.

For any category $\mathcal{D}$ with cofibrations and weak equivalences Waldhausen has constructed an injective map $\psi : Bw\mathcal{D} \to K(\mathcal{D})$, depending functorially on $\mathcal{D}$. Suppose for illustration that $\mathcal{D} = \mathcal{P}_R$ for some ring R, and that V is an object in $\mathcal{D}$. Then $K(\mathcal{D}) =: K(R)$, and the composition

$$B\,\mathrm{aut}(V) \xrightarrow{\subset} Bw\mathcal{D} \xrightarrow{\psi} K(\mathcal{D})$$

induces a homomorphism on π_1 which is of course the classical homomorphism $\mathrm{aut}(D) \to K_1(R)$.

Returning to an arbitrary exact category $\mathcal{M} \subset \mathcal{A}$, we introduce certain full subcategories of $\mathrm{ch}(\mathcal{M})$. Let $\mathrm{tch}(\mathcal{M})$ consist of the *trivial* chain complexes (with trivial differential), and let $\mathrm{sch}(\mathcal{M})$ consist of the very *special* chain complexes C whose homology groups H_iC belong to $\mathcal{M}$ for all i. Thus

$$\mathrm{tch}(\mathcal{M}) \subset \mathrm{sch}(\mathcal{M}) \subset \mathrm{ch}(\mathcal{M})$$

and we define $w\,\mathrm{tch}(\mathcal{M}) := \mathrm{tch}(\mathcal{M}) \cap w\mathcal{M}$, $w\,\mathrm{sch}(\mathcal{M}) := \mathrm{sch}(\mathcal{M}) \cap w\mathcal{M}$.

In the proposition just below, we regard the homology functor H_* as a functor from $\mathrm{sch}(\mathcal{M})$ to $\mathrm{tch}(\mathcal{M})$. We also use a *restricted* product Π' of pointed spaces. It consists of those points (x_i) in the honest product for which $x_i \neq *$ for only finitely many i.

2.2.3. Proposition. *The following diagram commutes up to homotopy:*

$$\begin{array}{ccccc} Bw\,\mathrm{sch}(\mathcal{M}) & \xrightarrow{H_*} & Bw\,\mathrm{tch}(\mathcal{M}) & \xrightarrow{\cong} & \prod'_{i\in\mathbb{Z}} Bw\mathcal{M} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\Pi'\psi} \\ K(\mathrm{ch}(\mathcal{M})) & \xrightarrow{=} & K(\mathrm{ch}(\mathcal{M})) & \xleftarrow{\text{altern. sum}} & \prod'_{i\in\mathbb{Z}} K(\mathrm{ch}(\mathcal{M}))\,. \end{array}$$

Proof. Given a chain complex C in $\mathrm{sch}(\mathcal{M})$, let P_kC be the k–the Postnikov approximation to C. Thus $(P_kC)_i = C_i$ for $i \le k$, $(P_kC)_{k+1} = \mathrm{im}(\partial : C_{k+1} \to C_k)$, and $(P_kC)_i = 0$ for $i > k+1$. Let Q_kC be the kernel of the canonical projection from P_kC to $P_{k-1}C$. Then

$$Q_kC \longrightarrow P_kC \longrightarrow P_{k-1}C$$

is a functorial cofibration sequence in $\mathrm{sch}(\mathcal{M})$. The projection

$$Q_kC \longrightarrow H_*(Q_kC)$$

is a weak equivalence in $\mathrm{sch}(\mathcal{M})$, and of course $H_*(Q_kC)$ is concentrated in degree k and equal to H_kC there. Using this and Waldhausen's additivity theorem [Wald3, 1.3.2(3)], one finds that the left–hand square in 2.2.3 commutes up to homotopy. Commutativity of the right–hand square is a consequence of [Wald3, 1.6.2] □

We note that 2.2.3 remains valid in the case $\mathcal{M} = \mathcal{P}_R$ if the chain complexes in $\mathrm{sch}(\mathcal{P}_R)$ and $\mathrm{ch}(\mathcal{R})$ are allowed to be chain complexes of projective left R–modules which are homotopy equivalent to finitely generated ones.

We return to $\mathfrak{p}$, p, E, B and V of 2.1.4 and 2.2.1. Let $H_i(\mathfrak{p}(c);V)$ be the i–th homology of $\mathfrak{p}(c)$ with twisted coefficients in V. Combining 2.2.3 with 2.2.1, we get the following result.

2.2.4. Proposition. *Suppose that each $H_i(\mathfrak{p}(c);V)$ has a finite length resolution by f.g. projective R–modules. Then the composition*

$$B \xrightarrow{\langle p\rangle} A_B(E) \longrightarrow A(E) \xrightarrow{\lambda} K(R)$$

is homotopic to the alternating sum $\Sigma(-1)^i \ldots$ of the maps

$$B \xrightarrow{k(i)} B\,\mathrm{iso}(\mathcal{NP}_R) \xrightarrow{\psi} K(R) \qquad (i \ge 0)$$

where $\mathcal{NP}_R$ consists of all left R–modules having a finite resolution by finitely generated projective ones, and $k(i)$ is induced by $c \mapsto H_i(\mathfrak{p}(c);V)$. (Recall that $B = B\mathcal{C}$.) □

The curious reader can go directly from here to §10 to see how 2.2.4 in conjunction with the formula (0–4) in the introduction implies the Bismut–Lott–Riemann–Roch formula [BiLo, Thm. 0.1].

2.3. Assembly

The motivation for the definitions which follow is that we want to refine the Euler characteristic $\langle Y\rangle \in A(Y)$ to a *microcharacteristic* $\langle\!\langle Y\rangle\!\rangle \in A^{\%}(Y)$. The price to be paid for this refinement is greater rigidity: the microcharacteristic is lax natural for homeomorphisms, but not for homotopy equivalences.

A functor $\boldsymbol{F}$ from spaces to CW–spectra is *homotopy invariant* if it takes homotopy equivalences to homotopy equivalences. A homotopy invariant $\boldsymbol{F}$ is *excisive* if $\boldsymbol{F}(\emptyset)$ is contractible and if $\boldsymbol{F}$ preserves homotopy pushout squares (alias homotopy cocartesian squares, see [Go1], [Go2]). The excision condition implies that $\boldsymbol{F}$ preserves finite coproducts, up to homotopy equivalence. Call $\boldsymbol{F}$ *strongly excisive* if it preserves arbitrary coproducts, up to homotopy equivalence.

2.3.1. Theorem [WWAs]**.** *For any homotopy invariant $\boldsymbol{F}$ from spaces to CW–spectra, there exist a strongly excisive (and homotopy invariant) $\boldsymbol{F}^{\%}$ from spaces to CW–spectra and a natural transformation*

$$\alpha = \alpha_{\boldsymbol{F}} : \boldsymbol{F}^{\%} \longrightarrow \boldsymbol{F}$$

such that $\alpha : \boldsymbol{F}^{\%}(*) \to \boldsymbol{F}(*)$ is a homotopy equivalence. Moreover, $\boldsymbol{F}^{\%}$ and $\alpha_{\boldsymbol{F}}$ can be made to depend functorially on $\boldsymbol{F}$, and

$$\boldsymbol{F}^{\%}(X) \simeq X_+ \wedge \boldsymbol{F}(*)$$

by a chain of natural homotopy equivalences.

2.3.2. Observation. *If $\boldsymbol{F}$ is already excisive, then*

$$\alpha : \boldsymbol{F}^{\%}(Y) \longrightarrow \boldsymbol{F}(Y)$$

is a homotopy equivalence for all finite Y, and if $\boldsymbol{F}$ is strongly excisive, then α is a homotopy equivalence for all Y.

Proof. By arguments going back to Eilenberg and Steenrod it is sufficient to verify that α is a homotopy equivalence for $Y =$ point. □

We want to show that $\alpha = \alpha_{\boldsymbol{F}}$ is the "universal" approximation (from the left) of $\boldsymbol{F}$ by a strongly excisive homotopy invariant functor. Suppose therefore that

$$\beta : \boldsymbol{E} \longrightarrow \boldsymbol{F}$$

is another natural transformation with strongly excisive and homotopy invariant $\boldsymbol{E}$. The commutative square

$$\begin{array}{ccc} \boldsymbol{E}^{\%} & \xrightarrow{\alpha_{\boldsymbol{E}}} & \boldsymbol{E} \\ \big\downarrow{\scriptstyle \beta^{\%}} & & \big\downarrow{\scriptstyle \beta} \\ \boldsymbol{F}^{\%} & \xrightarrow{\alpha_{\boldsymbol{F}}} & \boldsymbol{F} \end{array}$$

in which the upper horizontal arrow is a homotopy equivalence by 2.7, shows that β essentially factors through $\alpha_{\boldsymbol{F}}$.

Following [AnCoFePe] and [CaPe], [CaPeVo] we introduce control in order to get explicit models for assembly maps in A–theory. A *control space* is a pair consisting of a locally compact Hausdorff space $\bar{E}$ and an open dense subset E of $\bar{E}$. Let $p : X_1 \to E$ and $q : X_2 \to E$ be proper spaces over E (which means that X_1, X_2 are locally compact, and p, q are proper). A continuous proper map $f : X_1 \to X_2$ is a *controlled map* if it satisfies the following condition: Given $z \in \bar{E} \smallsetminus E$, and a neighborhood U of z in $\bar{E}$, there exists a smaller neighborhood U_0 of z in $\bar{E}$ such that $p(x) \in U_0$ implies $q(f(x)) \in U$, for all $x \in X_1$. It is straightforward to define controlled (proper) homotopies between controlled maps, and then controlled (proper) homotopy equivalences between proper spaces over E. Denote the homotopy category of proper spaces and controlled maps over E by $\mathcal{H}(E \lhd \bar{E})$.

We form the category of proper retractive ENR's over E, where the morphisms are maps over E and relative to E. Such a morphism is a *cofibration* if it is injective. Let $\boldsymbol{A}(E\lhd\bar{E})$ be the K–theory spectrum of this category with cofibrations, allowing as weak equivalences all those morphisms which become invertible in $\mathcal{H}(E\lhd\bar{E})$.

We shall also need *germs* near $\bar{E}\smallsetminus E$ of proper spaces over E. Such a germ is represented by a proper space over W, where $W\subset E$ is such that $E\smallsetminus W$ is closed in $\bar{E}$. Germs of controlled maps and germs of controlled homotopies are defined in the most obvious way. Hence there is a category $\mathcal{H}(E\lhd\bar{E})_\infty$ of germs near $\bar{E}\smallsetminus E$ of proper spaces over E and controlled homotopy classes of controlled map germs.

We form the category of germs near $\bar{E}\smallsetminus E$ of proper retractive ENR's over E, where the morphisms are germs of maps over E and relative to E. Such a morphism is a *cofibration* if it is the germ of an injection. Let $\boldsymbol{A}(E\lhd\bar{E})_\infty$ be the K–theory spectrum of this category with cofibrations, allowing as weak equivalences all those morphisms which become invertible in $\mathcal{H}(E\lhd\bar{E})_\infty$. The following is a reformulation of a special case of the main theorem of [Vo2]; see also [PeWei], [AnCoFePe], [Vo3], [Vo4]. *Notation:* Y is a *compact* ENR which we sometimes identify with $Y\times\{0\}$.

2.3.3. Theorem [Vo2]**,** [Vo4]**,** [CaPeVo]**.** *The functor* $Y\mapsto\boldsymbol{A}(Y\times[0,1)\lhd Y\times[0,1])_\infty$ *is homotopy invariant and excisive.*

2.3.4. Theorem [CaPeVo]**.** *The commutative square of inclusion maps*

$$\begin{array}{ccc}\boldsymbol{A}(Y)\cong\boldsymbol{A}(Y\times\{0\}) & \xrightarrow{\ \subset\ } & \boldsymbol{A}(Y\times[0,1)\lhd Y\times[0,1]) \\ \downarrow & & \downarrow{\scriptstyle\text{forget}} \\ * & \longrightarrow & \boldsymbol{A}(Y\times[0,1)\lhd Y\times[0,1])_\infty\end{array}$$

is a homotopy pullback square. If $Y=*$*, its upper right–hand vertex is contractible.*

Comments. Contractibility of $\boldsymbol{A}([0,1)\lhd[0,1])$ is proved using an Eilenberg swindle. The same Eilenberg swindle shows that $\boldsymbol{A}(Y\times[0,1)\lhd Y\times[0,1])$ is connected (but it is not always contractible). See the remark following 2.3.5 below. We mention this because Carlsson, Pedersen and Vogell use slightly bigger categories of retractive spaces than we do, with a view to "idempotent completeness". Consequently they obtain versions of A–theory with a potentially bigger π_0 than we do. However, when $Y=*$ there is no disagreement anywhere. Therefore 2.3.4 is correct when $Y=*$, and we can conclude that $\boldsymbol{A}([0,1)\lhd[0,1])_\infty$ is homotopy equivalent to $\mathbb{S}^1\wedge\boldsymbol{A}(*)$. (Note that this is the *coefficient spectrum* in 2.3.3.) Using 2.3.3 now, and assuming without loss of generality that Y is connected, we conclude that the maps

$$\begin{gathered}\boldsymbol{A}(Y)\to\boldsymbol{A}(*) \\ \boldsymbol{A}(Y\times[0,1)\lhd Y\times[0,1])\longrightarrow\boldsymbol{A}([0,1)\lhd[0,1]) \\ \boldsymbol{A}(Y\times[0,1)\lhd Y\times[0,1])_\infty\longrightarrow\boldsymbol{A}([0,1)\lhd[0,1])_\infty\end{gathered}$$

induced by $Y\to*$ are 1–connected, 1–connected and 2–connected, respectively. Hence 2.3.4 is correct for arbitrary Y, despite the missing components.

The category of proper retractive ENR's over $Y \times [0,1)$ has an endomorphism $\boldsymbol{t}$ induced by the shift map

$$Y \times [0,1) \to Y \times [0,1) \quad ; \quad (y, 1-u) \mapsto (y, 1-u/2)\,.$$

Using $\boldsymbol{t}$ we can manufacture other endomorphisms such as

$$\sum_{i \geq 0} k_i \boldsymbol{t}^i$$

where $k_0, k_2, k_3, \dots$ can be any sequence of positive integers and $k_i \boldsymbol{t}^i$ is short for a k_i–fold coproduct, and the sum sign also denotes a coproduct. A retractive map f between proper retractive ENR's over $Y \times [0,1)$ is a *microequivalence* if $\big(\sum_i k_i \boldsymbol{t}^i\big)(f)$ is invertible in $\mathcal{H}(Y \times [0,1) \lhd Y \times [0,1])$ for arbitrary positive integers $k_0, k_1, k_2, \dots$. For example, any isomorphism between proper retractive ENR's over $Y \times [0,1)$ is a microequivalence. Write $\boldsymbol{P}(Y)$ for the K-theory spectrum of the category of proper retractive ENR's over $Y \times [0,1)$, where cofibrations are defined as usual and the weak equivalences are the microequivalences. Then

$$\boldsymbol{P}(Y) \subset \boldsymbol{A}(Y \times [0,1) \lhd Y \times [0,1])\,.$$

In the following lemma, Y is still compact.

2.3.5. Lemma. *The spectrum $\boldsymbol{P}(Y)$ is contractible.*

Remark. Notice that $P(*)$ is all of $\boldsymbol{A}([0,1) \times [0,1])$, so 2.3.5 implies contractibility of $\boldsymbol{A}([0,1) \times [0,1])$. For arbitrary Y, the inclusion $\boldsymbol{P}(Y) \subset \boldsymbol{A}(Y \times [0,1) \lhd Y \times [0,1])$ induces a surjection on π_0 because the underlying inclusion of categories is a bijection on objects. Therefore 2.3.5 implies that $\boldsymbol{A}(Y \times [0,1) \lhd Y \times [0,1])$ is connected.

Proof of 2.3.5. Let $\boldsymbol{u} := \sum \boldsymbol{t}^i$ where the sum is taken over all $i \geq 0$. The endofunctors $\boldsymbol{t}$ and $\boldsymbol{u}$ take microequivalences to microequivalences and respect cofibrations. They therefore induce self–maps of $\boldsymbol{P}(Y)$. Write $[\boldsymbol{t}]$ and $[\boldsymbol{u}]$ for their homotopy classes. There exists another endofunctor $\boldsymbol{t}'$ respecting cofibrations and microequivalences, and there exist natural microequivalences

$$\boldsymbol{t}(X) \hookrightarrow \boldsymbol{t}'(X) \hookleftarrow X$$

where X is any retractive ENR over Y. (Details: $\boldsymbol{t}'(X) = g_*([0,1] \times X)$ where X is a retractive space over $Y \times [0,1)$, so that $[0,1] \times X$ is a retractive space over $[0,1] \times Y \times [0,1)$, and

$$g : [0,1] \times Y \times [0,1) \longrightarrow Y \times [0,1) \quad ; \quad (s, y, 1-u) \mapsto (y, 1-u/(2-s))\,.$$

Note that $X \cong g_*(\{1\} \times X)$ and $\boldsymbol{t}(X) \cong g_*(\{0\} \times X)$.) Therefore

$$[\mathrm{id}] = [\boldsymbol{t}] = [\boldsymbol{u}] - [\boldsymbol{t}\boldsymbol{u}] = [\boldsymbol{u}] - [\boldsymbol{t}][\boldsymbol{u}] = [\boldsymbol{u}] - [\boldsymbol{u}] = [*]. \quad \square$$

We are now ready for the microcharacteristic. Our model for $\boldsymbol{A}^{\%}(Y)$ will be the homotopy pullback of the diagram

$$\begin{array}{ccc} & & \boldsymbol{A}(Y) \\ & & \big\downarrow \subset \\ \boldsymbol{P}(Y) & \xrightarrow{\subset} & \boldsymbol{A}(Y\times[0,1)\lhd Y\times[0,1])\,. \end{array}$$

By 2.3.3, 2.3.4 and 2.3.5 this is indeed a homotopy invariant and excisive functor of the variable Y, and it comes with a projection to $\boldsymbol{A}(Y)$ which is a homotopy equivalence when $Y=*$. Hence, by the uniqueness result 2.3.2 and sequel, our notation $\boldsymbol{A}^{\%}(Y)$ is fully justified. We write $A^{\%}(Y)$ for the corresponding infinite loop space ; that, is $A^{\%}(Y)$ is the homotopy pullback of

$$\begin{array}{ccc} & & A(Y) \\ & & \big\downarrow \subset \\ P(Y) & \xrightarrow{\subset} & A(Y\times[0,1)\lhd Y\times[0,1])\,. \end{array}$$

Observe now that the Euler characteristic $\langle Y\rangle\in A(Y)$ actually lives in the subspace $A(Y)\cap P(Y)$ of $A(Y\times[0,1)\lhd Y\times[0,1])$. Hence it lives in $A^{\%}(Y)$. As such we denote it by $\langle\!\langle Y\rangle\!\rangle$.

2.3.6. Variation. It follows from 2.3.4 that the homotopy fiber of

$$P(Y)\xrightarrow{\subset} A(Y\times[0,1)\lhd Y\times[0,1])_\infty$$

is another possible model for $A^{\%}(Y)$. It receives a map from the model introduced above, so we can still use it as a receptacle for the microcharacteristic $\langle\!\langle Y\rangle\!\rangle$. Alternatively, we can construct $\langle\!\langle Y\rangle\!\rangle$ directly as an element of the *fiber* of

$$P(Y)\xrightarrow{\subset} A(Y\times[0,1)\lhd Y\times[0,1])_\infty\,.$$

Drawback: The relationship with $A(Y)$ and $\langle Y\rangle\in A(Y)$ is less direct.

In the sequel we write $\langle Y\rangle\in A(Y)$ and $\langle\!\langle Y\rangle\!\rangle\in A^{\%}(Y)$, suppressing any information about the specific models of $A(Y)$ and $A^{\%}(Y)$ that we had to use to make it all work. — It is easy to verify that $\langle\!\langle Y\rangle\!\rangle$ is lax natural for homeomorphisms. Although it is easy, it depends crucially on the fact that isomorphisms between compact retractive ENR's over $Y\cong Y\times\{0\}$ can be regarded as *microequivalences* between proper retractive ENR's over $Y\times[0,1)$. In fact, the same argument shows that $\langle\!\langle Y\rangle\!\rangle$ is lax natural for *cell–like* maps [La1], [La2], [La3] between ENR's.

2.4. The universal examples

As in [WW1] let $\mathrm{TOP}(M)$ be the topological group of homeomorphisms $M\to M$ which agree with the identity near ∂M. A difficult theorem due to [McD1] (see also [Se], [Math1], [Math2], [Thu]) asserts that the inclusion of $B\,\mathrm{TOP}^\delta(M)$ in

$B\,\mathrm{TOP}(M)$ is a homology equivalence for compact M, where $\mathrm{TOP}^\delta(M)$ is the underlying discrete group. (A map $f : X \to Y$ of connected spaces is a *homology equivalence* if it induces isomorphisms $f_* : H_*(X; J) \to H_*(Y; J)$ for any $\pi_1(Y)$–module J.) [Also, the inclusion of $B\,\mathrm{TOP}^\delta(M, \partial M)$ in $B\,\mathrm{TOP}(M, \partial M)$ is a homology equivalence ; $\mathrm{TOP}(M, \partial M)$ is the simplicial group of *all* topological automorphisms of M.]

We now set up diagram (0–7) in the introduction. Without going too much into details, we recall what this amounts to. Suppose that $p : E \to B$ is a fibration of the type considered in diagram (0–7), and γ is an n–microbundle on E. Then we should construct two other fibrations $q : A_B(E) \to B$ and $q^\% : A^\%_B(E) \to B$, with fibers $A(F_x)$ and $A^\%(F_x)$ over $x \in B$, respectively, where $F_x = p^{-1}(x)$. We should also construct a section $\langle p\rangle$ of q whose value at $x \in B$ is $\langle F_x\rangle$. If p is a fiber bundle with compact manifold fibers, then we should construct a section $\langle\!\langle p\rangle\!\rangle$ of $q^\%$ whose value at $x \in B$ is $\langle\!\langle F_x\rangle\!\rangle$. In this case we must ensure that $\alpha \cdot \langle\!\langle p\rangle\!\rangle$ is equal, or at least vertically homotopic by a canonical homotopy, to $\langle p\rangle$. Remember that α is the assembly, in this case from $A^\%$–theory to A–theory.

