POINCARÉ DUALITY AND STEINBERG'S THEOREM ON RINGS OF COINVARIANTS

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1. Introduction

Let k be a field, possibly of finite characteristic, V a finite dimensional vector space over k of dimension r, and $W \subset \mathrm{Aut}_k(V)$ a finite subgroup. There is a natural action of W on the k-dual $V^{\#}$ of V, as well as on the symmetric algebra $S = S(V^{\#})$. The algebra S is isomorphic to a polynomial algebra over k on r generators, and we are interested in the question of when the invariant algebra $R = S^W$ is also isomorphic to such a polynomial algebra. It is well–known (see for instance Serre [7]) that R is polynomial if W is generated by reflections and the characteristic of k is zero or prime to the order of W. In a slightly different direction, Steinberg [9] has shown that R is polynomial if k is the field of complex numbers and the quotient algebra $P_* = S \otimes_R k$ satisfies Poincaré duality (1.3). Steinberg's result was extended by Kane [3, 4] to other fields of characteristic zero, and by T.-C. Lin [5] to the case in which k is a finite field of characteristic prime to the order of W.

In this note we use elementary methods to prove Steinberg's result for fields of characteristic 0 or of characteristic prime to the order of W. This gives a new proof even in the characteristic zero case.

1.1. **Theorem.** *Let k be a field, V an r-dimensional k-vector space, and W a finite subgroup of $Aut_k(V)$. Let $S = S[V^{\#}]$ be the symmetric algebra on $V^{\#}$, and $R = S^W$ the ring of invariants of the natural action of W on S. Define P_* to be the quotient algebra $S \otimes_R k$. If the characteristic of k is zero or prime to the order of W and P_* satisfies Poincaré duality, then R is isomorphic to a polynomial algebra on r generators.*

In the context of 1.1, P_* is sometimes called the *ring of coinvariants* of W. This is a bit odd, since W acts on P_* in a way which is in general nontrivial (2.1) but the terminology is well–established [8].

1.2. *Remark.* We will follow commutative algebra conventions with respect to gradings and signs. Except in 4.4, the ring S is to be treated as

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a graded algebra in which the elements of $V^{\#}$ are given grading one; S is a strictly commutative algebra. (We do not consider algebras which are graded commutative in the sense of topology.) The invariant ring $R = S^W$ inherits a grading from S, and the homomorphism $R \to k$ which figures implicitly in the formula for P_* is the unique homomorphism which sends to zero all the elements in R of strictly positive degree.

1.3. *Remark.* Since the homomorphisms $R \to S$ and $R \to k$ preserve degree (where k has gradation zero), $P_* = S \otimes_R k$ has a natural grading. It is easy to see that P_* is finite dimensional over k; this amounts to observing that S is finitely generated as a module over R, which follows from the fact that S is integral over R and is finitely generated as an algebra. The statement that P_* *satisfies Poincaré duality* (of dimension n) then means that there is a number n such that

- $P_i = 0$ for $i > n$,
- $\dim_k\ P_n = 1$, and
- for $0 \le i \le n$, the product map $P_i \otimes_k P_{n-i} \to P_n$ is a nonsingular pairing.

Let R_+ denote the ideal in R generated by elements of strictly positive degree. In a direction converse to 1.1, it is well known that if R is a polynomial algebra, then $I = SR_+$ is a complete intersection ideal, so that $P_* = S/I$ is Gorenstein and hence (since P_* is finite-dimensional over k) satisfies Poincaré duality. See for example the books of Bruns-Herzog [1] or Kane [4].

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2. Some one dimensional $k[W]$ submodules of P_*

Since $|W|$ is a unit in k, the group algebra $k[W]$ is semisimple, and hence up to isomorphism there are only a finite number of irreducible $k[W]$–modules. Moreover, up to isomorphism each $k[W]$–module is a direct sum of these irreducible modules.

2.1. **Lemma.** *The trivial one dimensional module 1_W occurs as a summand in P_* only as P_0.*

Proof. Note that as $k[W]$–modules, both S and P_* have the averaging operator

$$AV(x) = |W|^{-1} \sum_{w \in W} w(x).$$

The projection map $S \to P_*$ commutes with AV, and so if $y \in P_s$ is W-invariant with preimage $y' \in S$, $AV(y')$ projects to $AV(y) = y$. If y' has degree $s > 0$, $AV(y')$ belongs to R_+, which implies that $AV(y')$ projects to zero in $P_* = S/R_+S$. □

2.2**. Proposition.** *Suppose that $P_* = S \otimes_R k$ satisfies Poincaré duality (1.3). Let $\rho = P_n$ as a $k[W]$-module. Then ρ is a one dimensional irreducible module and occurs as a summand of P_* only as P_n.*

Proof. The dimension and irreducibility of ρ are clear. The multiplication maps

$$P_i \otimes_k P_{n-i} \to P_n = \rho$$

are W-equivariant (where W acts diagonally on the tensor product) and so induce W-isomorphisms

$$P_i \cong P_{n-i}^{\#} \otimes_k \rho\,.$$

If P_{n-i} contains ρ as a summand, then P_i contains $\rho^{\#} \otimes_k \rho \cong 1_W$. By 2.1, this can occur only if $i = 0$, i.e., $n - i = n$. □

3. Proof of 1.1

We assume that P_* satisfies Poincaré duality of dimension n (1.3) and that $|W|$ is a unit in k. We first recall an elementary lemma.

3.1**