Finally, if all this is sufficiently natural, then diagram (0–7) follows by general nonsense. We can in fact simplify the task, and largely avoid the naturality issue, by concentrating on two universal examples. In the first, take B to be $B\mathfrak{G}_n$. Let p be the tautological quasifibration on B whose fiber over the point corresponding to an object (F, γ) in $\mathfrak{G}_n$ is F (something canonically homeomorphic to F, to be quite honest). In the second example, take B to be $B\mathfrak{homeo}_n$. Let p be the tautological fiber bundle on B with compact n–manifold fibers.

First universal example. Let $\mathfrak{G}^\delta_n$ be the discrete category underlying $\mathfrak{G}_n$. The inclusion of $B\mathfrak{G}^\delta_n$ in $B\mathfrak{G}_n$ is a homotopy equivalence (compare [DwKa], [Fie]). Hence we may take $B = B\mathfrak{G}^\delta_n$ instead of $B = B\mathfrak{G}_n$. Define $\mathfrak{p}, \mathfrak{q}, \mathfrak{q}^\%$ from $\mathfrak{G}^\delta_n$ to spaces by

$$\mathfrak{p}(F, \gamma) = F\,, \qquad \mathfrak{q}(F, \gamma) = A(F)\,, \qquad \mathfrak{q}^\%(F, \gamma) = A^\%(F)\,.$$

Let $p, q, q^\%$ be the corresponding quasifibrations. Euler characteristics $F \mapsto \langle F\rangle$ determine an element $\langle p\rangle$ of $\operatorname{holim}\mathfrak{q} \simeq \Gamma(q)$. We can also try the stable version $B\mathfrak{G}^\delta$ instead of $B\mathfrak{G}^\delta_n$. Then we still have $\mathfrak{q}$ and $\mathfrak{q}^\%$ on $\mathfrak{G}^\delta$, given by

$$(m, F, \gamma) \mapsto A(F)\,, \qquad (m, F, \gamma) \mapsto A^\%(F)\,.$$

(These are functors because a morphism $g : (m, F, \gamma) \to (n, F', \gamma')$ in $\mathfrak{G}$ induces

$$A(F) \xrightarrow{\times I^{n-m}} A(F \times I^{n-m}) \xrightarrow{g_*} A(F')$$
$$A^\%(F) \xrightarrow{\times I^{n-m}} A^\%(F \times I^{n-m}) \xrightarrow{g_*} A^\%(F')\,,$$

where $\times I^{n-m}$ is regarded as an exact functor from retractive spaces over F to retractive spaces over $F \times I^{n-m}$. Slight problems are caused by the fact that products in the category of sets are not strictly associative, but given any “category of sets” one can easily construct an equivalent category with an explicit and associative product.) We still have quasifibrations $q, q^{\%}$ on $B\mathfrak{G}^{\delta}$ and a distinguished $\langle p\rangle$ in $\operatorname{holim}\mathfrak{q} \simeq \Gamma(q)$, but the meaning of p is no longer clear.

Second universal example. Let $\mathfrak{homeo}_n^{\delta}$ be the discrete category underlying $\mathfrak{homeo}_n$. The inclusion $B\mathfrak{homeo}_n^{\delta} \hookrightarrow B\mathfrak{homeo}_n$ is a homology equivalence, by the McDuff–Mather–Thurston theorem mentioned earlier. It does not follow immediately that we may substitute $B\mathfrak{homeo}_n^{\delta}$ for $B\mathfrak{homeo}_n$, but we will do so and justify later. Define $\mathfrak{p}, \mathfrak{q}, \mathfrak{q}^{\%}$ from $\mathfrak{homeo}_n^{\delta}$ as before, restricting from $\mathfrak{G}_n^{\delta}$. Write $p, q, q^{\%}$ for the corresponding quasifibrations on $B\mathfrak{homeo}_n^{\delta}$. Euler characteristics $F \mapsto \langle F\rangle$ determine $\langle p\rangle$ in $\operatorname{holim}\mathfrak{q} \simeq \Gamma(q)$, and microcharacteristics $F \mapsto \langle\langle F\rangle\rangle$ determine $\langle\langle p\rangle\rangle$ in $\operatorname{holim}\mathfrak{q}^{\%} \simeq \Gamma(q^{\%})$. Since microcharacteristics lift Euler characteristics, we have $\alpha \cdot \langle\langle p\rangle\rangle = \langle p\rangle$, where α is the assembly (from $A^{\%}$–theory to A–theory).

Again, we may try the stable version $B\mathfrak{homeo}^{\delta}$ instead of $B\mathfrak{homeo}_n^{\delta}$. We can then still regard $\mathfrak{q}$ and $\mathfrak{q}^{\%}$ as functors on $\mathfrak{homeo}^{\delta}$, and we get distinguished elements $\langle p\rangle \in \operatorname{holim}\mathfrak{q}$ and $\langle\langle p\rangle\rangle \in \operatorname{holim}\mathfrak{q}^{\%}$, but again the meaning of p is not clear.

Justification. Concentrating on the stable case, we use the commutative square

$$\begin{array}{ccc} B\mathfrak{homeo}^{\delta} & \xrightarrow{\subset} & B\mathfrak{G}^{\delta} \\ \subset\big\downarrow & & \subset\big\downarrow\simeq \\ B\mathfrak{homeo} & \xrightarrow{\subset} & B\mathfrak{G}\ . \end{array}$$

We constructed *two* quasifibrations q, one on $B\mathfrak{homeo}^{\delta}$ and another on $B\mathfrak{G}^{\delta}$. However, one of these is the restriction of the other, so there is no serious ambiguity. We also constructed *two* quasifibrations $q^{\%}$, one on $B\mathfrak{homeo}^{\delta}$ and another on $B\mathfrak{G}^{\delta}$. But again, one of these is the restriction of the other.

Since the inclusion of simplicial sets $B\mathfrak{G}^{\delta} \hookrightarrow B\mathfrak{G}$ is a homotopy equivalence, the quasifibration q on $B\mathfrak{G}^{\delta}$ extends to a quasifibration on $B\mathfrak{G}$ which we still denote by q. (Construct the extension by induction over the relative skeleta.) In the same way, $q^{\%}$ extends to a quasifibration on $B\mathfrak{G}$, which we still denote by $q^{\%}$. Moreover, if we construct the extension of $q^{\%}$ carefully, then the morphism of quasifibrations $\alpha : q^{\%} \to q$ defined over $B\mathfrak{G}^{\delta}$ extends to one defined over $B\mathfrak{G}$. These extensions (of q, $q^{\%}$ and the morphism from $q^{\%}$ to q) are unique up to *contractible choice* ; details omitted.

Now we have a commutative diagram of spaces of sections

$$\begin{array}{ccccc} \Gamma(q^{\%} \mid B\mathfrak{homeo}^{\delta}) & \longrightarrow & \Gamma(q \mid B\mathfrak{homeo}^{\delta}) & \xleftarrow{\text{res}} & \Gamma(q \mid B\mathfrak{G}^{\delta}) \\ \big\uparrow \text{res} & & \big\uparrow \text{res} & & \big\uparrow \text{res} \\ \Gamma(q^{\%} \mid B\mathfrak{homeo}) & \longrightarrow & \Gamma(q \mid B\mathfrak{homeo}) & \xleftarrow{\text{res}} & \Gamma(q \mid B\mathfrak{G}) \end{array} \tag{2–3}$$

where $q^{\%} \mid B\mathfrak{homeo}$ denotes the restriction of $q^{\%}$ to $B\mathfrak{homeo}$, for example, and all arrows labelled res are restriction maps. The unlabelled arrows are induced by the morphism $q^{\%} \to q$. We know already that one of the vertical arrows is a homotopy equivalence ; we shall see that the other two are also homotopy equivalences. (This will complete the justification, because all we ever wanted was a point in the holim of the *lower* row of (2–1), and what we have constructed so far is a point in the holim of the *upper* row of (2–1).)

Suppose therefore that $f : X \to Y$ is a homology equivalence between CW–spaces, and that r is a fibration on Y with nilpotent fibers [HMR]. Then Postnikov technology implies that the pullback map from the section space $\Gamma(r)$ to $\Gamma(f^*r)$ is a homotopy equivalence. Example: f can be the inclusion of $B\mathfrak{homeo}^\delta$ in $B\mathfrak{homeo}$, and r can be q or $q^\%$ restricted to $B\mathfrak{homeo}$. In this case the fibers are nilpotent because they are infinite loop spaces. □

Remark. Note that all restriction maps in (2–1) are fibrations. Together with the preceding argument, this shows that the sections $\langle\!\langle p\rangle\!\rangle$ and $\langle p\rangle$ of $q^\%$ and q, defined over $B\mathfrak{homeo}^\delta$ and $B\mathcal{G}^\delta$, respectively, can be extended compatibly to $B\mathfrak{homeo}$ and $B\mathfrak{G}$, respectively. As usual, any choices made are contractible ; and we still use the symbols $\langle\!\langle p\rangle\!\rangle$ and $\langle p\rangle$ to denote these extended sections, although the meaning of p is even less clear than it was before.

We have now set up diagram (0–7) by reduction to universal examples (remember the interpretation of $\mathcal{S}^T(p,\gamma)$ given in 1.2, display (1–1)). It remains to show that (0–7) is a homotopy pullback square. Again this can be done at the "universal level", and the task is then to show that

(2–4)
$$\begin{array}{ccc} B\mathfrak{homeo} & \xrightarrow{\langle\!\langle p\rangle\!\rangle} & E(q^\%) \\ \big\downarrow{\scriptstyle\subset} & & \big\downarrow{\scriptstyle\alpha} \\ B\mathfrak{G} & \xrightarrow{\langle p\rangle} & E(q) \end{array}$$

is a homotopy pullback square (where $E(q)$ and $E(q^\%)$ are the *total spaces* of the quasifibrations $q^\%$ and q). Now every component of $B\mathfrak{G}$ contains a point corresponding to an object of the form (n, M^n, τ^M) where M is a compact n–manifold. The map of vertical homotopy fibers in (2–4), over such a point and its image point in $E(q)$, takes the form

(2–5)
$$\Omega\,\mathrm{Wh}^{\mathrm{TOP}}(M) \longrightarrow \mathrm{hofiber}_{\langle M\rangle}[A^\%(M) \to A(M)]\,,$$

by 1.5. The subscript $_{\langle M\rangle}$ means that the homotopy fiber (of the assembly map) must be taken over the point $\langle M\rangle$. (Note, however, that α from $A^\%(M)$ to $A(M)$ is a morphism of infinite loop spaces, so that all its homotopy fibers are canonically homotopy equivalent to that over the zero element.) We see that domain and codomain of (2–5) are "abstractly" homotopy equivalent by Waldhausen's theorem. But so far we do not know how the homotopy equivalence due to Waldhausen is related to the map (2–5). Moreover, before we can explore the relationship, we need a description of the Waldhausen map.

Fix M^n, with boundary ∂M, and let $\mathfrak{hcob}(\partial M)$ be the topological category of h–cobordisms over ∂M. Objects are (invertible) h–cobordisms $(W; \partial M, N)$ over ∂M, and morphisms are homeomorphisms relative to ∂M. Then

$$B\mathfrak{hcob}(\partial M) \cong \mathfrak{H}(\partial M)\,.$$

Also, let $\mathfrak{thick}(\partial M)$ be the topological category whose objects are pairs (X, γ) where X is a compact ENR containing ∂M, in such a way that the inclusion $\partial M \to X$ is a homotopy equivalence, and where γ is an n–microbundle on X extending the restriction of τ^M to ∂M. Then $B\mathfrak{thick}(\partial M)$ is contractible.

To an object (X, γ) in $\mathfrak{thick}(\partial M)$ we can associate its *relative* Euler characteristic: the point in $A(X)$ corresponding to the retractive space $X \amalg_{\partial M} X$ over X. This is lax natural (perhaps not in a continuous way, but we know how to handle such problems). To an h–cobordism $(W; \partial M, N)$ we can associate its *relative* microcharacteristic: the point in $A^{\%}(W)$ corresponding to the retractive space $W \amalg_{\partial M} W$ (see 2.8, 2.9, 2.10). Playing the same game as before, we obtain a diagram similar to (2–4):

$$\begin{array}{ccc} B\mathfrak{hcob}(\partial M) & \longrightarrow & E(q_1^{\%}) \\ \big\downarrow \subset & & \big\downarrow \alpha \\ B\mathfrak{thick}(\partial M) & \longrightarrow & E(q_1) \end{array} \tag{2–6}$$

where $q_1^{\%}$ and q_1 are the (quasi–)fibrations on $B\mathfrak{thick}(\partial M)$ determined by the functors $(X, \gamma) \mapsto A^{\%}(X)$ and $(X, \gamma) \mapsto A(X)$, respectively. The horizontal maps in (2–6) are essentially the sections determined by relative microcharacteristics and Euler characteristics, respectively, but again we have to use the McDuff–Mather–Thurston theorem to construct these. The map of vertical homotopy fibers in (2–6) takes the form

$$\mathfrak{H}(\partial M) \to \text{hofiber}[A^{\%}(\partial M) \to A(\partial M)] \tag{2–7}$$

and it is *the* Waldhausen map, by definition [Wald1–5]. As it stands, it is not a homotopy equivalence in general, but it turns into a homotopy equivalence under stabilization.

Now we are in a position to "relate", and we do it by introducing certain maps, from $B\mathfrak{thick}(\partial M)$ to $B\mathfrak{G}$, and from $B\mathfrak{hcob}(\partial M)$ to $B\mathfrak{homeo}$. The first of these, say ι, is induced by the functor

$$(X, \gamma) \mapsto (n, X \amalg_{\partial M} M, \gamma \cup \tau^M)\,.$$

The second is the restriction of the first. Now consider the map from left–hand vertical homotopy fiber to right–hand vertical homotopy fiber in the commutative diagram

$$\begin{array}{ccccc} B\mathfrak{hcob}(\partial M) & \longrightarrow & B\mathfrak{homeo} & \xrightarrow{\langle\langle p \rangle\rangle} & E(q^{\%}) \\ \big\downarrow \subset & & \big\downarrow \subset & & \big\downarrow \alpha \\ B\mathfrak{thick}(\partial M) & \xrightarrow{\ \iota\ } & B\mathfrak{G} & \xrightarrow{\langle p \rangle} & E(q) \end{array}$$

which becomes

$$\mathfrak{H}(\partial M) \longrightarrow \text{hofiber}_{\langle M \rangle}[A^{\%}(M) \to A(M)]\,. \tag{2–8}$$

Inspection and the additivity theorem in [Wald3] show that

$$\begin{array}{ccc} \mathfrak{H}(\partial M) & \xrightarrow{(2\text{–}8)} & \operatorname{hofiber}_{\langle M\rangle}[A^{\%}(M) \to A(M)] \\ \Big\downarrow{\scriptstyle (2\text{–}7)} & & \Big\uparrow{\scriptstyle +\langle\!\langle M\rangle\!\rangle} \\ \operatorname{hofiber}[A^{\%}(\partial M) \to A(\partial M)] & \xrightarrow{\subset} & \operatorname{hofiber}[A^{\%}(M) \to A(M)] \end{array}$$

commutes up to homotopy. Under stabilization (replacing M by $M \times I^k$ for large k), the upper horizontal arrow turns into (2–5), the left–hand vertical arrow turns into a homotopy equivalence (Waldhausen's theorem), and the remaining arrows are homotopy equivalences, essentially unchanged by stabilization. It follows that (2–5) is a homotopy equivalence. □

3. Poincaré Duality by Scanning

Let M^n be a closed topological manifold. For $y \in M$ we write M_y to mean the cofiber of the inclusion $M \smallsetminus \{y\} \hookrightarrow M$. Let $\boldsymbol{X}$ be any spectrum, and let $\Gamma(M; \boldsymbol{X})$ be the section space of the fibration on M whose fiber over $y \in M$ is $\Omega^\infty((M_y) \wedge \boldsymbol{X})$. For any $y \in M$ we have the inclusion $\jmath_y : \Omega^\infty(M_+ \wedge \boldsymbol{X}) \hookrightarrow \Omega^\infty((M_y) \wedge \boldsymbol{X})$, which we use to define

$$\begin{aligned} \wp : \Omega^\infty(M_+ \wedge \boldsymbol{X}) &\longrightarrow \Gamma(M; \boldsymbol{X}) \\ z \quad &\mapsto \quad \big(y \mapsto \jmath_y(z)\big)\,. \end{aligned} \tag{3–1}$$

3.1. Proposition. The map (3–1) is a homotopy equivalence.

Sketch Proof. The statement has a generalization to manifolds with countable basis (without boundary, but possibly noncompact): just replace $\Gamma(M; \boldsymbol{X})$ by a space $\Gamma_c(M; \boldsymbol{X})$ of sections with compact support. The generalized version is easier to prove. First use a direct limit argument to reduce to the case where M can be covered by finitely many charts. Then use excision properties of $\Omega^\infty((—)_+ \wedge \boldsymbol{X})$ and $\Gamma_c(—; \boldsymbol{X})$ to reduce to the case where M is open in $\mathbb{R}^n$. Triangulate M and use excision properties once more to reduce to the case where $M = \mathbb{R}^n$. Then use inspection. □

We would like to call $\wp$ a Poincaré duality map (from homology to cohomology). A Poincaré duality map should however admit a description in homotopy theoretic terms, and it is not clear that this one does. (Try to define (3–1) assuming merely that M is a Poincaré duality space.)

Let ν^k be a normal bundle for M, with Thom space $\operatorname{thom}(\nu)$ and collapse map ρ from $\mathbb{S}^{n+k}$ to $\operatorname{thom}(\nu)$. Recall that Milnor–Poincaré duality is the homotopy equivalence

$$\mu : \operatorname{map}(\operatorname{thom}(\nu)\,,\, \Sigma^{n+k}\boldsymbol{X}) \longrightarrow \Omega^\infty(M_+ \wedge \boldsymbol{X})$$

which to a (pointed) map $g : \operatorname{thom}(\nu) \to \Sigma^{n+k}\boldsymbol{X}$ associates the composition

$$\mathbb{S}^{n+k} \xrightarrow{\rho} \operatorname{thom}(\nu) \xrightarrow{\text{diag.}} M_+ \wedge \operatorname{thom}(\nu) \xrightarrow{\operatorname{id}\wedge g} M_+ \wedge \Sigma^{n+k}\boldsymbol{X}\,.$$

(To be quite precise here: Ω^∞ of a spectrum *is* the space of maps from $\mathbb{S}^0$ to that spectrum, and Σ^{n+k} is the shift operator in the category of spectra.) We shall construct a homotopy from

$$\wp\mu : \mathrm{map}(\mathrm{thom}(\nu)\,,\, \Sigma^{n+k}\boldsymbol{X}) \longrightarrow \Gamma(M;\boldsymbol{X}) \tag{3–2}$$

to a map which has a description in homotopy theoretic terms. The homotopy is essentially contained in the following lemma.

3.2. Lemma. *For every $y \in M$, the composition*

$$\mathbb{S}^{n+k} \xrightarrow{\rho} \mathrm{thom}(\nu) \xrightarrow{\text{diag.}} M_+ \wedge \mathrm{thom}(\nu) \xrightarrow{\mathrm{id}\wedge g} (M_y) \wedge \mathrm{thom}(\nu)$$

is canonically homotopic to a map f_y with image contained in $(M_y) \wedge \mathrm{thom}(\nu_{|\{y\}})$. Moreover, f_y is a homotopy equivalence from $\mathbb{S}^{n+k}$ to $(M_y) \wedge \mathrm{thom}(\nu_{|\{y\}})$.

The proof is an exercise. *Digression:* The fiber bundle on M with fiber M_y over $y \in M$ is a spherical fibration with canonical section. It is obviously fiber homotopy equivalent to the fiberwise one–point compactification of "the" tangent bundle of M, and it is therefore not surprising that it should be Whitney inverse to the normal bundle ν, as a spherical fibration. *End of digression.*

Using the lemma, and the notation of the lemma, we find that $\wp\mu$ in (3–2) is homotopic by a canonical homotopy to the map

$$g \quad \mapsto \quad \Big(y \;\mapsto\; ((\mathrm{id}\wedge g)f_y\Big) \tag{3–3}$$

where g is any map from $\mathrm{thom}(\nu)$ to $\Sigma^{n+k}\boldsymbol{X}$, and $((\mathrm{id}\wedge g)f_y)$ is the composition

$$\mathbb{S}^{n+k} \xrightarrow{f_y} (M_y) \wedge \mathrm{thom}(\nu_{|\{y\}}) \xrightarrow{\mathrm{id}\wedge g} (M_y) \wedge \Sigma^{n+k}\boldsymbol{X}\,.$$

The description (3–3) is in homotopy theoretic terms. Indeed, suppose that M is merely a Poincaré space with Spivak normal fibration $\bar\nu$ and with a reduction $\rho : \mathbb{S}^{n+k} \to \mathrm{thom}(\bar\nu)$. Define the Spivak tangent fibration $\bar\tau$ as "the" stable inverse of $\bar\nu$. More precisely, choose a spherical fibration $\bar\tau$ on M and a trivialization of $\bar\tau \oplus \bar\nu$. The trivialization will consist of pointed homotopy equivalences

$$f_y : \mathbb{S}^i \to \mathrm{thom}(\bar\tau_{|\{y\}}) \wedge \mathrm{thom}(\bar\nu_{|\{y\}})$$

for suitable i and all $y \in M$. Use these to write down (3–3), replacing M_y by $\mathrm{thom}(\bar\tau_{|\{y\}})$ throughout.

3.3. Remark. Proposition 3.1 and Lemma 3.2 can be generalized to compact manifolds M with boundary as follows. Define M_y exactly as before, but note that M_y is contractible for $y \in \partial M$ and $M_y \simeq \mathbb{S}^n$ if $y \notin \partial M$. Define $\Gamma(M, \partial M; \boldsymbol{X})$ as the space of sections of the *fibration pair* with base pair $(M, \partial M)$ whose fiber over $y \in M$ is $\Omega^\infty(M_y \wedge \boldsymbol{X})$. Here a *fibration pair* is a map of pairs

$$(Z_1, Z_2) \longrightarrow (Z_3, Z_4)$$

having the homotopy lifting property with respect to maps of pairs with codomain (Z_1, Z_2). Finally

$$\wp : \Omega^\infty(M_+ \wedge \boldsymbol{X}) \longrightarrow \Gamma(M, \partial M; \boldsymbol{X})$$

can be defined as before, by "scanning". It is a homotopy equivalence, and it can be described in homotopy theoretic terms. We omit the details.

4. Discrete models

Fix a topological manifold M^n. Let $\mathcal{C}$ be the set of embeddings $f : \mathbb{R}^n \to M$. For $f, g \in \mathcal{C}$, a *morphism* from f to g is an embedding $\lambda : \mathbb{R}^n \to \mathbb{R}^n$ such that $f = g\lambda$. Such a morphism from f to g is of course unique if it exists. At any rate, $\mathcal{C}$ is a category, and our goal here is to prove

$$B\mathcal{C} \simeq M\,.$$

For a more precise statement, we introduce an open subset $E \subset B\mathcal{C} \times M$. We say $(x, y) \in E$ if the (open) cell of $B\mathcal{C}$ containing x corresponds to a nondegenerate simplex (diagram in $\mathcal{C}$)

$$f_0 \to f_1 \to \cdots \to f_{k-1} \to f_k$$

such that the image of f_k contains y.

4.1. Proposition. *The projections $B\mathcal{C} \longleftarrow E \longrightarrow M$ are homotopy equivalences.*

The proof is a double application of the following lemma, which is only a mild improvement on [Se, App.].

4.2. Lemma. *A microgibki map (explanation follows) with contractible fibers is a Serre fibration.*

Explanation. A map $q : X \to Y$ is *microgibki* if it has the *homotopy micro–lifting* property of [Gro]): For every space W, every map $f : W \to X$ and every homotopy $h : W \times I \to Y$ with $h(w, 0) = qf(w)$ for all $w \in W$, there exists a map $\bar{h}$ from a neighbourhood of $W \times \{0\}$ in $W \times I$ to X such that $q\bar{h}$ agrees with h where defined.

Example: The projections $E \to B\mathcal{C}$ and $E \to M$ in 4.1 are microgibki, since E is open in the product $B\mathcal{C} \times M$. It is obvious that $E \to B\mathcal{C}$ has contractible fibers. We shall verify that the fibers of the second projection, $q : E \to M$, are also contractible. Fix $y \in M$, and consider the subspaces

$$B\mathcal{C}_y \subset E_y$$

of $B\mathcal{C}$ defined as follows: $B\mathcal{C}_y$ is the union of all (open) cells corresponding to nondegenerate simplices

$$f_0 \to f_1 \to \cdots \to f_{k-1} \to f_k$$

where $y \in \mathrm{im}(f_0)$, and E_y is the union of all cells corresponding to nondegenerate simplices

$$f_0 \to f_1 \to \cdots \to f_{k-1} \to f_k$$

where $y \in \mathrm{im}(f_k)$. Note the following:

1. E_y is homeomorphic to $q^{-1}(y)$.
2. $B\mathcal{C}_y$ is a deformation retract of E_y (details follow).
3. $B\mathcal{C}_y$ is the classifying space of a full subcategory $\mathcal{C}_y \subset \mathcal{C}$. The subcategory consists of all objects f such that $y \in \mathrm{im}(f)$.

(For the deformation retraction in (2), suppose that x in E_y belongs to a cell corresponding to a simplex

$$f_0 \to f_1 \to \cdots \to f_{k-1} \to f_k$$

with $y \in \mathrm{im}(f_k)$. Let $(x_0, x_1, \ldots, x_k)$ be the barycentric coordinates of x, all $x_i > 0$, and let $j \le k$ be the least integer such that $y \in \mathrm{im}(f_j)$. Let

$$\begin{aligned} h_{1-t}(x) :=& (tx_{\mathrm{no}} + x_{\mathrm{yes}})^{-1}(tx_0, tx_1, \ldots, tx_{j-1}, x_j, \ldots, x_k) \\ x_{\mathrm{no}} :=& \sum_{i<j} x_i \qquad x_{\mathrm{yes}} := \sum_{i \ge j} x_i \end{aligned}$$

for $t \in [0,1]$, using barycentric coordinates in the same simplex.)

It only remains to prove that $B\mathcal{C}_y$ is contractible: For any finite collection of objects $f_1, f_2, \ldots, f_k$ in $\mathcal{C}_y$, there exists another object $f_0 \in \mathcal{C}_y$ such that f_0 is an initial object in the full subcategory generated by $f_0, f_1, \ldots, f_k$. (Just make sure that $\mathrm{im}(f_0) \subset \mathrm{im}(f_i)$ for $1 \le i \le k$.)

We conclude that the two projections in 4.1 are *weak* homotopy equivalences. But open subspaces of CW–spaces are homotopy equivalent to CW–spaces [Mil2], so that E is homotopy equivalent to a CW–space. Hence 4.2 implies 4.1. □

Proof of 4.2. Suppose throughout this proof that $p : X \to Y$ is microgibki. Then $p^I : X^I \to Y^I$ is also microgibki (mapping spaces with the compact–open topology). This uses the adjunctions

$$\mathrm{mor}(W \times I, X) \cong \mathrm{mor}(W, X^I)\,, \qquad \mathrm{mor}(W \times I, Y) \cong \mathrm{mor}(W, Y^I)$$

where mor denotes *sets* of continuous maps. See [MaL, VII.8]. Recall now that a map is a *Serre fibration* if it has the homotopy lifting property for maps from cubes I^k, for any $k \ge 0$. Hence it is sufficient to prove the following.

(1) *If the fibers of p are weakly contractible, then p has the homotopy lifting property for maps from a point. That is, any path $\omega : I \to Y$ has a lift $\bar\omega : I \to X$, and $\bar\omega(0)$ can be prescribed arbitrarily in the fiber over $\omega(0)$.*

(2) *If the fibers of p are weakly contractible (explanation follows), then the fibers of p^I are weakly contractible also.*

A space is *weakly contractible* if it is nonempty, and any map from a sphere to it is homotopic to a constant map. For the proof of (1) we need the following observation:

(3) *For continuous $\mu : I \to X$, any vertical homotopy of $\mu|\{0\}$ can be extended to a vertical homotopy of μ.*

(A *vertical homotopy* is a homotopy which turns into a constant homotopy when composed with $p : X \to Y$.) Note: in (3), we do not assume that the fibers are weakly contractible. The proof of (3) is easy, so we concentrate on (1) and (2).

Assume that the fibers of p are weakly contractible. Given $\omega : I \to Y$, we can find a subdivision

$$0 = t_0 < t_1 \cdots < t_{k-1} < t_k = 1\,,$$

of I such that the restriction of ω to $[t_i, t_{i+1}]$ has a lift to X, for $0 \le i < k$. This uses only the microgibki property, and the fact that the fibers of p are nonempty. Using the weak contractibility of the fibers over the division points t_i, and (3), we can then construct a lift $\bar{\omega} : I \to X$ of ω, with any prescribed value at $t_0 = 0$. This proves (1).

Let Φ be the fiber of $p^I : X^I \to Y^I$ over some $\omega : I \to Y$. We know already that $\Phi \neq \emptyset$. For $t \in I$, we have the evaluation map

$$\varepsilon_t : \Phi \longrightarrow p^{-1}(\omega(t)) \quad ; \quad \bar{\omega} \mapsto \bar{\omega}(t)\,.$$

Given $f : \mathbb{S}^{n-1} \to \Phi$, we must try to extend f to a map $\mathbb{D}^n \to \Phi$. Such extensions may be regarded as sections of another microgibki map r with target I, whose fiber over $t \in I$ is the space of extensions to $\mathbb{D}^n$ of $\varepsilon_t f$. The fibers of r are again weakly contractible, and the base is I, so we know from (1) that r admits a section. □

Let $\mathcal{D}$ be the discrete monoid of embeddings $\mathbb{R}^n \to \mathbb{R}^n$. A monoid may be regarded as a category with one object, so $\mathcal{D}$ is a category. Define a functor

$$\tan : \mathcal{C} \longrightarrow \mathcal{D}$$

by sending a morphism $f \to g$ in $\mathcal{C}$ to "itself", that is, to the unique $\lambda : \mathbb{R}^n \to \mathbb{R}^n$ such that $f = g\lambda$. The functor induces a map of spaces

(4–1) $$|\tan| : B\mathcal{C} \to B\mathcal{D}$$

which we may regard as the classifying map for the tangent bundle of M. Indeed, we have verified that $B\mathcal{C} \simeq M$, and [McD2, Cor.of Thm.A], [Se] show that

(4–2) $$B\mathcal{D} \simeq B\,\mathrm{TOP}(\mathbb{R}^n)\,.$$

However, we should also verify that tan is the right map. Of course, we cannot do it without explaining to some extent why (4–2) holds. Suppose that X is a simplicial set, and that

$$f : |X| \longrightarrow B\mathcal{D}$$

is the geometric realization of a simplicial map. Suppose also that $|X|$ is a simplicial complex (equivalently, any nondegenerate i–simplex in X is determined by its $i+1$ vertices, which are all distinct). Using f we construct

$$E(f) := \coprod_v \mathrm{star}(v) \times \mathbb{R}^n \Big/ \sim$$

where star(v) is the open star of the vertex v in $|X|$. The relations depend on f and are as follows: Any edge in $|X|$ from a vertex v_1 to a vertex v_2 determines under f a morphism (=element) in $\mathcal{D}$, say $\lambda : \mathbb{R}^n \to \mathbb{R}^n$. Identify (x, y) in star$(v_1) \times \mathbb{R}^n$ with $(x, \lambda(y))$ in star$(v_2) \times \mathbb{R}^n$.

The projection $E(f) \to |X|$ is a microgibki map with fibers homeomorphic to $\mathbb{R}^n$, but it need not be a microbundle. But it is a homotopy equivalence by 4.2, and if we are willing to replace $|X|$ by $E(f)$ we have a microbundle

$$E(f) \times_{|X|} E(f) \longrightarrow E(f) \quad ; \quad (z_1, z_2) \mapsto z_1 \tag{4–3}$$

with a preferred section (the diagonal). This is how maps to $B\mathcal{D}$ give rise to microbundles on the domain, and it is not a very serious objection that we had to make special assumptions on the domain and, in the end, had to modify the domain in order to see a microbundle on it. In this way, $B\mathcal{D}$ becomes a classifying space for n–microbundles.

Returning to (4–1), suppose that f is the geometric realization of $\tan : \mathcal{C} \to \mathcal{D}$. Then $E(f)$ is what we previously called E. We now have to find an isomorphism from the microbundle (4–3) on $E(f) = E$ to the pullback of τ^M under the projection $E \to M$. This is easy. □

4.3. Remark. So far we have not paid much attention to the boundary ∂M. This is however easy to do if we assume that M is equipped with a collar, by which we understand an embedding

$$\partial M \times [-\infty, +\infty) \longrightarrow M$$

extending the identification $\partial M \times \{-\infty\} \cong \partial M$. Writing $\mathcal{C}(\partial M)$ and $\mathcal{C}(M)$ as well as $\mathcal{D}(\mathbb{R}^{n-1})$ and $\mathcal{D}(\mathbb{R}^n)$ for better distinction, we then have a rather obvious commutative diagram of categories and functors

$$\begin{array}{ccc} \mathcal{C}(\partial M) & \xrightarrow{\subset} & \mathcal{C}(M) \\ \big\downarrow{\scriptstyle \tan} & & \big\downarrow{\scriptstyle \tan} \\ \mathcal{D}(\mathbb{R}^{n-1}) & \xrightarrow{\subset} & \mathcal{D}(\mathbb{R}^n)\,. \end{array} \tag{4–4}$$

In addition, there are homotopy equivalences of pairs

$$\begin{aligned} (M, \partial M) &\simeq (B\mathcal{C}(M), B\mathcal{C}(\partial M)) \\ (B\,\mathrm{TOP}(\mathbb{R}^n), B\,\mathrm{TOP}(\mathbb{R}^{n-1})) &\simeq (B\mathcal{D}(\mathbb{R}^n), B\mathcal{D}(\mathbb{R}^{n-1})) \end{aligned}$$

and so we may think of the diagram of nerves obtained from (4–4) as the (pairwise) classifying map for the tangent bundle pair $(\tau^M, \tau^{\partial M})$.

5. Microcharacteristics for noncompact spaces

In §2, especially 2.3.6, we defined $\boldsymbol{A}^{\%}(Y)$ and $\langle\!\langle Y \rangle\!\rangle \in A^{\%}(Y)$, assuming that Y is a *compact* ENR. These definitions make perfectly good sense for noncompact Y. In the noncompact case, however, we write $\boldsymbol{A}^{\%}_{\ell f}(Y)$ instead of $\boldsymbol{A}^{\%}(Y)$ for reasons given in 5.1 and 5.2. In technical terms then,

$$\boldsymbol{A}^{\%}_{\ell f}(Y) := \mathrm{hofiber}\big[\boldsymbol{P}(Y) \longrightarrow \boldsymbol{A}(Y \times [0,\infty) \lhd Y \times [0,\infty])_{\infty}\big]$$

and $\boldsymbol{P}(Y)$ is contractible (by the appropriate generalization of 2.3.4). What follows is the appropriate generalization of 2.3.3, justifying the notation.

5.1. Theorem [CaPe], [CaPeVo]. *The functor*

$$Y \mapsto \boldsymbol{F}(Y) := \boldsymbol{A}(Y \times [0,\infty) \lhd Y \times [0,\infty])_\infty$$

on the category of ENR's and proper maps is homotopy invariant and pro–excisive (details follow).

Details. The meaning of *homotopy invariance* in 5.1 is that $\boldsymbol{F}$ takes proper homotopy equivalences to homotopy equivalences. The meaning of *pro–excisive* in 5.1 is as follows. Consider a commutative diagram of ENR's and proper continuous maps

$$\begin{array}{ccc} Y_1 & \longrightarrow & Y_2 \\ \downarrow & & \downarrow \\ Y_3 & \longrightarrow & Y_4 \end{array} \tag{5–1}$$

Let Y_0 be the homotopy pushout of $Y_3 \leftarrow Y_1 \rightarrow Y_2$. This is an ENR by [Hu]. We say that (5–1) is a *proper* homotopy pushout square if the map from Y_0 to Y_4 which it determines is a proper homotopy equivalence (=invertible up to proper homotopies). The functor $\boldsymbol{F}$ takes proper homotopy pushout squares of the form (5–1) to homotopy pushout squares of spectra, and it takes $\emptyset$ to a contractible spectrum. Finally, the canonical graded homomorphism

$$\pi_* \boldsymbol{F}(\mathbb{N}) \longrightarrow \prod_{i \in \mathbb{N}} \pi_* \left[\boldsymbol{F}(\mathbb{N} \smallsetminus \{i\}) \to \boldsymbol{F}(\mathbb{N}) \right]$$

is an isomorphism.

For the proof of 5.1, we urge the reader to take a look at the first chapter of [CaPe], and especially Thm. 1.40 of [CaPe] which is the K–theory version of 5.1 above. The A–theory result can be found in [CaPeVo], but it must be pieced together from Lemma 3.1, Thm. 2.21 and Prop. 2.18 in [CaPeVo] (and we can only hope that the numbering is final). We suggest choosing $X = *$, $F = Y \cup *$ (one–point compactification) and $C =$ point at infinity in Prop. 2.18 of [CaPeVo]. □

So far we have allowed proper maps as morphisms between our ENR's, but we can be more generous. Namely, any proper map $Y \to Y'$ gives rise to a continuous pointed map $Y \cup * \to Y' \cup *$ of the one–point compactifications, but not every continuous pointed map $Y \cup * \to Y' \cup *$ comes from a proper map $Y \to Y'$. In the sequel we call a pointed continuous map $Y \cup * \to Y' \cup *$ a *morphism* from Y to Y'. Equivalently, a morphism from Y to Y' is a proper continuous map f from an *open subset* X of Y to Y'. In particular, the inclusion of an open subset $X \subset Y$ is a morphism from $Y \to X$ (take $Y' = X$ and $f = \mathrm{id}_X$).

It is surprising, but easy to verify, that $\boldsymbol{A}^{\%}_{\ell f}(Y)$, with the technical definition given above, is functorial in Y for *morphisms*. Showing this amounts to defining a restriction map

$$\boldsymbol{A}^{\%}_{\ell f}(Y) \longrightarrow \boldsymbol{A}^{\%}_{\ell f}(X)$$

corresponding to any open subset $X \subset Y$. Now a germ near $Y\times\{\infty\}$ of retractive spaces over $Y\times[0,\infty)$ can be restricted to a germ near $X\times\{\infty\}$ of retractive spaces over $X\times[0,\infty)$. Just take the appropriate inverse images, etc. etc.— it all works out. In particular, restriction takes weak equivalences between germs of proper retractive ENR's to weak equivalences. Similarly, restriction takes microequivalences between proper retractive ENR's over $Y\times[0,\infty)$ to microequivalences.

5.2. Proposition. *Let $\boldsymbol{F}$ be any functor from ENR's and their morphisms to CW–spectra. Suppose that $\boldsymbol{F}$ is pro–excisive. Suppose or arrange that $\boldsymbol{F}(*)$ is an Ω–spectrum. Then there exists a chain of natural weak homotopy equivalences*

$$\boldsymbol{F}(Y) \simeq \cdots \simeq (Y\cup *) \wedge \boldsymbol{F}(*)$$

where $Y\cup *$ is the one–point compactification. Informally, we call $\boldsymbol{F}$ a "locally finite homology theory".

Proof. See [WWPro].

5.3. Observation. *The microcharacteristic is lax natural for open embeddings $X\to Y$ of ENR's.*

(Recall: an open embedding $X\to Y$ is a morphism $Y\to X$.)

We proceed to the construction of characteristic classes etc. for euclidean bundles, by taking the microcharacteristics of their fibers. Fix a space V homeomorphic to $\mathbb{R}^n$ for some n. As in §4, let $\mathcal{D}=\mathcal{D}(V)$ be the discrete monoid of embeddings V, so that $B\mathcal{D}\simeq B\,\mathrm{TOP}(V)$. Note that $\mathcal{D}^{\mathrm{op}}$, not $\mathcal{D}$, acts on $A^{\%}_{\ell f}(V)$. The action determines a functor $\mathfrak{q}$ from $\mathcal{D}^{\mathrm{op}}$ to spaces, taking the unique object to $A^{\%}_{\ell f}(V)$. Lax naturality of the microcharacteristic $\langle\!\langle V\rangle\!\rangle$, as in 5.3, determines a point $\mathfrak{e}$ in $\mathrm{holim}\,\mathfrak{EU}\simeq\Gamma(\mathfrak{EU})$ where $\mathfrak{EU}:\mathrm{hocolim}\,\mathfrak{q}\to B\mathcal{D}^{\mathrm{op}}$ is the projection (see 2.1.2 for notation). Informally, $\mathfrak{e}$ is a section of $\mathfrak{EU}$. In §8 we shall see that $\mathfrak{e}$ can be identified with the *Euler section* of the universal euclidean bundle on $B\,\mathrm{TOP}(V)$. (See the introduction.)

5.4. Remark. For any V as above, product with $\mathbb{R}$ defines a map

$$\mathbb{R}\times\,:\boldsymbol{A}^{\%}_{\ell f}(V)\longrightarrow\boldsymbol{A}^{\%}_{\ell f}(\mathbb{R}\times V) \tag{5–2}$$

taking $\langle\!\langle\mathbb{R}^n\rangle\!\rangle$ to $\langle\!\langle\mathbb{R}\times V\rangle\!\rangle$. *Do not confuse* (5–2) *with the map induced by a certain inclusion* $V\to\mathbb{R}\times V$. Here are some more details: The product of $\mathbb{R}$ with a proper retractive ENR over $V\times[0,\infty)$ is a proper retractive ENR over $\mathbb{R}\times V\times[0,\infty)$; similarly for germs, weak equivalences, microequivalences, and so on.

The following facts will be quite important in the sequel:

- (5–2) commutes with the right actions of $\mathcal{D}(V)$, the discrete monoid of embeddings $V\to V$;
- it is equivariantly nullhomotopic, loosely speaking.

We prove the second of these—the first is obvious. Let $\mathbb{R}_{[} = [-\infty, +\infty)$. There is a factorization of (5–2) of the form

$$\boldsymbol{A}^{\%}_{\ell f}(V) \xrightarrow{\lambda} \boldsymbol{A}^{\%}_{\ell f}(\mathbb{R}_{[} \times V) \xrightarrow{\omega} \boldsymbol{A}^{\%}_{\ell f}(\mathbb{R} \times V)$$

where λ is product with the space $\mathbb{R}_{[}$, and ω is induced by the "morphism" from $\mathbb{R}_{[} \times V$ to $\mathbb{R} \times V \cong \mathbb{R} \times V$ which corresponds to the inclusion

$$\mathbb{R} \times V \hookrightarrow \mathbb{R}_{[} \times V\,.$$

Now 5.1, 5.2 imply that $\boldsymbol{A}^{\%}_{\ell f}(\mathbb{R}_{[} \times V)$ is contractible. □

5.5. Lemma. *Let $r : \mathbb{R} \times V \to \mathbb{R} \times V$ be the reflection at V. The following is a homotopy pullback square:*

$$\begin{array}{ccc} \boldsymbol{A}^{\%}_{\ell f}(V) & \xrightarrow{\lambda} & \boldsymbol{A}^{\%}_{\ell f}(\mathbb{R}_{[} \times V) \\ \lambda\big\downarrow & & \omega\big\downarrow \\ \boldsymbol{A}^{\%}_{\ell f}(\mathbb{R}_{[} \times V) & \xrightarrow{r_*\omega} & \boldsymbol{A}^{\%}_{\ell f}(\mathbb{R} \times V)\,. \end{array}$$

Proof. Let $\mathbb{R}_{[]} = [-\infty, +\infty]$. We can factor λ as

$$\boldsymbol{A}^{\%}_{\ell f}(V) \longrightarrow \boldsymbol{A}^{\%}_{\ell f}(\mathbb{R}_{[]} \times V) \longrightarrow \boldsymbol{A}^{\%}_{\ell f}(\mathbb{R}_{[} \times V) \tag{5–3}$$

where the first map is given by product with the space $\mathbb{R}_{[]}$, and the second is induced by the "morphism" from $\mathbb{R}_{[]} \times V$ to $\mathbb{R}_{[} \times V$ corresponding to the inclusion of $\mathbb{R}_{[} \times V$ in $\mathbb{R}_{[]} \times V$. The first arrow in (5–3) is clearly a homotopy equivalence. Hence it is enough to show that

$$\begin{array}{ccc} \boldsymbol{A}^{\%}_{\ell f}(\mathbb{R}_{[]} \times V) & \longrightarrow & \boldsymbol{A}^{\%}_{\ell f}(\mathbb{R}_{[} \times V) \\ \big\downarrow & & \omega\big\downarrow \\ \boldsymbol{A}^{\%}_{\ell f}(\mathbb{R}_{[} \times V) & \xrightarrow{r_*\omega} & \boldsymbol{A}^{\%}_{\ell f}(\mathbb{R} \times V)\,. \end{array} \tag{5–4}$$

is a homotopy pullback square (all maps induced by "morphisms" corresponding to inclusions in the opposite direction). Square (5–4) is the target of a map from another commutative square

$$\begin{array}{ccc} \boldsymbol{A}^{\%}_{\ell f}(V) & \longrightarrow & \boldsymbol{A}^{\%}_{\ell f}([0,\infty) \times V) \\ \big\downarrow & & \omega\big\downarrow \\ \boldsymbol{A}^{\%}_{\ell f}((-\infty,0] \times V) & \longrightarrow & \boldsymbol{A}^{\%}_{\ell f}(\mathbb{R} \times V)\,. \end{array} \tag{5–5}$$

(all maps in (5–5) induced by proper inclusions, no reversal of direction). The various maps connecting vertices of (5–4) with vertices of (5–5) are induced by proper inclusions and proper homotopy equivalences, so by 5.1 they are themselves homotopy equivalences. It is therefore enough to show that (5–4) is a homotopy pullback square. But this follows from 5.1. □

6. The index theorem

In this section we give a *provisional* proof of the index theorem 0.4. The proviso is that the Euler sections e constructed in the introduction agree with the sections $\mathfrak{e}$ constructed by entirely different methods in §5, sequel of 5.3. This will be verified later, in §7. (The verification is much harder than our provisional proof of the index theorem here.)

Recall that the plan is to prove that microcharacteristics of compact manifolds are Poincaré dual to the Euler sections of their tangent bundles. Of course, this needs to be proved for families of compact manifolds. We therefore consider several cases:

(1) one closed manifold at a time ;
(2) a "flat" family (= fiber bundle with discrete structure group) of closed manifolds ;
(3) an arbitrary family (= fiber bundle) of closed manifolds
(4) one compact manifold with boundary ;
(5) a "flat" family (= fiber bundle with discrete structure group) of compact manifolds with boundary.
(6) an arbitrary family (= fiber bundle) of compact manifolds with boundary.

In cases (1) and (4), the (provisional) index theorem is essentially true by inspection. Cases (2) and (5) follow by dint of naturality, while the McDuff–Segal–Mather–Thurston theory is needed to make the step from (2) to (3) and from (5) to (6).

Case (1). Let M^n be closed, with tangent bundle τ. Define $\mathcal{C}$ and $\mathcal{D}$ as in §4. Then $B\mathcal{C}$ is a model for the homotopy type of M, and

$$|\tan| : B\mathcal{C} \longrightarrow B\mathcal{D}$$

is a model for the classifying map of τ. What are the appropriate models for Euler fibration and Euler section ? We saw (sequel of 5.3) that $\mathcal{D}^{\mathrm{op}}$ acts on $A^{\%}_{\ell f}(\mathbb{R}^n)$ by homotopy automorphisms. The action gives a functor $\mathfrak{q}$ from $\mathcal{D}^{\mathrm{op}}$ to spaces. Our model for the Euler fibration of τ is the quasifibration

$$\operatorname{hocolim}(\mathfrak{q}\cdot\tan) \longrightarrow B\mathcal{C}^{\mathrm{op}},$$

also denoted by $\mathfrak{EU}(\tau)$, and our model for the section space of $\mathfrak{EU}(\tau)$ is $\operatorname{holim}(\mathfrak{q}\cdot\tan)$ (see 2.1.2). Lax naturality of $\langle\!\langle \mathbb{R}^n \rangle\!\rangle$ gives rise to a point $\mathfrak{e}(\tau)$ in $\operatorname{holim}(\mathfrak{q}\cdot\tan)$. This is our model for the Euler section of τ. What is the appropriate model for Poincaré duality ? Of course, we take the map $\wp$ from (3–1). In our setup, $\boldsymbol{X} = \boldsymbol{A}(*)$ and we may replace $\Omega^\infty(M_+ \wedge \boldsymbol{X})$ by $A^{\%}(M)$ as defined in §2. We may also replace $\Gamma(M;\boldsymbol{X})$ by $\operatorname{holim}(\mathfrak{q}\cdot\tan)$. Then $\wp$ becomes the "scanning" map
(6–1)

$$A^{\%}(M) = A^{\%}_{\ell f}(M) \longrightarrow \lim_{f:\mathbb{R}^n\hookrightarrow M} A^{\%}_{\ell f}(\mathbb{R}^n) \;\subset\; \operatorname*{holim}_{f:\mathbb{R}^n\hookrightarrow M} A^{\%}_{\ell f}(\mathbb{R}^n) \;=\; \operatorname{holim}(\mathfrak{q}\cdot\tan)$$

induced by the various $f : \mathbb{R}^n \to M$, which one must remember are "morphisms" from M to $\mathbb{R}^n$ in the category of ENR's. Lax naturality of the microcharacteristic provides a canonical path from the image of $\langle\!\langle M \rangle\!\rangle$ under this map to $\mathfrak{e}(\tau)$ as defined above. $\square$

Case (2). It is enough to consider the universal situation

$$p : ET \times_T M \longrightarrow BT \tag{6–2}$$

where $T = \mathrm{TOP}^\delta(M)$. As before, we must translate everything into categories, nerves, and so on. In particular we need the category $\mathcal{ET}$ with object set T, and exactly one morphism between any two objects. T acts on $\mathcal{ET}$ by left translation. We still have $\mathcal{C}$ as in Case (1). Then

$$\mathcal{ET} \times_T \mathcal{C} \;=\; (\mathcal{ET} \times \mathcal{C})/T$$

is a category whose classifying space is our model for $ET \times_T M$. Note that the object set of $\mathcal{ET} \times_T \mathcal{C}$ is canonically identified with that of $\mathcal{C}$, but a morphism in $\mathcal{ET} \times_T \mathcal{C}$ from $f : \mathbb{R}^n \to M$ to $g : \mathbb{R}^n \to M$ is a pair (λ, h) where $h \in T$, $\lambda : \mathbb{R}^n \to \mathbb{R}^n$ is an embedding, and $f = hg\lambda$. Composition of morphisms is defined by $(\lambda_1, h_1)(\lambda_2, h_2) = (\lambda_2\lambda_1, h_1h_2)$. The rule $(\lambda, h) \mapsto \lambda$ is a functor from $\mathcal{ET} \times_T \mathcal{C}$ to the monoid $\mathcal{D}$ of embeddings from $\mathbb{R}^n \to \mathbb{R}^n$. We denote it by $\tan_v$ to stress the analogy with Case (1). The induced map of classifying spaces,

$$B\big(\mathcal{ET} \times_T \mathcal{C}\big) \longrightarrow B\mathcal{D}\,, \tag{6–3}$$

is our model for the map classifying the *vertical tangent bundle* τ_v of the manifold bundle (6–2). As before, $\mathcal{D}^{\mathrm{op}}$ acts on $A^{\%}_{\ell f}(\mathbb{R}^n)$ by homotopy automorphisms, which gives rise to a functor $\mathfrak{q}$ on $\mathcal{D}^{\mathrm{op}}$; our model for the Euler fibration of τ_v is

$$\mathrm{hocolim}(\mathfrak{q} \cdot \tan_v) \longrightarrow B\big(\mathcal{ET} \times_T \mathcal{C}\big)$$

and our model for the Euler section is a certain point $\mathfrak{e}(\tau_v)$ in $\mathrm{holim}(\mathfrak{q} \cdot \tan_v)$ which we get from the lax naturality of $\langle\!\langle \mathbb{R}^n \rangle\!\rangle$. Finally our model for fiberwise Poincaré duality is essentially *still* the scanning map (6–1). But now we think of it as a T–map, or as a map between bundles on BT, and focus on the induced map between section spaces. Using homotopy limits as models for section spaces, this takes the form

$$\Big(A^{\%}(M)\Big)^{hT} \longrightarrow \mathrm{holim}(\mathfrak{q} \cdot \tan_v)$$

or equivalently

$$\Big(A^{\%}(M)\Big)^{hT} \longrightarrow \big(\mathrm{holim}(\mathfrak{q} \cdot \tan)\big)^{hT}$$

because the homotopy limit of a group action is a space of homotopy fixed points. Lax naturality provides a canonical path from the image of the fiberwise microcharacteristic section, now denoted $\langle\!\langle p \rangle\!\rangle$, to the Euler section $\mathfrak{e}(\tau_v)$. $\square$

Case (3). Keeping the notation from Case (2), we let $T' = \mathrm{TOP}(M)$ and

$$p' : ET' \times_{T'} M \longrightarrow BT'\,,$$

with vertical tangent bundle τ'. Think of T' as a simplicial group, so that BT' is a bisimplicial set containing T as its vertical 0–skeleton. Recall the McDuff–Mather–Thurston theorem to the effect that the inclusion $BT \subset BT'$ is a homology equivalence. Suppose for the moment that the actions of T on $A^{\%}(M)$ and on holim$(\cdot\tan)$ can be extended in a canonical way to T'. Then an easy obstruction theory argument shows that the forgetful vertical maps in the would–be square

$$\begin{array}{ccc} \big(A^{\%}(M)\big)^{hT} & \xrightarrow{\ \wp\ } & \big(\mathrm{holim}(\mathfrak{q}\cdot\tan)\big)^{hT} \\ \uparrow{\scriptstyle\phi_1} & & \uparrow{\scriptstyle\phi_2} \\ \big(A^{\%}(M)\big)^{hT'} & & \big(\mathrm{holim}(\mathfrak{q}\cdot\tan)\big)^{hT'} \end{array}.$$

Hence the missing lower horizontal arrow can be filled in, and this is our definition of the fiberwise Poincaré duality map for the manifold bundle p'. In the same spirit, we define the microcharacteristic section $\langle\!\langle p'\rangle\!\rangle$ of p' as the essentially unique point in the homotopy fiber of ϕ_1 over $\langle\!\langle p\rangle\!\rangle$, and we define the Euler section of the vertical tangent bundle of p' as the essentially unique point in the homotopy fiber of ϕ_2 over the Euler section of the vertical tangent bundle of p. Then by definition, fiberwise Poincaré duality takes the microcharacteristic section $\langle\!\langle p'\rangle\!\rangle$ to the Euler section of the vertical tangent bundle of p', up to a canonical homotopy of sections. In short, the problem has been defined away.

It remains to be seen why the actions of T on $A^{\%}(M)$ and on holim$(\cdot\tan)$ can be extended to T'. As one might expect, this requires a "change of models". The forgetful map

$$\mathrm{map}_T(T', A^{\%}(M)) \longrightarrow \mathrm{map}_T(T, A^{\%}(M)) \cong A^{\%}(M)$$

is a homotopy equivalence by obstruction theory. (Its homotopy fiber is a space of sections of a fiber bundle with acyclic base T'/T and nilpotent fiber $A^{\%}(M)$; moreover the action of π_1 of the base on the homotopy groups of the fiber is trivial.) The domain of this forgetful map is our new model for $A^{\%}(M)$, and clearly it comes with an action of T'. We can subject the T–space holim$(\cdot\tan)$ to exactly the same treatment. □

6.1. Notation. Suppose that (X, Y) is a pair of spaces, and that q, ∂q are fibrations or just quasifibrations on X, Y, respectively, related by a map $t : \partial q \to q_{|Y}$ over Y. We call $(q, \partial q)$ a pair of quasifibrations on (X, Y), and define the space of sections $\Gamma(q, \partial q)$ as the holim (= homotopy pullback) of

$$\Gamma(\partial q) \xrightarrow{t_*} \Gamma(\imath^* q) \longleftarrow \Gamma(q)$$

where $\imath : Y \to X$ is the inclusion. (This is in the notation of 2.1.2.) Note that the homotopy pullback contains the strict pullback.

Case (4). Here we assume that M is compact and equipped with a collar (details as in 4.3). Recall the categories $\mathcal{C}(M)$ and $\mathcal{C}(\partial M)$ from 4.3. Our model for the "relative" Euler fibration associated with the tangent bundle pair $\tau = (\tau^M, \tau^{\partial M})$ is a pair of quasifibrations $(\mathfrak{EU}(\tau), \partial\mathfrak{EU}(\tau))$ on the pair $(B\mathcal{C}(M), B\mathcal{C}(\partial M))$. Here $\mathfrak{EU}(\tau)$ is constructed from the functor

$$\begin{aligned} \mathcal{C}(M) &\longrightarrow \text{spectra} \\ (f : \mathbb{R}^n \hookrightarrow M) &\mapsto A^{\%}_{\ell f}(\mathbb{R}^n) \end{aligned}$$

as in Case (1), and $\partial\mathfrak{EU}(\tau)$ is constructed from the functor

$$\begin{aligned} \mathcal{C}(\partial M) &\longrightarrow \text{ spectra} \\ (g:\mathbb{R}^{n-1}\hookrightarrow \partial M) &\mapsto A^{\%}_{\ell f}(\mathbb{R}_{[}\times\mathbb{R}^{n-1}) \end{aligned}$$

(notation of 5.4). *Note:* In general, despite appearances, none of these two functors is constant—they do not take all morphisms to the identity morphism. We are dealing with a *pair* of quasifibrations because there is a map t from $\partial\mathfrak{EU}(\tau)$ to $\mathfrak{EU}(\tau)$ covering the inclusion of $B\mathcal{C}(\partial M)$ in $B\mathcal{C}(M)$. It is induced by the inclusion $\mathrm{int}(\mathbb{R}_{[}\times\mathbb{R}^{n-1})\to\mathbb{R}_{[}\times\mathbb{R}^{n-1}$ which one must remember is a "morphism" from $\mathbb{R}_{[}\times\mathbb{R}^{n-1}$ to $\mathrm{int}(\mathbb{R}_{[}\times\mathbb{R}^{n-1})$ in the category of ENR's.

Lax naturality of the microcharacteristics $\langle\!\langle\mathbb{R}^n\rangle\!\rangle$ and $\langle\!\langle\mathbb{R}_{[}\times\mathbb{R}^{n-1}\rangle\!\rangle$ leads to a quasisection $\mathfrak{e}(\tau)$ of the pair $(\mathfrak{EU}(\tau),\partial\mathfrak{EU}(\tau))$. Our model for relative Poincaré duality is $\wp$ from 3.3, which in our categorical set–up becomes a certain map

$$A^{\%}(M)\longrightarrow\Gamma(\mathfrak{EU}(\tau),\partial\mathfrak{EU}(\tau))$$

(much as in Case (1)). By inspection, this takes $\langle\!\langle M\rangle\!\rangle$ to $\mathfrak{e}(\tau)$, up to a canonical homotopy of sections. ☐

Case (5). This is of course very much like Case (2), but there is one little point we must worry about. We used a collar in Case (4). How can we equip M with a collar that is invariant under the group of homeomorphisms $M\to M$? The answer is simple: We fix a closed collar

$$\partial M\times[-\infty,+\infty]\hookrightarrow M \tag{6–6}$$

and, instead of allowing arbitrary homeomorphisms $M\to M$, allow only those which commute with (6–6). The classifying space of the discrete group of these homeomorphisms still maps to $B\,\mathrm{TOP}(M)$ by a homology equivalence ; hence it is good enough for our purposes, that is, good enough for Case (6). ☐

Case (6). Left to the reader.

Remark. We hope to have made it clear that the index theorem is a tautology from the point of view that we have taken in this §. However, it is surprisingly hard to verify the crucial proviso: "The two types of Euler sections associated with Euclidean bundles agree". This is done in the next §.

7. Euler sections revisited

This §is about a translation. The purpose is to show that the Euler sections $e=e^T$ associated in the introduction with euclidean bundles carry exactly the same information as the Euler sections $\mathfrak{e}$ constructed in §5.

7.1. Euler systems

One crucial property of Euler sections of euclidean bundles, say with fibers homeomorphic to $\mathbb{R}^n$, is of course that they are nullhomotopic when the euclidean bundles in question admit a reduction from structure group $\mathrm{TOP}(n)$ to structure group $\mathrm{TOP}(n-1)$. For the Euler sections e, we were able to verify this directly in the introduction. For the competing Euler sections $\mathfrak{e}$, we verified it in §5. The crucial property is needed to show that the Euler section of the tangent bundle of a manifold M with boundary is canonically nullhomotopic over the boundary. It is a direct consequence of two more basic properties as follows. Let γ be a Euclidean n–bundle on some space X, and form γ' by adding a trivial line bundle.

(1) There are maps over X of the form $\mathrm{EU}(\gamma) \to \mathrm{EU}(\gamma')$ and $\mathfrak{EU}(\gamma) \to \mathfrak{EU}(\gamma')$, taking $e(\gamma)$ and $\mathfrak{e}(\gamma)$ to $e(\gamma')$ and $\mathfrak{e}(\gamma')$, respectively.
(2) These maps are fiberwise nullhomotopic.

Actually the maps in (2) are fiberwise nullhomotopic in two preferred ways, giving a map from $\mathrm{EU}(\gamma)$ to $\Omega_X \mathrm{EU}(\gamma')$ which happens to be a fiberwise homotopy equivalence, and another map from $\mathfrak{EU}(\gamma)$ to $\Omega_X \mathfrak{EU}(\gamma)$ which again is a fiberwise homotopy equivalence. These maps are clearly important to us, in any situation involving stabilization of euclidean bundles, such as Theorem 0.3. (The fact that they are homotopy equivalences is less so, since that can always be enforced by a stabilization process.) We do not want to lose them in the translation process. This leads to the following abstraction.

7.1.1. First provisional abstraction. An *Euler system* on a filtered space $Z = \cup_{i\ge 0} Z_i$ consists of a map of filtered spaces $Y \to Z$, where $Y = \cup_i Y_i$; two sections $Z \to Y$ respecting the filtrations, denoted by "zero section" and "Euler section" (or just e), respectively ; and finally, maps

$$\Sigma_{Z_i} Y_i \to Y_{i+1}$$

over Z_{i+1}, extending the inclusions $Y_i \subset Y_{i+1}$. (Use the zero section to make Y_i into a retractive space over Z_i.)

Notice that a filtered space is a special kind of *functor* from the poset $\mathbb{N}$ to spaces—special because it takes all morphisms in $\mathbb{N}$ to embeddings. Actually most of the filtrations that we shall see are *coordinate free* (explanation follows), and this is a feature that we want to insist on because it will make it easier to construct morphisms between Euler systems.

Let $\mathcal{J}$ be the category of finite dimensional real vector spaces $U, V, W, \ldots$ with inner product (positive definite). The space of morphisms from U to V is the Stiefel manifold of linear maps $U \to V$ respecting the inner product ; it is empty if and only if $\dim(U) > \dim(V)$. Thus $\mathcal{J}$ is a topological category with a discrete class of objects. A *coordinate free filtered space* is a covariant continuous functor Z from $\mathcal{J}$ to spaces, taking all morphisms to embeddings. *Continuity* of Z means that the evaluation maps $\mathrm{mor}(U, V) \times Z(U) \to Z(V)$ are continuous. Note that Z gives rise to an ordinary filtered space with $Z_i = Z(\mathbb{R}^i)$.

In practice we do not care whether or not Z takes all morphisms to embeddings. We therefore drop this condition and arrive at the following.

7.1.2 Second provisional abstraction. Let Z be a continuous functor from $\mathcal{J}$ to spaces. An *Euler system* on Z consists of another continuous functor Y from $\mathcal{J}$ to spaces, a natural transformation $Y \to Z$ with two sections $Z \to Y$, denoted "zero section" and "Euler section", and finally a natural map

$$\Sigma_{Z(V)} Y(V) \longrightarrow Y(V \times \mathbb{R})$$

over $Z(V\times\mathbb{R})$, extending the natural map $Y(V) \to Y(V\times\mathbb{R})$ induced by the inclusion $V \to V\times\mathbb{R}$.

7.1.3. Example. We define $Z(V) = *$ for all i and $Y(V) = \mathrm{TOP}(\mathbb{R}\times V)/\,\mathrm{TOP}(V)$. Zero section and Euler section e from $Z(V)$ to $Y(V)$ are then fully determined if we specify them for $i = 0$, and in that case we specify them by taking the unique point to the cosets [id] and $[-\,\mathrm{id}]$, respectively. The map $Y(V) \to Y(V\times\mathbb{R})$ induced by $V \subset V\times\mathbb{R}$ has two nullhomotopies $\{h_t^\pm\}$, where $h_t^\pm$ for $0 \le t \le \infty$ is given by

$$[f] \mapsto [(f\times \mathrm{id}_1)\rho_t^\pm]$$

and $\rho_t^\pm$ is any linear isometry of $\mathbb{R}\times V\times\mathbb{R}$ taking the vector $(1, 0, \dots, 0)$ to a suitable positive scalar multiple of $(1, 0, \dots, 0, \pm t)$. The two nulhomotopies taken together define a map $\Sigma Y(V) \to Y(V\times\mathbb{R})$.

7.1.4. Example. Here again $Z(V) = *$ for all V, but we shall write $sY(V)$, not $Y(V)$, to avoid confusion with 7.1.3. In fact, this example is meant to be the "stabilization" of the previous one. We have seen that for each V the spaces $Y(V\times\mathbb{R}^i)$ in 7.1.3 form a spectrum $\boldsymbol{Y}(V)$. Let $sY(V) = \Omega^{\infty+V}\Sigma^V\boldsymbol{Y}(V)$ be the space of maps (in the sense of Boardman) from the suspension spectrum of the one–point compactification V^c to $V^c \wedge \boldsymbol{Y}(V)$. (*Remark*: $sY(V)$ is a colimit of mapping spaces $\Omega^{n+V}\Sigma^V Y(V\times\mathbb{R}^n)$, for $n \to \infty$, and one has to work in a convenient category of spaces where colimits of the type above are homotopy equivalent to the corresponding homotopy colimits (telescopes). The category of simplicial sets would qualify, but we suggest using virtual spaces as in [WW1].) Then sY is a continuous functor, and it is clear that the maps $sY(V) \to sY(V\times\mathbb{R})$ extend naturally to $\Sigma sY(V)$.

Notation. From now on a $\backslash\!\!\backslash$ is used to denote homotopy orbit spaces (Borel construction) ; thus $G\backslash\!\!\backslash X$ is the homotopy orbit space of a left action of the group or monoid G on the space X. Often G is a topological group or monoid.

7.1.5. Example. Return to 7.1.3. Note that the group $\mathrm{TOP}(V)$ acts on the left of each $Y(V)$ and leaves the base point and the "Euler point" fixed. The actions are compatible with the nullhomotopies $\{h_t^\pm\}$ to the extent that

$$\begin{aligned} \bar{Z}(V) &:= \mathrm{TOP}(V)\backslash\!\!\backslash Z(V) = B\,\mathrm{TOP}(V) \\ \bar{Y}(V) &:= \mathrm{TOP}(V)\backslash\!\!\backslash Y(V) \\ \bar{e} &:= \mathrm{TOP}(V)\backslash\!\!\backslash e \end{aligned}$$

is a new Euler system. Similarly, $\mathrm{TOP}(V)$ acts on $sY(V)$ (notation of 7.1.4), giving a new Euler system

$$\begin{aligned} \bar{Z}(V) &:= \mathrm{TOP}(V)\backslash\!\!\backslash Z(V) = B\,\mathrm{TOP}(V) \\ s\bar{Y}(V) &:= \mathrm{TOP}(V)\backslash\!\!\backslash sY(V) \\ \bar{e} &:= \mathrm{TOP}(V)\backslash\!\!\backslash e\,. \end{aligned}$$

In example 7.1.3, the suspension map $\Sigma Y(V) \to Y(V\times\mathbb{R})$ has a special property: it is equivariant with respect to certain canonical involutions on domain and codomain. (For the domain, use reflection in the suspension coordinate ; for the codomain, use the map induced by reflection at the hyperplane $V \subset V\times\mathbb{R}$.) This equivariance property is also present in examples 7.1.4 and 7.1.5 and leads us to revise abstraction 7.1.2.

7.1.6. Third provisional abstraction. Let Z be a continuous functor from $\mathcal{J}$ to spaces. An *Euler system* on Z consists of another continuous functor Y from $\mathcal{J}$ to spaces, a natural transformation $Y \to Z$ with two sections $Z \to Y$, denoted "zero section" and "Euler section", and finally a vertical nullhomotopy, over $Z(V\times\mathbb{R})$, of the map $Y(V) \longrightarrow Y(V\times\mathbb{R})$ induced by the inclusion $V \to V\times\mathbb{R}$.

Comment. Go from 7.1.6 to 7.1.2 by viewing the fiberwise suspension $\Sigma_{Z(V)}Y(V)$ as a union of two mapping cylinders, switched by the involution. The vertical nullhomotopy in 7.1.6 is a map whose domain is one of the mapping cylinders.

In the next abstraction, which shall be the last, there are two new features. The first, which can hardly give rise to objections, is that we replace "vertical nullhomotopy" over $Z(V\times\mathbb{R})$ by "factorization thru a space over $Z(V\times\mathbb{R})$ with contractible homotopy fibers". We denote this space by $Z^\sharp(V\times\mathbb{R})$; it is supposed to depend functorially and continuously on V. Note that this is abusive notation, in that $Z^\sharp(V)$ is not defined. The other new feature is that we discard the "zero sections", partly because they have become nearly redundant (see the comment following 7.1.7) and partly because that eases the burden of proof (see theorem 7.2.1).

7.1.7. Fourth and final abstraction. Let Z be a continuous functor from $\mathcal{J}$ to spaces. An *Euler system* on Z consists of another continuous functor Y from $\mathcal{J}$ to spaces, a natural transformation $p : Y \to Z$ with a section $e : Z \to Y$, called "Euler section", and finally, a natural factorization

$$Y(V) \xrightarrow{\jmath} Z^\sharp(V\times\mathbb{R}) \xrightarrow{\zeta} Y(V\times\mathbb{R})$$

of the map $Y(V) \to Y(V\times\mathbb{R})$ induced by the inclusion $V \to V\times\mathbb{R}$. Condition: the composition $p\jmath$ from $Z^\sharp(V\times\mathbb{R})$ to $Z(V\times\mathbb{R})$ is a homotopy equivalence.

Comment. In view of the "condition", we may think of the natural map ζ as a substitute for the abandoned zero section. Drawback: $Z^\sharp(W)$ and the substitute zero section $\jmath : Z^\sharp(W) \to Y(W)$ are only defined when a splitting $W \cong V\times\mathbb{R}$ has been selected. For us this is not a serious loss.

The next two examples are formalizations of the end of §5.

7.1.8. Example. For V in $\mathcal{J}$ let $Z(V) = *$ and $Y(V) = A^{\%}_{\ell f}(V)$. A morphism $V \to W$ in $\mathcal{J}$ determines a projection $W \to V$, and pullback with this is an exact functor between the appropriate categories of retractive spaces which induces $Y(V) \to Y(W)$. In this way Y becomes a functor.

Warning 1: The morphism $V \to W$ also determines a proper inclusion of locally compact spaces $V \to W$, which in turn leads to a much more obvious map $A^{\%}(V) \to A^{\%}(W)$. But this is not the map we want—explanation in a moment.

Warning 2: We must use a model of the category of sets with a canonical choice of pullbacks such that pullbacks are associative—otherwise Y will not be a functor. Details are given in the remark just below.

Let $Z^\sharp(V\times\mathbb{R}) = A^{\%}(V\times\mathbb{R}_{[})$ and define $\imath : Y(V) \to Z^\sharp(V\times\mathbb{R})$ using the exact functor induced by pullback with the projection $V\times\mathbb{R}_{[} \to V$. Define $\zeta : Z^\sharp(V\times\mathbb{R}) \to Z(V\times\mathbb{R})$ as the restriction map associated with the open embedding $V\times\mathbb{R} \hookrightarrow V\times\mathbb{R}_{[}$.

The Euler section $e : Z(V) \to Y(V)$ will be fully specified if we specify it for $V = 0$, and in this case we send the unique point in $Z(0)$ to the microcharacteristic $\langle\!\langle 0\rangle\!\rangle = \langle\!\langle *\rangle\!\rangle \in A(*) = Y(0)$. For arbitrary V, we may then call the unique point in the image of $e : Z(V) \to Y(V)$ the "microcharacteristic" of V, without contradicting earlier definitions too shamelessly. At this point the reader may be able to appreciate our definition of Y as a *functor*.

One thing is conspicuously absent in this example: continuity. The functors Y, and $Z^\sharp(\text{—}\times\mathbb{R})$ are clearly not continuous. However, it is explained in Appendix B how continuity can be enforced. A key observation is that Y and $Z^\sharp(\text{—}\times\mathbb{R})$ can be extended from $\mathcal{J}$ to a larger category $\mathcal{J}^{\mathrm{top}}$ which has the same objects as $\mathcal{J}$. A morphism $V \to W$ in $\mathcal{J}^{\mathrm{top}}$ is an equivalence class of homeomorphisms $h : U\times V \cong W$, where U is another object in $\mathcal{J}$; two such, say $h_1 : U_1\times V \cong W$ and $h_2 : U_2\times V \cong W$, are equivalent if there exists a homeomorphism $g : U_1 \to U_2$ such that $h_2(g\times\mathrm{id}) = h_1$. Note that a morphism $V \to W$ in $\mathcal{J}^{\mathrm{top}}$ still determines a projection $W \to V$; it does not determine an inclusion $V \to W$, which makes "Warning 1" above even more appropriate.— In B.7, we replace $\mathcal{J}^{\mathrm{top}}$ by the equivalent full subcategory with objects $\mathbb{R}^i$ for $i \geq 0$, so that the set of morphisms from $\mathbb{R}^i$ to $\mathbb{R}^j$ is $\mathrm{TOP}(\mathbb{R}^j)/\,\mathrm{TOP}(\mathbb{R}^{j-i})$ made discrete.

Remark. Let $\mathcal{S}$ be the full subcategory of the category of sets consisting of all von Neumann ordinals [Va]. A diagram

$$\begin{array}{ccc} P & \longrightarrow & S_2 \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ S_1 & \xrightarrow{\ f\ } & S_3 \end{array}$$

in $\mathcal{S}$ is a *special pullback* if the two maps issuing from P induce an *order preserving* bijection from P to

$$\{(a,b) \in S_1\times S_2 \mid f(a) = p(b)\}$$

where $S_1 \times S_2$ has the lexicographic ordering determined by the orderings of S_1 and S_2. In this case, and only in this case, we denote the left–hand vertical arrow in the square by f^*p. One finds that f^*p is uniquely determined by p and f (and it *exists* when only p and f are given). The following properties are easily verified: $(fg)^*p = g^*(f^*p)$, and $f^*p = p$ if f is an identity morphism. This is what we mean by "associative pullbacks". Note that our associative pullbacks are not commutative: the domains of f^*p and p^*f are not identical, only isomorphic.

7.1.9. Example. For V in $\mathcal{J}$ let $\mathcal{D}(V)$ be the discrete monoid of embeddings $V \to V$. Then $\mathcal{D}(V)$ acts on the right of the spaces $Z(V)$, $Y(V)$ and $Z^\sharp(V\times\mathbb{R})$ in example 7.1.8. We therefore obtain a new Euler system (in the sense of 7.1.7) by using 7.1.8 and letting

$$\begin{aligned} \bar{Z}(V) &:= \mathcal{D}(V)^{\mathrm{op}}\backslash\!\!\backslash Z(V) \cong B\mathcal{D}(V) \\ \bar{Y}(V) &:= \mathcal{D}(V)^{\mathrm{op}}\backslash\!\!\backslash Y(V) \\ \bar{Z}^{\sharp}(V\times\mathbb{R}) &:= \mathcal{D}(V)^{\mathrm{op}}\backslash\!\!\backslash Z^{\sharp}(V\times\mathbb{R}) \\ \bar{e} &:= \mathcal{D}(V)^{\mathrm{op}}\backslash\!\!\backslash e \end{aligned}$$

Note that $\bar{e}$ is essentially $\mathfrak{e}$ from §5. We cannot always be consistent about notation. Can you read letters like $\mathfrak{Z}$? Remarks about continuity or absence of continuity made at the end of 7.1.8 apply also here.

7.1.10. Theorem. *The Euler systems 7.1.5 and 7.1.9 are equivalent.*

Explanation. "Euler system" is to be understood as in 7.1.7. There is an obvious notion of *morphism* between Euler systems ; such a morphism is an *equivalence* if all the maps that it consists of, say $Z_1(V) \to Z_2(V)$, $Y_1(V) \to Y_2(V)$ and $Z_1^{\sharp}(V\times\mathbb{R}) \to Z_2^{\sharp}(V\times\mathbb{R})$, are homotopy equivalences. Two Euler systems are *equivalent* if they can be related by a chain of equivalences.

Implicit in 7.1.10 is the claim that $B\,\mathrm{TOP}(V)$ can be related to $B\mathcal{D}(V)$ by a chain of homotopy equivalences, natural in V ; but we know this already from §4.

The proof of 7.1.10 is given in the next subsection, 7.2, and appendix C. One property shared by the Euler systems 7.1.5 and 7.1.9 that will help us to establish their equivalence is *stability.* To define this notion, we suppose for simplicity that Z in 7.1.7 is a (continuous) functor from $\mathcal{J}$ to *connected pointed* spaces. Let $Y_0(V)$ and $Z_0^{\sharp}(V\times\mathbb{R})$ be the homotopy fibers of

$$p : Y(V) \to Z(V)\,, \qquad p\zeta : Z_0^{\sharp}(V\times\mathbb{R}) \to Z(V\times\mathbb{R})\,,$$

respectively. Then Y_0 and $Z_0^{\sharp}$ constitute an Euler system on Z_0 where $Z_0(V) = *$.

7.1.11. Definition. The Euler system 7.1.7 is *stable* if

$$\begin{array}{ccc} Y_0(V) & \xrightarrow{\ \imath\ } & Z_0^{\sharp}(V\times\mathbb{R}) \\ {\scriptstyle \imath}\big\downarrow & & \big\downarrow{\scriptstyle \zeta} \\ Z_0^{\sharp}(V\times\mathbb{R}) & \xrightarrow{\ \rho_*\zeta\ } & Y_0(V\times\mathbb{R}) \end{array}$$

is a homotopy pullback square for every V. Here $\rho : V\times\mathbb{R} \to V\times\mathbb{R}$ is the reflection at the hyperplane V.

It is clear that 7.1.4 is stable ; therefore 7.1.5 is stable. We verified stability of 7.1.8 in 5.5. Therefore 7.1.9 is also stable.

7.2. The canonical Euler system

7.2.1. Theorem. *Let $Z(V) = B\,\mathrm{TOP}(V)$. Up to equivalence relative to Z, there exists a unique stable Euler system on Z such that*

$$\lim_{n\to\infty} c_n - n = \infty$$

where c_n is the connectivity of the map between vertical homotopy fibers in the commutative square (notation of 7.1.7) where $\dim(V) = n$,

$$\begin{array}{ccc} Z(V) & \xrightarrow{\imath\cdot e} & Z^\sharp(V\times\mathbb{R}) \\ \downarrow & & \downarrow\zeta \\ Z(V\times\mathbb{R}) & \xrightarrow{e} & Y(V\times\mathbb{R})\,. \end{array}$$

Proof and explanation. See Appendix C for the proof. Often in an equivalence between Euler systems, the base functor Z is kept fixed. Such an equivalence would be called *relative to Z*. The meaning of *up to equivalence relative to Z* is: there exists a chain of equivalences ... , relative to Z.

We must now verify that examples 7.1.5 and 7.1.9 satisfy the connectivity assumption in 7.2.1. In the case of 7.1.5, this is very easy. Modulo straightforward identifications, the map between vertical homotopy fibers in question becomes the stabilization map

$$\mathrm{TOP}(\mathbb{R}\times V)/\,\mathrm{TOP}(V) \longrightarrow \operatorname*{hocolim}_{k>0} \Omega^k \left(\mathrm{TOP}(\mathbb{R}\times V\times\mathbb{R}^k)/\,\mathrm{TOP}(V\times\mathbb{R}^k)\right)\,.$$

Its connectivity is approximately $(4/3)\cdot\dim(V)$ by [Ig], plus Morlet smoothing theory. See also [Wal2, §2].

7.2.2. Lemma. *Example 7.1.9 also satisfies the connectivity assumption in 7.2.1.*

Preliminaries and notation. We assume $V = \mathbb{R}^n$ but continue to write V because it is more convenient. Let $Y^\sharp(V\times\mathbb{R})$ be the homotopy pushout of

$$\begin{array}{ccc} Z(V) & \xrightarrow{\imath\cdot e} & Z^\sharp(V\times\mathbb{R}) \\ \downarrow & & \\ Z(V\times\mathbb{R}) & & \end{array}$$

so that we have a canonical map $Y^\sharp(V\times\mathbb{R}) \to Y(V\times\mathbb{R})$, from the diagram in 7.2.1. Think of it as a map over $Z(V\times\mathbb{R})$, and take homotopy fibers over the base point: this gives a map

$$\beta : Y_0^\sharp(V\times\mathbb{R}) \to Y_0(V\times\mathbb{R})\,.$$

Note that the domain of β is homotopy equivalent to the join

$$\mathbb{S}^0 * \mathrm{TOP}(V\times\mathbb{R})/\,\mathrm{TOP}(V)\,,$$

and the codomain is homotopy equivalent to an $(n+1)$–fold delooping of $A(*)$, where $n = \dim(V)$. *It is enough to show that the connectivity of β exceeds n by a quantity which goes to infinity with n.*

Proof of 7.2.1. Let $\boldsymbol{a}(*)$ be the suspension spectrum generated in degree $n+1$ by $Y_0^\sharp(V\times\mathbb{R})$. By Waldhausen, and by Igusa stability, we know that $\boldsymbol{a}(*)$ is j–equivalent to $\boldsymbol{A}(*)$ where $j = 4n/3$ approximately. Let F be any compact CW–space, equipped with the trivial euclidean bundle ε^{n+1} with fiber $V\times\mathbb{R}$. Recall from the introduction the space $\mathcal{S}^T_{n+1}(F,\varepsilon^{n+1})$ of compact manifold structures on F. Actually this is not exactly the notation used in the introduction, so let's be a little more precise: A 0–simplex in $\mathcal{S}^T_{n+1}(F,\varepsilon^{n+1})$ is a compact manifold M^{n+1} with a homotopy equivalence to F, covered by a microbundle map from τ^M to ε^{n+1}. A 1–simplex is a "continuous family" of such things, parametrized by Δ^1, and so on.

We can now compare three maps

$$\mathcal{S}^T_{n+1}(F,\varepsilon^{n+1}) \xrightarrow{(1)} \Omega^\infty(F_+\wedge \mathbf{a}(*))$$
$$\mathcal{S}^T_{n+1}(F,\varepsilon^{n+1}) \xrightarrow{(2)} \Omega^\infty(F_+\wedge \boldsymbol{A}(*))$$
$$\mathcal{S}^T_{n+1}(F,\varepsilon^{n+1}) \xrightarrow{(3)} \Omega^\infty(F_+\wedge \boldsymbol{A}(*))\,.$$

Map (2) for example goes like this: A 0–simplex in $\mathcal{S}^T_{n+1}(F,\varepsilon^{n+1})$ is a compact manifold M^{n+1} with a homotopy equivalence to F, and so on ; we construct the (relative) Euler section $\mathfrak{e}(\tau)$ as in Case (1) or Case (4) of §6, think of it as a map from $M/\partial M$ to an $(n+1)$–fold delooping of $A(*)$, and apply Poincaré duality to land in $\Omega^\infty(M_+\wedge \boldsymbol{A}(*)) \simeq \Omega^\infty(F_+\wedge \boldsymbol{A}(*))$. Map (1) is a much more primitive construction: Starting with the same 0–simplex in $\mathcal{S}^T_{n+1}(F,\varepsilon^{n+1})$ corresponding to M^{n+1} with a homotopy equivalence to F, and so on, we note that near ∂M the structure group of τ^M has a canonical reduction from $\mathrm{TOP}(V\times\mathbb{R})$ to $\mathrm{TOP}(V)$. This in conjunction with the trivialization of τ^M leads to a pointed map from $M/\partial M$ to the join $\mathbb{S}^0 * (\mathrm{TOP}(V\times\mathbb{R})/\,\mathrm{TOP}(V))$ taking the complement of a collar about ∂M to the nontrivial point in $\mathbb{S}^0$. Apply Poincaré duality to land in $\Omega^\infty(M_+\wedge \mathbf{a}(*)) \simeq \Omega^\infty(F_+\wedge \mathbf{a}(*))$. Note that Poincaré duality is easy because we are dealing with stably parallelized manifolds. Last not least, map (3) is the microcharacteristic map that we know from (0–7), in a much more general form, and from §2. By the index theorem, to the extent that we have it in §6, the maps (2) and (3) are homotopic.

Taking a sufficiently hard look at the commutative square in 7.2.1, we find that map (2) factors through map (1) up to homotopy. More precisely, (2) is the composition of (1) with Ω^∞ of

$$\mathrm{id}_F \wedge \boldsymbol{\beta} : F_+\wedge \mathbf{a}(*) \longrightarrow F_+\wedge \boldsymbol{A}(*)$$

where $\boldsymbol{\beta}$ is the map β from the preliminaries, now regarded as a map of spectra.

We seem to know nothing about $\boldsymbol{\beta}$. But try taking $F = *$. Then it is clear that map (3) maps the unique component of $\mathcal{S}^T_{n+1}(F,\varepsilon^{n+1})$ to the unit component of $\Omega^\infty(F_+\wedge \boldsymbol{A}(*))$. Hence, by the relationship just observed, $\boldsymbol{\beta}$ from $\mathbf{a}(*)$ to $\boldsymbol{A}(*)$ is surjective on π_0. So we know something about $\boldsymbol{\beta}$ after all.

Next, take $F = \mathbb{S}^k$. Choose $n \gg k$. There is then an obvious inclusion of the h–cobordism space $\mathfrak{H}(F\times\mathbb{S}^{n-k})$ in $\mathcal{S}^T_{n+1}(F,\varepsilon^{n+1})$, and the following diagram is homotopy commutative up to addition of a constant map with constant value $\langle\!\langle F\rangle\!\rangle$:

$$\begin{array}{ccc} \mathfrak{H}(F\times\mathbb{S}^{n-k}) & \xrightarrow{\subset} & \mathcal{S}^T_{n+1}(F,\varepsilon^{n+1}) \\ \big\downarrow{\scriptstyle \text{Waldhausen}} & & \big\downarrow{\scriptstyle (3)} \\ A^{\%}(F\times\mathbb{S}^{n-k}) & \longrightarrow & A^{\%}(F) \simeq \Omega^\infty(F_+ \wedge \boldsymbol{A}(*))\,. \end{array}$$

(The arrow labelled *Waldhausen* is Waldhausen's forgetful map, compare (2–5) and (2–7), and the lower horizontal arrow is induced by the projection.) Furthermore,

$$A^{\%}(F) = A^{\%}(\mathbb{S}^k) \simeq \Omega^\infty \boldsymbol{S}^0 \times \Omega^{\infty-k}\boldsymbol{S}^0 \times \Omega^\infty(\boldsymbol{A}(*)/\boldsymbol{S}^0) \times \Omega^{\infty-k}(\boldsymbol{A}(*)/\boldsymbol{S}^0)\,.$$

It is well known [Wald2], [Ig] that the composite map, from upper left in the diagram to lower right via lower left, and onwards to $\Omega^{\infty-k}(\boldsymbol{A}(*)/\boldsymbol{S}^0)$, is split onto on π_i for $i < 2k$ approximately. Combining this with the previous observations, we see that

$$\boldsymbol{\beta} : \boldsymbol{a}(*) \longrightarrow \boldsymbol{A}(*) \simeq \boldsymbol{S}^0 \vee (\boldsymbol{A}(*)/\boldsymbol{S}^0)$$

induces epimorphisms in π_i for $i < k$ approximately. Moreover, as we have noted, $\pi_i(\boldsymbol{a}(*))$ is abstractly isomorphic to $\pi_i(\boldsymbol{A}(*))$ (for $i < 4n/3$ approximately) and finitely generated by [Dw]. Hence $\boldsymbol{\beta}$ is approximately k–connected. Since k can be arbitrary provided n is big enough, the proof is complete. □

7.3. A subtlety

We seem to have achieved the goal of the section, but one little question remains. Let $B = B\,\mathrm{O}(n)$, large n. With the universal n–dimensional vector bundle ξ on B, we can associate *three* Euler fibrations:

$$\mathrm{EU}^D(\xi)\,, \quad \mathrm{EU}^T(\xi)\,, \quad \mathfrak{EU}(\xi)\,.$$

These are the differentiable Euler fibration from the introduction, with fibers homotopy equivalent to $\Omega^{\infty-n}\boldsymbol{S}^0$, the topological Euler fibration from the introduction and this §, with fibers homotopy equivalent to $\Omega^{\infty-n}\boldsymbol{A}(*)$, and the categorical Euler fibration from §5 and §6, again with fibers homotopy equivalent to $\Omega^{\infty-n}\boldsymbol{A}(*)$. To be quite honest, $\mathfrak{EU}(\xi)$ is a quasifibration, pulled back from $B\,\mathrm{TOP}(n) \simeq B\mathcal{D}(\mathbb{R}^n)$; see 5.3 and sequel. We know now that $\mathrm{EU}(\xi)$ and $\mathfrak{EU}(\xi)$ are related by a preferred fiber homotopy equivalence, respecting the zero sections. Using this as an identification, we write the inclusion $\mathrm{EU}^D(\xi) \longrightarrow \mathrm{EU}^T(\xi)$ in the form

$$\lambda : \mathrm{EU}^D(\xi) \longrightarrow \mathfrak{EU}(\xi)\,.$$

To make this even more explicit, we note that $\mathrm{EU}^D(\xi)$ is canonically fiber homotopy equivalent to the projection (with zero section)

(7–1) $$E\,\mathrm{O}(n) \times_{\mathrm{O}(n)} \Omega^\infty(\mathbb{S}^n \wedge \boldsymbol{S}^0) \longrightarrow B\,\mathrm{O}(n)$$

and $\mathfrak{EU}(\xi)$ is canonically fiber homotopy equivalent to the projection (with zero section)

$$E\,\mathrm{O}(n) \times_{\mathrm{O}(n)} \Omega^\infty(\mathbb{S}^n \wedge \boldsymbol{A}(*)) \longrightarrow B\,\mathrm{O}(n) \tag{7–2}$$

(by §5, especially 5.2). Here the obvious base point preserving action of $\mathrm{O}(n)$ on $\mathbb{S}^n = (\mathbb{R}^n \cup *)$ is understood. These descriptions, (7–1) and (7–2), are quite essential to us since they allow us to say that the Poincaré dual of the Euler section (e^D or $\mathfrak{e}$) of the tangent bundle of a smooth or topological compact manifold M is a point in $\Omega^\infty(M_+ \wedge \boldsymbol{S}^0)$, or in $\Omega^\infty(M_+ \wedge \boldsymbol{A}(*))$, as appropriate. For similar reasons, we want to be sure that λ is compatible with (7–1) and (7–2). We want to be sure that λ is fiberwise homotopic to

$$E\,\mathrm{O}(n) \times_{\mathrm{O}(n)} \Omega^\infty(\mathbb{S}^n \wedge \boldsymbol{\imath})$$

where $\boldsymbol{\imath} : \boldsymbol{S}^0 \to \boldsymbol{A}(*)$ is the unit map. Only then can we claim to have established diagram (0–13) in the introduction.

This is easily translated into another question, as follows. Let $p : E \to B$ be the unit disk bundle associated with ξ. The fiberwise tangent bundle of E determines a relative Euler section $e^D(p)$, equal to zero on the fiberwise boundary ∂E. Is it true that the fiberwise Poincaré dual of $\lambda_*(e^D(p))$ equals $\boldsymbol{\imath}_*$ of the Poincaré dual of $e^D(p)$?

Now the answer is easy: by the index theorem, §6, and by inspection, both are homotopic to the unit section of the trivial fiber bundle

$$A(*) \times B \longrightarrow B\,. \quad \square$$

8. Smoothing Theory and Euler Sections

Let ξ denote the universal fiber bundle with fibers $\cong \mathbb{R}^n$ on $B\,\mathrm{TOP}(n)$. Write $\mathrm{EU}^T(\xi)$ for the Euler fibration (introduction or 7.1.5). Recall from §7 that the square

$$\begin{array}{ccc} B\,\mathrm{TOP}(n-1) & \longrightarrow & B\,\mathrm{TOP}(n) \\ \big\downarrow \subset & & \big\downarrow \text{zero section} \\ B\,\mathrm{TOP}(n) & \xrightarrow{e(\xi)} & \text{total space of } \mathrm{EU}(\zeta) \end{array} \tag{8–1}$$

is commutative up to a preferred homotopy, and as such approximately $(4n/3)$–cartesian (i.e., the resulting map from $B\,\mathrm{TOP}(n-1)$ to the homotopy pullback of the other three terms is approximately $(4n/3)$–connected).

Notation: Let η be a fiber bundle with structure group G, say on a CW–space Z. Suppose that $H \subset G$ is a subgroup. We denote the space of reductions of the structure group of η from G to H by

$$\mathcal{R}^G_H(\eta)\,.$$

It can also be described as the homotopy fiber of $BH^Z \hookrightarrow BG^Z$ over $c(\eta)$, the classifying map for η.

8.1. Lemma. *If F is a CW–space of dimension k, with a euclidean n–bundle η, then the map resulting from* (8–1),

$$\mathcal{R}^{\mathrm{TOP}(n)}_{\mathrm{TOP}(n-1)}(\xi) \longrightarrow \{\textit{nullhomotopies of } e(\xi)\}\,,$$

is approximately $((4n/3)-k)$*–connected.* □

Next, suppose that M^n is a compact topological manifold with boundary, where $n \neq 4, 5$, and that $\tau = \tau^M$ is equipped with a vector bundle structure υ. (The structure group of τ has been reduced from $\mathrm{TOP}(n)$ to $\mathrm{O}(n)$.) We want to understand the space $\mathrm{sm}(M, \upsilon)$ of smooth structures on M inducing the given vector bundle structure on τ. More precisely, there is a forgetful map from the space of smooth structures on M to the space of vector bundle structures on τ, and we want to analyse its homotopy fiber over the point υ. Morlet's sliced smoothing theory says that this homotopy fiber is contractible if $\partial M = \emptyset$ [Mor1], [KiSi], [BuLa]. In general, sliced smoothing theory still says that everything can be expressed in terms of bundles and reductions: the homotopy fiber in question maps by a homotopy equivalence to the homotopy fiber of

$$\mathcal{R}^{\mathrm{O}(n)}_{\mathrm{O}(n-1)}(\tau_{|\partial M}) \longrightarrow \mathcal{R}^{\mathrm{TOP}(n)}_{\mathrm{TOP}(n-1)}(\tau_{|\partial M})$$

over the base point. Combining this with 8.1 (and its analogue for vector bundles), we have an approximately $(n/3)$–cartesian square

$$\begin{array}{ccc} \mathrm{sm}(M,\upsilon) & \longrightarrow & * \\ \big\downarrow & & \big\downarrow \\ \{\text{nullhomotopies of } e^D(\tau_{|\partial M})\} & \xrightarrow{\ \subset\ } & \{\text{nullhomotopies of } e^T(\tau_{|\partial M})\} \end{array} \tag{8–2}$$

where e^D and e^T are the smooth and topological Euler sections, respectively. Finally we note that the spaces in the lower row of (8–2) can be identified with the homotopy fibers of

$$\begin{array}{ccc} \Gamma_{\mathrm{in}}(\mathrm{EU}^D(\tau)) & \longrightarrow & \Gamma(\mathrm{EU}^D(\tau))\,, \\ \Gamma_{\mathrm{in}}(\mathrm{EU}^T(\tau)) & \longrightarrow & \Gamma(\mathrm{EU}^T(\tau)) \end{array} \tag{8–3}$$

over $e^D(\tau)$ and $e^T(\tau)$, respectively. Here Γ denotes spaces of sections (with support in the interior of M in the case of Γ_{in}). If $M \simeq F$, where F is a CW–space of dimension $< 2n/3$, then the right–hand terms in (8–3) are still $n/3$–connected. Hence we may replace the lower row in (8–2) by

$$\Gamma_{\mathrm{in}}(\mathrm{EU}^D(\tau)) \longrightarrow \Gamma_{\mathrm{in}}(\mathrm{EU}^T(\tau))$$

and we have proved the following result.

8.2. Lemma. *The homotopy commutative square*

$$
\begin{array}{ccc}
\mathrm{sm}(M,\upsilon) & \longrightarrow & * \\
\Big\downarrow{\scriptstyle e^D} & & \Big\downarrow{\scriptstyle e^T(\tau)} \\
\Gamma_{\mathrm{in}}(\mathrm{EU}^D(\tau)) & \longrightarrow & \Gamma_{\mathrm{in}}(\mathrm{EU}^T(\tau))
\end{array}
\tag{8–4}
$$

is approximately $(n/3)$*–cartesian if* $M \simeq F$ *where* $\dim(F) < 2n/3$.

Using 7.3 we can further simplify the lower row in (8–4):

8.3. Lemma. *The homotopy commutative square*

$$
\begin{array}{ccc}
\mathrm{sm}(M,\upsilon) & \longrightarrow & * \\
{\scriptstyle \text{Poincaré dual of } e^D}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{Poincaré dual of } e^T(\tau)} \\
Q(M_+) & \xrightarrow{\;\imath\;} & A^{\%}(M)
\end{array}
$$

is approximately $(n/3)$*–cartesian if* $M \simeq F$ *where* $\dim(F) < 2n/3$.

Returning now to the notation of the introduction, around diagram (0–13), we assume that $p : E \to B$ is a fibration with compact connected B and fibers homotopy equivalent to some compact F. We also fix a vector bundle γ^n on E. If $\dim(F) < 2n/3$, then by 8.3 the diagram

$$
\begin{array}{ccc}
\mathcal{S}^D_n(p,\gamma) & \xrightarrow{\;\subset\;} & \mathcal{S}^T_n(p,\gamma) \\
{\scriptstyle \text{Poincaré dual of } e^D}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{Poincaré dual of } e^T} \\
\Gamma\big(Q_B(E_+) \to B\big) & \xrightarrow{\;\iota\cdot\;} & \Gamma\big(A^{\%}_B(E) \longrightarrow B\big)
\end{array}
\tag{8–5}
$$

is approximately $((n/3) - \dim(B))$–cartesian. Here the vertical maps are defined as follows: Fix $x \in B$. Each ϕ in $\mathcal{S}^T_n(p,\gamma)$ determines a manifold M^n and a homotopy equivalence $M \to F_x$ where F_x is $p^{-1}(x)$. The Poincaré dual of the Euler section of τ^M is an element in $A^{\%}(M)$ which we push forward to $A^{\%}(F_x)$ using $M \to F_x$. This element in $A^{\%}(F_x)$ is the value on x of the section associated to ϕ.

We leave it to the reader to deduce from §7 that (8–5) can be stabilized with respect to n (simultaneously, γ must be stabilized). This amounts to understanding how the Euler section of the tangent bundle of a smooth or topological manifold M determines the Euler section of the tangent bundle of $M \times [0,1]$. Both sections should be regarded as sections vhich vanish near the boundary.

Since $((n/3) - \dim(B))$ tends to infinity with n, the stabilized version of (8–5) is cartesian (is a homotopy pullback square). It is also identical with diagram (0–13).

9. The Lott Challenge

Let $p : E \to B$ be a fibration where B is a connected polyhedron, and the fibers are homotopy equivalent to compact CW spaces. Fix also a bundle V of finitely generated projective left R–modules on E, where R is a ring. Assume that for each

$i \geq 0$, the i–th homology group of each fiber F_x of p with coefficients in V is *nearly f. g. projective* over R, which means: has a finite length resolution by f. g. projective R–modules. This is the situation of 2.2.4, where we obtained a description of the composite map

$$B \xrightarrow{\langle p \rangle} A_B(E) \to A(E) \xrightarrow{\lambda} K(R). \qquad (9\text{–}1)$$

It was found to belong to the homotopy class

$$\sum_i (-1)^i [V_i]$$

where $[V_i]$ classifies the bundle V_i of nearly finitely generated projective R–modules $H_i(F_x; V)$ on B.

Suppose now that p is a smooth fiber bundle with, with smooth *compact* manifold fibers. (See the introduction for terminology.) Then we have another way to understand (9–1), namely, by using formula (0–4) or Theorem 0.1 in the introduction. This tells us that the composition

$$B \xrightarrow{\langle p \rangle} A_B(E) \longrightarrow A(E)$$

is homotopic to the Becker–Gottlieb transfer $\mathrm{tr} : B \to Q(E_+)$ followed by the standard map $\imath : Q(E_+) \to A(E)$. We now make the crucial observation that the composite map from $Q(E_+)$ to $K(R)$, via $A(E)$, is a map of infinite loop spaces. Hence it is determined up to homotopy by its restriction to E. By construction, the restricted map $E \to K(R)$ is the map $[V]$ classifying the bundle V. Briefly, (9–1) is in the homotopy class $\mathrm{tr}^*[V]$.

9.1. Theorem.

$$\mathrm{tr}[V] \; = \; \sum_i (-1)^i [V_i] \, .$$

9.2. Remarks. (i) Assume $R = \mathbb{C}$. There exist characteristic classes c_k in $H^k(K(\mathbb{C}); \mathbb{R})$, for odd k, with the following property. For a flat complex vector bundle V on E, the class $c_k[V]$ is the k–th secondary characteristic class for V defined by Kamber and Tondeur (see [KamTon] and [Du]). Bismut and Lott [BiLo] have used a local version of the Atiyah-Singer index theorem for families of elliptic operators to prove that the equation in 9.1 holds after c_k has been applied to both sides. Lott's challenge was to give a purely topological proof of their result.

To be quite precise, the left–hand side of the Bismut–Lott formula is $\mathrm{tr}^*(c_k[V])$, not $c_k(\mathrm{tr}^*[V])$. Hence one needs to know that each c_k is classified by an infinite loop space map from $K(\mathbb{C})$ to the appropriate rational Eilenberg–MacLane space. This is indeed the case because the classes c_k are *primitive* [BiLo, Prop. 1.13]

(ii) There exist fiber bundles with closed *topological* manifold fibers for which the equation in 9.1 is violated. This is a consequence of Theorem 0.3. We refer to a (short) forthcoming paper for details.

(iii) We could have been a little more generous in 9.1 at no extra cost, by allowing V to be a bundle of *nearly* f.g. projective R–modules on E.

Appendix A: Becker–Gottlieb transfer for smooth fiber bundles

Here we verify that for a smooth fiber bundle $p : E \to B$ with compact fibers, the Becker–Gottlieb transfer agrees with the fiberwise Poincaré dual of the Euler class $[e^D]$ of the tangent bundle along the fibers.

First we must recall the Becker–Gottlieb transfer. This is defined for any fibration $p : E \to B$ where B has the homotopy type of a CW–space and each fiber $F_b = p^{-1}(b)$ has the homotopy type of a compact CW–space. There is no smoothness assumption anywhere, but initially we shall make another assumption which may appear drastic: $B = *$. Thus we are dealing with a single fiber, F. In this case the Becker–Gottlieb transfer is defined as the composition

$$\text{(A–1)}\qquad \begin{array}{ccccc} \boldsymbol{S}^0 & \xrightarrow{\ \eta\ } & F_+ \wedge \boldsymbol{D}(F_+) & \xrightarrow{\Delta\wedge\mathrm{id}} & (F_+ \wedge F_+) \wedge \boldsymbol{D}(F_+) \\ & & & & \big\downarrow\cong \\ & & & & F_+ \wedge (F_+ \wedge \boldsymbol{D}(F_+)) \xrightarrow{\mathrm{id}\wedge\eta^*} F_+ \wedge \boldsymbol{S}^0\,. \end{array}$$

Details: $\boldsymbol{D}(F_+)$ is a finite spectrum which is Spanier–Whitehead 0–dual to the pointed space F_+. Therefore it comes with a map $\eta : \boldsymbol{S}^0 \to F_+ \wedge \boldsymbol{D}(F_+)$ such that for any finite spectrum $\boldsymbol{L}$, slant product with $[\eta]$ is an isomorphism

$$[\boldsymbol{S}^0 \wedge F_+, \boldsymbol{L}] \longrightarrow [\boldsymbol{S}^0 \wedge \boldsymbol{S}^0, \boldsymbol{L} \wedge \boldsymbol{D}(F_+)]$$

where $[...,...]$ denotes homotopy classes. The spectrum $\boldsymbol{J} := F_+ \wedge \boldsymbol{D}(F_+)$ is Spanier–Whitehead *self–dual*. That is, there exists a map of *bispectra*

$$\mu : \boldsymbol{S}^0 \wedge \boldsymbol{S}^0 \longrightarrow \boldsymbol{J} \wedge \boldsymbol{J}$$

such that slant product with μ is an isomorphism

$$[\boldsymbol{J}, \boldsymbol{L}] \longrightarrow [\boldsymbol{S}^0 \wedge \boldsymbol{S}^0, \boldsymbol{L} \wedge \boldsymbol{J}]$$

for any finite spectrum $\boldsymbol{L}$. Namely, μ can be defined as

$$\boldsymbol{S}^0 \wedge \boldsymbol{S}^0 \xrightarrow{\sigma} \boldsymbol{S}^0 \wedge \boldsymbol{S}^0 \xrightarrow{\eta\wedge\eta} \boldsymbol{J} \wedge \boldsymbol{J} \xrightarrow{\sigma'} \boldsymbol{J} \wedge \boldsymbol{J}$$

where the σ' on the right is the anti–map switching the two copies of $D(F_+)$, and the σ on the left is the anti–map switching the two copies of $\boldsymbol{S}^0$. (We call these maps *anti* because they interchange left and right suspension. Both are necessary to ensure that the composition is an honest map of bispectra.)

Since $\boldsymbol{J}$ is self–dual, and $\boldsymbol{S}^0$ is also self–dual, the *transpose* η^* of η is defined as a map from $\boldsymbol{J} = F_+ \wedge \boldsymbol{D}(F_+)$ to $\boldsymbol{S}^0$. We emphasize that η^* is determined up to contractible choice, like everything else so far. To explain this, suppose that $f : \boldsymbol{U} \to \boldsymbol{V}$ is a map between finite spectra. Let

$$\kappa : \boldsymbol{S}^0 \wedge \boldsymbol{S}^0 \longrightarrow \boldsymbol{U} \wedge \boldsymbol{D}(\boldsymbol{U})\,, \qquad \lambda : \boldsymbol{S}^0 \wedge \boldsymbol{S}^0 \longrightarrow \boldsymbol{V} \wedge \boldsymbol{D}(\boldsymbol{V})$$

be Spanier–Whitehead dualities. Choose the transpose f^* together with a homotopy h between the two maps

$$\boldsymbol{S}^0 \wedge \boldsymbol{S}^0 \xrightarrow{\kappa} \boldsymbol{U} \wedge \boldsymbol{D}(\boldsymbol{U}) \xrightarrow{f \wedge \mathrm{id}} \boldsymbol{V} \wedge \boldsymbol{D}(\boldsymbol{U}) ,$$
$$\boldsymbol{S}^0 \wedge \boldsymbol{S}^0 \xrightarrow{\lambda} \boldsymbol{V} \wedge \boldsymbol{D}(\boldsymbol{V}) \xrightarrow{\mathrm{id} \wedge f^*} \boldsymbol{V} \wedge \boldsymbol{D}(\boldsymbol{U}) .$$

Then (f, h) can be interpreted as an element in the homotopy fiber of a certain map between certain spaces of stable maps, and the homotopy fiber is easily seen to be contractible.

So much for the Becker-Gottlieb transfer when B is a point ; when B is not a point, these constructions must be done “fiberwise”. Thus we start with the construction of a fibered spectrum $\boldsymbol{U}$ on B whose fiber $\boldsymbol{U}_b$ over $b \in B$ is Spanier-Whitehead 0–dual to F_b, and equipped with explicit duality maps

$$\eta : \boldsymbol{S}^0 \to (F_b)_+ \wedge \boldsymbol{U} \ , \qquad , \eta^* : (F_b)_+ \wedge \boldsymbol{U} \to \boldsymbol{S}^0 \, .$$

Then (A–1) gives a section of the fibration on B with fibers $Q((F_b)_+)$. This is the Becker–Gottlieb transfer. Existence and essential uniqueness of $\boldsymbol{U}$ is established in [BeGo] and [Cla]. *(End of details.)*

Back to the case of a single fiber F : if F is a compact smooth n–manifold, then we can make “geometric” choices for $\boldsymbol{D}(F_+)$, η and η^*. Embed the pair $(F, \partial F)$ in some $(\mathbb{R}^{k,+}, \mathbb{R}^{k-1})$, with normal bundle ν. Here $\mathbb{R}^{k,+}$ is the closed upper half space, and we assume the embedding is transverse to the boundary $\mathbb{R}^{k-1}$. Let $\boldsymbol{D}(F_+)$ be the k–fold desuspension of $t(\nu)/t(\partial\nu)$ where $t(...)$ denotes Thom spaces and $\partial\nu$ is the restriction of ν to ∂F. Let η be the k–fold desuspension of

$$\mathbb{S}^k \xrightarrow{\text{collapse}} t(\nu)/t(\partial\nu) \xrightarrow{\text{diagonal}} F_+ \wedge (t(\nu)/t(\partial\nu)) \, .$$

Let η^* be the k–fold desuspension of the map

$$\text{(A–2)} \qquad F_+ \wedge (t(\nu)/t(\partial\nu)) \longrightarrow \mathbb{D}^{k,+}/\partial\mathbb{D}^{k,+} \quad ; \quad (x, y) \mapsto (y - x)/\delta \, .$$

To make sense of this formula, assume that the total space (pair) of $(\nu, \partial\nu)$ is embedded in $(\mathbb{R}^{k,+}, \mathbb{R}^{k+1})$ as a tubular neighborhood of F, in such a way that each $x \in F$ has distance $> \delta$ from the complement of the tubular neighborhood. Also, identify the codomain of (A–2) with $\mathbb{S}^k$. And finally, verify that η^* is indeed the transpose of η. (Remember that this amounts to specifying a certain homotopy.)

Let us now take a look at the composition (A–1), with these choices of $\boldsymbol{D}(F_+)$, η and η^*. What we get is the k–fold desuspension of

$$\text{(A–3)} \qquad \mathbb{S}^k \xrightarrow{\text{collapse}} t(\nu)/t(\partial\nu) \xrightarrow{\iota} t(\varepsilon^k)/t(\varepsilon^{k,-})$$

where τ is the tangent bundle, ε^k is a trivial k–dimensional vector bundle on F, $\varepsilon^{k,-}$ is the corresponding trivial bundle of closed lower half spaces on ∂F, and ι is induced by the inclusion of ν in $\tau \oplus \nu \cong \varepsilon^k$. Finally $t(\varepsilon^{k,-})$ is just the one point compactification of the total space of $\varepsilon^{k,-}$. We have to verify that (A–3) is the Poincaré dual of the Euler class.

CAP PRODUCTS. Suppose that we have a pair of parametrized spectra [Be], [ClaPu] over a space F,

$$\begin{aligned} \boldsymbol{U} &= \{r_i : U_i \to F, s_i : F \to U_i, f_i : \Sigma_B U_i \to U_{i+1}\} \\ \boldsymbol{U}' &= \{r_i' : U_i' \to F, s_i' : F \to U_i', f_i' : \Sigma_B U_i' \to U_{i+1}'\}\,. \end{aligned}$$

(The r_i and the r_i' are fibrations.) Suppose $g : F \to U_j'$ represents a cohomology class $[g] \in H^j(F;\boldsymbol{U})$. Define

$$\bar{g} : U_\ell/F \longrightarrow (U_j' \wedge_F U_\ell)/F$$

by $u \mapsto (g \cdot r_\ell(u), u)$ for $u \in U_\ell$. Suppose further that $f : \mathbb{S}^{\ell+i} \to U_\ell/F$ represents a homology class $[f] \in H_i(f;\boldsymbol{U})$. Then $[f] \cap [g]$ is represented by the composition

$$\mathbb{S}^{\ell+i} \xrightarrow{f} U_\ell/F \xrightarrow{\bar{g}} (U_j' \wedge_F U_\ell)/F.$$

POINCARÉ DUALITY. Suppose ξ is a j-dimensional vector bundle over F. Let $\boldsymbol{U}(\xi)$ be the parametrized spectrum over F where $U(\xi)_{j+i}$ is the fiberwise Thom space $t_F(\xi\oplus\varepsilon^i)$. If F^n is a closed, smooth manifold embedded in R^k with normal bundle ν, then the universal fundamental class of F lives in $H_n(F;\boldsymbol{U}(\nu))$ and is represented by the collapse map $c : \mathbb{S}^k \to t_F(\nu)/F = t(\nu)$. The Euler class $[e] = [e^D(\xi)]$ in $H^j(F;\boldsymbol{U}(\xi))$ is represented by the "zero section" map $e : F \to t_F(\xi)$. Thus the Poincare dual of $[e^D(\xi)]$ is represented by the composition

$$\mathbb{S}^k \xrightarrow{c} t(\nu) \xrightarrow{\bar{e}} t(\nu \oplus \xi)$$

where $\bar{e}$ is obtained from the inclusion map of vector bundles $\nu \to \nu \oplus \xi$. More generally, if F is any compact smooth manifold, then c must be written as a map $\mathbb{S}^k \to t(\nu)/t(\partial\nu)$. On the other hand, if ξ has a distinguished nonzero section over ∂F, then the Euler class $[e^D(\xi)]$ is in $H^j(F,\partial F;\boldsymbol{U}(\xi))$. In this case, by a mild generalization of the arguments above, the Poincaré dual of $[e^D(\xi)]$ is represented by a composition of the form

$$\mathbb{S}^k \xrightarrow{c} t(\nu)/t(\partial\nu) \xrightarrow{\bar{e}} t(\nu \oplus \xi)\,.$$

For $\xi = \tau$, the tangent bundle of F, this agrees with (A–3), showing that (A–3) is indeed the Poincaré dual of the Euler class. □

Again, for a smooth fiber bundle $p : E \to B$ with compact fibers, the above argument goes through fiberwise, showing that the fiberwise Poincaré dual of the fiberwise Euler class is the Becker–Gottlieb transfer. We omit the details.

Appendix B: Automatic Continuity

Let $\gamma : \mathcal{C} \to \mathcal{D}$ be a functor between small categories, and let F be a functor from $\mathcal{C}$ to spaces. We ask whether F has an extension to, or factorization thru, $\mathcal{D}$. To be precise, we are willing to replace F by any other functor from $\mathcal{C}$ to spaces which is related to F by a chain of natural weak homotopy equivalences.

For questions of this type one has the theory of Kan extensions, which we now recall briefly. For $D \in \mathcal{D}$ we form the "comma" category $(D \downarrow \gamma)$ whose objects are pairs (C, f) where C is in $\mathcal{C}$ and $f : D \to F(C)$. A morphism from (C_1, f_1) to $C_2, f_2)$ is a morphism from C_1 to C_2 making the appropriate triangle commute. Let P_D be the forgetful functor from $(D \downarrow \gamma)$ to $\mathcal{C}$. Let

$$\operatorname{Kan}^{\gamma} F(D) = \lim F \cdot P_D$$

(inverse limit). Then $\operatorname{Kan}^{\gamma} F$ is a functor on $\mathcal{D}$. There is a natural transformation $(\operatorname{Kan}^{\gamma} F)\gamma \to F$ which has a suitable universal property. $\operatorname{Kan}^{\gamma} F$ is the *right Kan extension of F along γ.*

B.1. Variation. Let $\text{h–Kan}^{\gamma} F(D) = \operatorname{holim} F \cdot P_D$. (Homotopy inverse limits will be recalled in a moment ; see [BK].) This would be the *homotopy right Kan extension of F along γ*. Drawback: there is no obvious natural transformation $(\text{h–Kan}^{\gamma} F)\gamma \to F$. But there are canonical natural transformations

$$(\text{h–Kan}^{\gamma} F)\gamma \to \text{h–Kan}^{\mathrm{id}} F \leftarrow F$$

and the second of these is a natural weak homotopy equivalence.

B.2. Variation. Suppose that $\mathcal{D}$ above is a *topological* category, whereas $\mathcal{C}$ is discrete as before. In more detail, we assume that $\mathcal{D}$ has a discrete set of objets, but the morphism sets are equipped with a topology and composition of morphisms is continuous. Then it is appropriate to replace the homotopy limits above by *topological homotopy limits*. To explain this, we recall homotopy inverse limits. Namely, the homotopy inverse limit of a functor G from $\mathcal{A}$ to spaces is the totalization of the cosimplicial space

$$m \mapsto \prod_{Z:[m]\to\mathcal{A}} GZ(0)$$

where the product is taken over the set of all *contravariant* functors Z from the poset $[m] := \{0, 1, \ldots, m\}$ to $\mathcal{A}$. (At this point we have to make clear what we mean by *spaces*, since one clearly needs a convenient category of spaces which is cartesian closed [MaL]. [BK] use simplicial sets, [HoVo] use Kelley spaces, and we use virtual spaces as in [WW1].) When $\mathcal{A} = (D \downarrow \gamma)$ and $G = FP_D$, then $Z : [m] \to \mathcal{A}$ is equivalent to a pair consisting of a contravariant $Y : [m] \to \mathcal{C}$ and an $f : D \to \gamma Y(m)$, and our cosimplicial set becomes

$$m \quad \mapsto \quad \prod_Y \prod_f FY(0) = \prod_{Y:[m]\to\mathcal{C}} FY(0)^{\operatorname{mor}(D,\gamma Y(m))}\,.$$

But here two interpretations of $FY(0)^{\operatorname{mor}(D,\gamma Y(m))}$ are possible: as the space of maps from $\operatorname{mor}(D, \gamma Y(m))$ with the discrete topology to $FY(0)$, or as the space of continuous maps from $\mathrm{mor}(D, \gamma Y(m))$ to $FY(0)$. Here we use the second interpretation, totalize and get something we call th–Kan$^\gamma F(D)$. There is a canonical and rather forgetful transformation

$$\text{th–Kan}^\gamma F(D) \longrightarrow \text{h–Kan}^\gamma F(D)\,.$$

B.3. Variation. Note that the set of all functors $Y : [m] \to \mathcal{C}$ could be regarded as a category in which the morphisms are the natural *isomorphisms.* The rule

$$Y \mapsto FY(0)^{\mathrm{mor}(D,\gamma Y(m))}$$

could be regarded as a functor on that category. Hence

$$\operatorname*{holim}_{Y:[m]\to\mathcal{C}} FY(0)^{\mathrm{mor}(D,\gamma Y(m))}$$

is defined and maps forgetfully to

$$\prod_{Y:[m]\to\mathcal{C}} FY(0)^{\mathrm{mor}(D,\gamma Y(m))}\,.$$

We now define the *big* topological homotopy Kan extension bth–Kan$^\gamma F(D)$ as the totalization of

$$m \quad\mapsto\quad \operatorname*{holim}_{Y:[m]\to\mathcal{C}} FY(0)^{\mathrm{mor}(D,\gamma Y(m))}\,.$$

B.4. Proposition. *The forgetful map* bth–Kan$^\gamma F(D) \to$ th–Kan$^\gamma F(D)$ *is a homotopy equivalence.*

Proof. From the definition, bth–Kan$^\gamma F(D)$ is the totalization of a bi–cosimplicial set, say $(m, n) \mapsto X_{mn}$. The forgetful map to th–Kan$^\gamma F(D)$ is the forgetful map from the totalization of $X_{\bullet\bullet}$ to that of $X_{\bullet 0}$. Hence it is sufficient to show that the degeneracy $X_{\bullet i} \to X_{\bullet 0}$ is a homotopy equivalence for every $i \geq 0$. However, this is a map between topological homotopy limits induced by a "change of indexing category" which happens to be an equivalence of topological categories. □

Specializing a little more now, suppose that $\gamma : \mathcal{C} \to \mathcal{D}$ is a bijection on objects and on morphism sets, so that $\mathcal{C}$ is simply the discrete category underlying $\mathcal{D}$. We are looking for criteria which ensure that the forgetful map

$$\text{bth–Kan}^\gamma F(D) \to \text{bth–Kan}^{\mathrm{id}} F(D)$$

is a homotopy equivalence for every object D (in $\mathcal{D}$ or in $\mathcal{C}$, which amounts to the same thing). Note that bth–Kan$^\gamma F$ is a continuous functor on $\mathcal{D}$, and that there is a canonical homotopy equivalence from $F(D)$ to bth–Kan$^{\mathrm{id}} F(D)$ for every D. It will be enough to ensure that for each D and m, the forgetful map

$$\operatorname*{holim}_{Y:[m]\to\mathcal{C}} FY(0)^{\mathrm{mor}(D,\gamma Y(m))} \quad\to\quad \operatorname*{holim}_{Y:[m]\to\mathcal{C}} FY(0)^{\delta\mathrm{mor}(D,\gamma Y(m))}$$

is a homotopy equivalence, where δmor denotes the (discrete) morphism sets in $\mathcal{C}$. To understand this condition better, recall that the homotopy inverse limit of a functor from a *groupoid* to spaces can be identified with the space of sections of the projection from the homotopy *direct* limit to the nerve. Thus the forgetful map under investigation takes the form

$$\Gamma\begin{pmatrix}\operatorname*{hocolim}_{Y:[m]\to\mathcal{C}} FY(0)^{\mathrm{mor}(D,\gamma Y(m))} \\ \Big\downarrow \\ \operatorname*{hocolim}_{Y:[m]\to\mathcal{C}} *\end{pmatrix} \longrightarrow \Gamma\begin{pmatrix}\operatorname*{hocolim}_{Y:[m]\to\mathcal{C}} FY(0)^{\delta\mathrm{mor}(D,\gamma Y(m))} \\ \Big\downarrow \\ \operatorname*{hocolim}_{Y:[m]\to\mathcal{C}} *\end{pmatrix}$$

or equivalently

$$\Gamma\begin{pmatrix}\operatorname*{hocolim}_{Y:[m]\to\mathcal{C}} FY(0)\times\mathrm{mor}(D,\gamma Y(m)) \\ \Big\downarrow p \\ \operatorname*{hocolim}_{Y:[m]\to\mathcal{C}} \mathrm{mor}(D,\gamma Y(m))\end{pmatrix} \to \Gamma\begin{pmatrix}\operatorname*{hocolim}_{Y:[m]\to\mathcal{C}} FY(0)\times\delta\mathrm{mor}(D,\gamma Y(m)) \\ \Big\downarrow v^*p \\ \operatorname*{hocolim}_{Y:[m]\to\mathcal{C}} \delta\mathrm{mor}(D,\gamma Y(m))\end{pmatrix}$$

where $v : \mathrm{hocolim}_Y\, \delta\mathrm{mor}(D,\gamma Y(m)) \to \mathrm{hocolim}_Y\, \mathrm{mor}(D,\gamma Y(m))$ is obvious.

B.5. Observation. If the map v is an abelian homology equivalence (explanation follows), if all values of F (hence the fibers of p) are nilpotent spaces, and if u^*p is a trivial bundle for every nullhomologous u from $\mathbb{S}^1$ to the base space of p, then the map $\Gamma(p) \to \Gamma(v^*p)$ is a homotopy equivalence.

The proof is by obstruction theory. A map $g : X_1 \to X_2$ between pointed connected spaces is an *abelian homology equivalence* [McD1] if it induces isomorphisms in homology $H_*(X_1;\Lambda) \to H_*(X_2;\Lambda)$ for any module Λ over $\mathbb{Z}[H_1(X_2)]$.

B.6. Corollary. *If the conditions in B.5 are satisfied for every D and $m \geq 0$, then for every D the canonical map*

$$\text{th–Kan}^{\gamma} F(D) \to \text{th–Kan}^{\mathrm{id}} F(D)$$

is a homotopy equivalence.

Note: there is another canonical map $F(D) \to \text{th–Kan}^{\mathrm{id}} F(D)$ which is always a homotopy equivalence, without any conditions. Note also that

$$\text{th–Kan}^{\gamma} F$$

is a functor from $\mathcal{D}$ to spaces. Hence, if the conditions in B.5 are satisfied, our functor F does have a factorization through $\mathcal{D}$, up to equivalence.

B.7. Example. We shall use the abbreviations $Q(i) = \mathrm{TOP}(i)$, $P(i) = \mathrm{TOP}^{\delta}(i)$. Let $\mathcal{D}$ be the topological category with objects $0, 1, 2, \ldots$ where $\mathrm{mor}(i,j)$ is the group–theoretic quotient $Q(j)/Q(j-i)$. In other words, a morphism $i \to j$ is an equivalence class of homeomorphisms $\mathbb{R}^i \times \mathbb{R}^{j-i} \to \mathbb{R}^j$, two such being equivalent if they are in the same orbit under the action of $Q(j-i)$. Warning: A morphism $i \to j$ does not determine an embedding $\mathbb{R}^m \to \mathbb{R}^n$ (because elements in $Q(j-i)$ are not required to preserve the origin of $\mathbb{R}^{j-i}$), but it does determine a fiber bundle projection $\mathbb{R}^j \to \mathbb{R}^i$.

Let $\mathcal{C}$ be the underlying discrete category. One of the conditions in B.5 ("the map v is an abelian homology equivalence") does not depend on any functor, so we can verify it right away. Fix an object i in $\mathcal{C}$ (we should call it D to be consistent) and fix $m \geq 0$. Each isomorphism class of contravariant functors $Y : [m] \to \mathcal{C}$ has a preferred representative

$$j_m \to j_{m-1} \to \cdots \to j_0$$

(a string in $\mathcal{C}$) where $j_0 \geq j_1 \geq \cdots \geq j_m$ and the morphism $j_k \to j_{k-1}$ is the standard one (trivial coset). The automorphism group of this representative is

$$P(j_m) \times \prod_{k=1}^{m} P(j_{k-1} - j_k)\,.$$

It acts on

$$\mathrm{mor}(i, j_m) = Q(j_m)/Q(j_m - i)$$

via projection to $P(j_m)$, and then translation of cosets. Therefore domain and codomain of v in B.5 are homotopy equivalent to disjoint unions of pieces of the form

$$P(j_m)\backslash\!\!\backslash P(j_m)/P(j_m - i) \times \prod_{k=1}^{m} BP(j_{k-1} - j_k)\ ,$$
$$P(j_m)\backslash\!\!\backslash Q(j_m)/Q(j_m - i) \times \prod_{k=1}^{m} BP(j_{k-1} - j_k)$$

respectively, where $\backslash\!\!\backslash$ denotes a homotopy orbit construction. (Thus $G\backslash\!\!\backslash X$ is the homotopy orbit space of a left action of the group G on the space X.) Parentheses have been deliberately omitted to suggest associativity ; this is easily justified and we therefore obtain a commutative diagram

$$\begin{array}{ccc}
P(j_m)\backslash\!\!\backslash P(j_m) & \longrightarrow & P(j_m)\backslash\!\!\backslash Q(j_m) \\
\downarrow & & \downarrow \\
P(j_m)\backslash\!\!\backslash P(j_m)/P(j_m - i) & \longrightarrow & P(j_m)\backslash\!\!\backslash Q(j_m)/Q(j_m - i) \\
\downarrow & & \downarrow \\
BP(j_m - i) & \longrightarrow & BQ(j_m - i)
\end{array}$$

where the columns are fibration sequences up to homotopy. The lower horizontal arrow is a homology equivalence by [McD1]. The space $P(j_m)\backslash\!\!\backslash P(j_m)$ is contractible, and $P(j_m)\backslash\!\!\backslash Q(j_m)$ is acyclic, again by [McD1]. Therefore the middle horizontal arrow is an abelian homology equivalence. Therefore v is an abelian homology equivalence.

B.8. More of the same. With $\mathcal{C}$ and $\mathcal{D}$ as in B.7, let F from $\mathcal{C}$ to spaces be the functor $F(i) = \boldsymbol{A}^{\%}(\mathbb{R}^i)$. Induced morphisms are defined as follows: We have seen that a morphism $i \to j$ in $\mathcal{C}$ determines a projection $\mathbb{R}^j \to \mathbb{R}^i$, and one uses pullback with this to define an exact functor from the category of retractive spaces over $\mathbb{R}^i \times [0,1)$ to the category of retractive spaces over $\mathbb{R}^j \times [0,1)$. The exact functor induces $F(i) \to F(j)$. For us this is the "correct" map because it takes the microcharacteristic of $\mathbb{R}^i$ to that of $\mathbb{R}^j$.

We now check that the remaining conditions in B.5, those which depend on a functor, are satisfied for this particular functor. Clearly all values of F are nilpotent, since they are infinite loop spaces. Fix i and $m \geq 0$ as in B.6, and recall that the codomain of the map v discussed in B.6 is homotopy equivalent to a disjoint union of pieces of the form

$$X \; = \; P(j_m)\backslash\!\backslash Q(j_m)/Q(j_m - i) \times \prod_{k=1}^{m} \cdots .$$

Note that X comes with a canonical principal $P(j_m)$–bundle, say ζ. This is relevant here because the fiber bundle p under investigation (notation of B.5) is simply the bundle with structure group $P(j_m)$ and fiber $F(j_m)$ associated to ζ. Elements in the commutator subgroup of $P(j_m)$ act by automorphisms of $F(j_m)$ which are homotopic to the identity. This follows easily from §5, especially 5.1 and 5.2. Therefore all conditions are satisfied. □

B.9. Uniqueness. Suppose that F from $\mathcal{C}$ to spaces is of the form $\bar{F}\gamma$, where $\bar{F}$ is from $\mathcal{D}$ to spaces. In other words, the factorization problem is already solved. Then we would like to know that the "artificial" factorization or extension th–Kan$^\gamma F$ agrees with $\bar{F}$ up to equivalence. In fact there is a natural transformation $\bar{F} \to$ th–Kan$^\gamma F$, obvious from the definition of th–Kan$^\gamma F$, making the following diagram of natural transformations commutative:

$$\begin{array}{ccc} \bar{F}\gamma & \longrightarrow & (\text{th–Kan}^{\gamma} F)\gamma \\ \big\downarrow{=} & & \big\downarrow \\ F & \longrightarrow & \text{th–Kan}^{\text{id}} F\,. \end{array}$$

In the situation of B.6, the diagram shows that $\bar{F}(D) \longrightarrow$ th–Kan$^\gamma \bar{F}$ is a homotopy equivalence for every D in $\mathcal{D}$.

Appendix C: Orthogonal Calculus and Euler Systems

The goal is to prove theorem 7.2.1. A certain familiarity with [We] will be assumed. $\mathcal{J}$ denotes the category of finite dimensional real vector spaces $U, V, W, \ldots$ with inner product (as in 7.1, sequel of 7.1.1) and $\mathcal{E}$ denotes the category of continuous functors from $\mathcal{J}$ to spaces. "Spaces" are assumed to be homotopy equivalent to CW–spaces ; if this raises questions, see the appendix to [We]. A morphism $F \to E$ in $\mathcal{E}$ is an *equivalence* if $F(V) \to E(V)$ is a homotopy equivalence for all V in $\mathcal{J}$. Objects in $\mathcal{E}$ are *equivalent* if they can be related by a chain of equivalences. We use letters E, F but also X, Y, Z for objects in $\mathcal{E}$. Recall now [We, Def. 5.1] which we repeat here for convenience:

C.1. Definition. An object E in $\mathcal{E}$ is *polynomial of degree* $\leq n$ if

$$\rho : E(V) \longrightarrow \operatorname*{holim}_{0 \neq U \subset \mathbb{R}^{n+1}} E(U \times V)$$

is a homotopy equivalence for all V.

Beware that the indexing category for the homotopy limit is a topological category (a partially ordered *space* where the order relation is closed) and the homotopy limit is really a *topological homotopy limit.* Compare Appendix B, but see [We, §5] for details. Here we need definition C.1 only when $n = 0$ or $n = 1$. A functor which is polynomial of degree 0 is of course equivalent to a constant functor.

C.2. Definition. An object E in $\mathcal{E}$ is *good* if $\lim_n b_n - n = \infty$, where b_n is the connectivity of

$$\rho : E(V) \longrightarrow \operatorname*{holim}_{0 \neq U \subset \mathbb{R}^{2}} E(U \times V)$$

for n–dimensional V.

In particular, an E which is polynomial of degree ≤ 1 is good. However, goodness is a much more "generic" property. By [We3] there is a fibration sequence up to homotopy

$$E^{(i+1)}(V) \xrightarrow{u} E(V) \xrightarrow{\rho} \operatorname*{holim}_{0 \neq U \subset \mathbb{R}^{i+1}} E(U \times V)$$

and another, from the introduction of [We],

$$E^{(i+1)}(V) \to E^{(i)}(V) \to \Omega^i E^{(i)}(\mathbb{R} \times V)$$

and $E^{(0)} = E$. In particular $E^{(1)}(V)$ is the homotopy fiber of $E(V) \to E(\mathbb{R} \times V)$, and $E^{(2)}(V)$ is the homotopy fiber of a certain stabilization map $V^c \to \Omega(\mathbb{R} \times V)^c$. Hence, if the connectivity of this stabilization map exceeds $\dim(V)$ by a quantity which tends to infinity with $\dim(V)$, then E is good.

C.3. Example. Let $E(V) = B\,\mathrm{O}(V)$. Then E is good. Indeed, the stabilization map turns out to be the usual one. It is $(2n-1)$–connected by Freudenthal. Therefore $b_n \geq 2n - 1$.

C.4. Example. Let $Z(V) = B\,\mathrm{TOP}(V)$. Here, instead of Freudenthal's theorem, we have the estimate due to [Ig]: the connectivity of the stabilization map

$$\mathrm{TOP}(\mathbb{R} \times V)/\,\mathrm{TOP}(V) \longrightarrow \Omega\,(\mathrm{TOP}(\mathbb{R} \times V \times \mathbb{R})/\,\mathrm{TOP}(V \times \mathbb{R}))$$

is $\geq 4\dim(V)/3 - k$ for a (small) constant k. Therefore $b_n \geq 4n/3 - k$ in this case, and we conclude that Z is good.

Recall further that each E in $\mathcal{E}$ has a "best approximation" from the right by an object which is polynomial of degree $\leq n$:

$$\eta_n : E \longrightarrow T_n E\,.$$

This is the n–th Taylor approximation [We,§6]. It is functorial in E. The objects in $\mathcal{E}$ which are polynomial of degree $\leq n$ can be characterized as those objects for which η_n is an equivalence. This motivates [We, Def. 8.1] which we also recall, even though we shall use it for $n \leq 1$ only.

C.5. Definition. A morphism $g : E \to F$ in E is *polynomial of degree* $\leq n$ if

$$\begin{array}{ccc} E & \xrightarrow{g} & F \\ \downarrow{\scriptstyle \eta_n} & & \downarrow{\scriptstyle \eta_n} \\ T_n E & \xrightarrow{T_n(g)} & T_n F \end{array}$$

is a homotopy pullback square.

It is not hard to see that T_n respects fibration sequences up to homotopy, so that we have the following criterion.

C.6. Observation. *Suppose that E in C.5 is a continuous functor from $\mathcal{J}$ to pointed connected spaces. Then g is polynomial of degree $\leq n$ if and only if the functor ϕ defined by*

$$\phi(V) := \text{hofiber}\,[g : F(V) \to E(V)]$$

is polynomial of degree $\leq n$.

There is a very satisfactory classification theory for objects and morphisms in $\mathcal{E}$ which are polynomial of degree $\leq n$, particularly when $n = 1$. See [We, §8,§9, §10]. Hence the following result is an important ingredient in the proof of 7.2.1.

C.7. Lemma. *Let $Z(V) = B\,\mathrm{TOP}(V)$. If $(Y, Z^\sharp, \dots)$ is an Euler system on Z satisfying the hypotheses in 7.2.1, then $p : Y \to Z$ is polynomial of degree ≤ 1.*

Proof. From the commutative square in 7.2.1, we see that the goodness of Z established in C.4 implies goodness of the homotopy fiber of ζ, and hence goodness of ϕ, the homotopy fiber of $p : Y \to Z$. On the other hand, stability of the Euler system gives a chain of natural homotopy equivalences

$$\phi(V\times\mathbb{R}) \simeq \cdots \simeq \Omega^k \phi(V\times\mathbb{R}^{k+1})\,.$$

This shows that the connectivity $b_n = b_n(\phi)$ in C.2 satisfies $b_n \geq b_{n+k} - k$. Since also $\lim_n b_n - n = \infty$, we must have $b_n = \infty$ for all n. $\square$

Using the functoriality of the construction T_1, we can now reduce 7.2.1 to the following statement. (The reduction will be explained in detail below.)

C.8. Proposition. *Let Z in $\mathcal{E}$ be polynomial of degree ≤ 1. Up to equivalence relative to Z, there exists a unique stable Euler system on Z such that the following is a homotopy pullback square for all V in $\mathcal{J}$, and Y is polynomial of degree ≤ 1:*

$$\begin{array}{ccc} Z(V) & \xrightarrow{\imath\cdot e} & Z^\sharp(V\times\mathbb{R}) \\ \downarrow & & {\scriptstyle \zeta}\downarrow \\ Z(V\times\mathbb{R}) & \xrightarrow{e} & Y(V\times\mathbb{R})\,. \end{array}$$

Reduction. With $Z, Y, Z^\sharp$ as in 7.2.1 let $\bar{Z} = T_1 Z$, $\bar{Y} = T_1 Y$ and $\bar{Z}^\sharp = T_1 Z^\sharp$. Since T_1 respects homotopy pullback squares (also easy from the definition) we see that $\bar{Z}$, $\bar{Y}$, $\bar{Z}^\sharp$ etc. form a *stable* Euler system. (It is necessary to know, and easy to verify, that the endofunctor $T_1 : \mathcal{E} \to \mathcal{E}$ commutes with the endofunctor $\lambda : \mathcal{E} \to \mathcal{E}$ given by $\lambda E(V) = E(V \times \mathbb{R})$.) The square

$$\begin{array}{ccc} \bar{Z}(V) & \xrightarrow{\bar{\imath}\cdot\bar{e}} & \bar{Z}^\sharp(V\times\mathbb{R}) \\ \big\downarrow & & \big\downarrow{\scriptstyle\bar{\zeta}} \\ \bar{Z}(V\times\mathbb{R}) & \xrightarrow{\bar{e}} & \bar{Y}(V\times\mathbb{R})\,. \end{array}$$

is a homotopy pullback square because the map of vertical homotopy fibers which it determines is obtained from the map of vertical homotopy fibers in the square of 2.2.1 by applying T_1 ; but the latter is an *approximation of order* 1 [We, 5.16], so that T_1 of it is an equivalence [We 5.15]. Thus the Euler system $\bar{Z}$, $\bar{Y}$, $\bar{Z}^\sharp$ satisfies the hypotheses of C.8. Hence, if C.8 holds, it is unique up to suitable equivalence.

To complete the reduction we must verify that Y, $Z^\sharp$ etc. can be recovered from (are determined by) $\bar{Y}$, $\bar{Z}^\sharp$ etc. and the morphism $\eta_1 : Z \to \bar{Z}$. In fact by C.7 we can recover $Y(V)$ as the homotopy pullback of

$$Z(V) \xrightarrow{\eta_1} \bar{Z}(V) \xleftarrow{\bar{p}} \bar{Y}(V)$$

and we can recover $Z^\sharp(V\times\mathbb{R})$ as the homotopy pullback of

$$Z(V\times\mathbb{R}) \xrightarrow{\eta_1} \bar{Z}(V\times\mathbb{R}) \xleftarrow{\bar{p}\cdot\bar{\zeta}} \bar{Z}^\sharp(V\times\mathbb{R})\,.$$

In the process we have recovered p and ζ as well. Recovery of e and $\imath$ from $\bar{e}$ and $\bar{\imath}$ is also easy. □

Let $\mathcal{E}(1) \subset \mathcal{E}$ be the full subcategory consisting of all objects of degree ≤ 1. According to [We], the following category $\mathcal{X}$ is a good "substitute" for $\mathcal{E}(1)$.

An object of $\mathcal{X}$ is a triple $(B, \mathbf{\Theta}, s)$ where B is a space, $\mathbf{\Theta}$ is a parametrized Ω–spectrum [Be], [ClaPu], with involution over B, and s is a possibly nontrivial section of $\Omega^\infty_B(\mathbf{\Theta}_{h\mathbb{Z}/2})$, a retractive space over B. Further details can be found below in C.13. Morphisms in $\mathcal{X}$ are defined in the most obvious way. A morphism $(B, \mathbf{\Theta}, s) \to (B', \mathbf{\Theta}', s')$ is an *eqivalence* if the underlying map $f : B \to B'$ is a homotopy equivalence, and the resulting map $\mathbf{\Theta} \to f^*\mathbf{\Theta}'$ is a homotopy equivalence between parametrized Ω–spectra over B. Now the "substitute" statement is as follows. There exist functors $\alpha : \mathcal{E}(1) \to \mathcal{X}$ and $\beta : \mathcal{X} \to \mathcal{E}(1)$ such that $\beta\alpha$ and $\alpha\beta$ can be related to the respective identity functors by chains of natural equivalences. We describe β only. Given $(B, \mathbf{\Theta}, s)$ in $\mathcal{X}$ let $\beta(B, \mathbf{\Theta}, s)$ be the functor taking V in $\mathcal{J}$ to the homotopy pullback of

$$B \xrightarrow{\text{zero}} \Omega^\infty_B\left((V^c_\rho \wedge_B \mathbf{\Theta})_{h\mathbb{Z}/2}\right) \xleftarrow{js} B$$

where V^c_ρ is the one point compactification V^c equipped with the involution $-\mathrm{id}$, the (fiberwise) smash product $V^c_\rho \wedge_B \mathbf{\Theta}$ has the diagonal involution, and j from

$\Omega_B^\infty\big((0^c \wedge_B \mathbf{\Theta}_{h\mathbb{Z}/2})\big)$ to $\Omega_B^\infty\big((V_\rho^c \wedge_B \mathbf{\Theta})_{h\mathbb{Z}/2}\big)$ is the inclusion (and of course $0^c \cong \mathbb{S}^0$).

We can say that C.8 is a statement about Euler systems in $\mathcal{E}(1)$. Under the above correpondence $\mathcal{E}(1) \longleftrightarrow \mathcal{X}$, Euler systems in $\mathcal{E}(1)$ correspond to what we should call Euler systems in $\mathcal{X}$, and we would like to spell out what these Euler systems in $\mathcal{X}$ are. Here a crucial observation is that the functor $\lambda : \mathcal{E}(1) \to \mathcal{E}(1)$ given by $(\lambda E)(V) = E(V \times \mathbb{R})$ corresponds to the functor

$$\mu : \mathcal{X} \to \mathcal{X} \quad ; \quad (B, \mathbf{\Theta}, s) \mapsto (B, \mathbb{S}^1_\rho \wedge_B \mathbf{\Theta}, js)$$

where $\mathbb{S}^1_\rho$ is $\mathbb{S}^1$ with the reflection involution ρ having fixed point set $\mathbb{S}^0$, and where j is the usual inclusion $\mathbb{S}^0 \wedge_B \mathbf{\Theta} \to \mathbb{S}^1_\rho \wedge_B \mathbf{\Theta}$. On $\mathbb{S}^1_\rho \wedge \mathbf{\Theta}$ we use the diagonal involution.

Another crucial observation is that the functor $(B, \mathbf{\Theta}, s) \mapsto B$ on $\mathcal{X}$ corresponds to the functor $E \mapsto E(\mathbb{R}^\infty)$ on $\mathcal{E}(1)$, where $E(\mathbb{R}^\infty = \mathrm{hocolim}_n\, E(\mathbb{R}^n)$. In a situation such as C.8, all functors in sight (from $\mathcal{J}$ to spaces) have the same value at $\mathbb{R}^\infty$, up to canonical homotopy equivalences ; therefore we may keep our B fixed. We arrive at the following.

C.9. First reformulation. An Euler system on an object $(B, \mathbf{\Theta}, s)$ in $\mathcal{X}$ consists of a map $p : \mathbf{\Phi} \to \mathbf{\Theta}$ of parametrized spectra with involution over B, with a section $e : \mathbf{\Theta} \to \mathbf{\Phi}$ respecting the involutions, and a factorization

$$\mathbf{\Phi} \xrightarrow{\imath} \Psi \xrightarrow{\zeta} \mathbb{S}^1_\rho \wedge_B \mathbf{\Phi}$$

of the inclusion $\mathbf{\Phi} \to \mathbb{S}^1_\rho \wedge_B \mathbf{\Phi}$. In this factorization, $\mathbf{\Psi}$ is another parametrized Ω–spectrum over B with involution, and the maps respect involutions. Condition: $(\mathbb{S}^1_\rho \wedge_B p)\zeta$ from $\mathbf{\Psi}$ to $\mathbb{S}^1_\rho \wedge_B \mathbf{\Theta}$ is a homotopy equivalence. The Euler system is *stable* if, in addition,

$$\begin{array}{ccc} \mathbf{\Phi} & \xrightarrow{\ \imath\ } & \mathbf{\Psi} \\ {\scriptstyle \imath}\big\downarrow & & \big\downarrow{\scriptstyle \zeta} \\ \mathbf{\Psi} & \xrightarrow{(\rho\wedge\mathrm{id})\zeta} & \mathbb{S}^1 \wedge_B \mathbf{\Phi} \end{array}$$

is a homotopy pullback square. Here $\rho : \mathbb{S}^1 \to \mathbb{S}^1$ is still the usual reflection with fixed point set $\mathbb{S}^0$.

Remark. The pair $(B, \mathbf{\Phi})$ can be completed in a unique way to an object $(B, \mathbf{\Phi}, s')$ in $\mathcal{X}$ such that $e : \mathbf{\Theta} \to \mathbf{\Phi}$ together with $\mathrm{id} : B \to B$ is a morphism in $\mathcal{X}$ from $(B, \mathbf{\Theta}, s)$ to $(B, \mathbf{\Phi}, s')$. Similarly $(B, \mathbf{\Psi})$ can be uniquely completed to an object $(B, \mathbf{\Psi}, s'')$ such that $\imath e$ is a morphism. For this reason s' and s'' do not appear explicitly in C.9, and as a result s is also redundant. This should explain how an Euler system as in C.9 gives rise to an Euler system as in C.8.

Note that the endofunctor $\mathbf{\Theta} \mapsto \mathbb{S}^1_\rho \wedge_B \mathbf{\Theta}$ on the category of parametrized spectra with involution over B has an inverse up to equivalence. By *equivalence* we mean here an equivariant map which is an ordinary homotopy equivalence. This means that the zero section ζ in C.9 can be regarded as an inverse (up to suitable equivalence) of $p : \mathbf{\Phi} \to \mathbf{\Theta}$. We use this to split $\mathbf{\Phi}$, up to equivalence, discard the summand $\mathbf{\Theta}$, call the other summand Φ', and obtain the following drastic simplification of C.9.

C.10. Second reformulation. An Euler system on $(B, \mathbf{\Theta}, s)$ consists of a map

$$e' : \mathbf{\Theta} \to \mathbf{\Phi}'$$

between parametrized Ω–spectra with involution over B, and a factorization

$$\mathbf{\Phi}' \longrightarrow \mathbf{\Psi}' \longrightarrow \mathbb{S}^1_\rho \wedge_B \mathbf{\Phi}'$$

of the inclusion $\mathbf{\Phi}' \to \mathbb{S}^1 \wedge_B \mathbf{\Phi}'$. In this factorization, $\mathbf{\Psi}'$ is another parametrized spectrum over B with involution, and the maps respect involutions. Condition: $\mathbf{\Psi}'$ is contractible. The Euler system is *stable* if

$$\begin{array}{ccc} \mathbf{\Phi}' & \xrightarrow{\;\imath\;} & \mathbf{\Psi}' \\ {\scriptstyle \imath}\big\downarrow & & \big\downarrow{\scriptstyle \zeta} \\ \mathbf{\Psi}' & \xrightarrow{\;\rho_*\zeta\;} & \mathbb{S}^1_\rho \wedge_B \mathbf{\Phi}' \end{array}$$

is a homotopy pullback square.

Remark. It is also important to reformulate the notion of *equivalence*. In the setting of C.10, it is clear what a morphism between Euler systems on $(B, \mathbf{\Theta}, s)$ is ; such a morphism, say from $e' : \mathbf{\Theta} \to \mathbf{\Phi}'$ etc. to $e'' : \mathbf{\Theta} \to \mathbf{\Phi}''$ etc., is an *equivalence* if the underlying (equivariant) map from $\mathbf{\Phi}'$ to $\mathbf{\Phi}''$ is an ordinary homotopy equivalence. Since we are interested in a classification up to (specified) equivalence, it follows that we lose nothing by smashing $\mathbf{\Theta}$ and $\mathbf{\Phi}'$ fiberwise with $E\mathbb{Z}/2_+$. Then the involutions are sufficiently free so that the equivariant factorization through a contractible object may be replaced by an equivariant nullhomotopy.

Remark. Note that C.10 implicitly mentions *two* commuting involutions on $\mathbb{S}^1 \wedge \mathbf{\Phi}'$, one that acts on both $\mathbb{S}^1$ and $\mathbf{\Phi}'$, and another which appears under the name ρ_* in the commutative square and only acts on the first factor.

These remarks lead to the next reformulation.

C.11. Third reformulation. A stable Euler system on $(B, \mathbf{\Theta}, s)$ consists of a map

$$e' : \mathbf{\Theta} \to \mathbf{\Phi}'$$

between parametrized Ω–spectra with involution over B, and a homotopy equivalence

$$\kappa : \mathbb{S}^1 \wedge_B \mathbf{\Phi}' \longrightarrow \mathbb{S}^1 \wedge_B \mathbf{\Phi}'$$

which is a $\mathbb{Z}/2{\times}\mathbb{Z}/2$ map for the following actions. On the domain $(a, b) \in \mathbb{Z}/2{\times}\mathbb{Z}/2$ acts by $\rho^a \wedge \tau^b$ where τ is the involution on $\mathbf{\Phi}'$, and on the codomain (a, b) acts by $\rho^{a+b} \wedge \tau^b$.

The assumptions on $\mathbf{\Phi}'$ in C.11 imply that $\mathbf{\Phi}'$ is *co–induced*: there exist a parametrized Ω–spectrum $\mathbf{\Lambda}$ over B, without involution, and a map $f : \mathbf{\Phi}' \to \mathbf{\Lambda}$ such that $(f, f\tau)$ from $\mathbf{\Phi}'$ to $\mathbf{\Lambda} \times_B \mathbf{\Lambda}$ is a homotopy equivalence. Prove this by thinking of κ as a map from $(\mathbb{S}^1/\mathbb{S}^0) \wedge_B \mathbf{\Phi}'$ to itself, and then taking homotopy orbits for the actions of the diagonal subgroup of $\mathbb{Z}/2\times\mathbb{Z}/2$; this shows that $\mathbb{S}^1 \wedge_B \mathbf{\Phi}'$ is co–induced, and it follows that $\mathbf{\Phi}'$ itself is co–induced.

Conversely, if we only have $f : \mathbf{\Phi}' \to \mathbf{\Lambda}$ with the above properties, then we can recover κ. Consequently nothing is lost by composing e' in C.11 with f, and so we get our final reformulation.

C.12. Fourth and last reformulation. A stable Euler system on $(B, \mathbf{\Theta}, s)$ is a map $\mathbf{\Theta} \to \Lambda$, where Λ is another parametrized Ω–spectrum over B, without involution.

Recall now the extra condition in C.8, the one which requires that a certain square be a homotopy pullback square. In the language and notation of C.12, the corresponding requirement is that $\mathbf{\Theta} \to \Lambda$ be a homotopy equivalence. The details of this translation are left to the reader. Clearly, up to equivalence, there is only one stable Euler system on $(B, \mathbf{\Theta}, s)$ satisfying the extra condition. This completes the proof of C.8, and that of 7.2.1. □

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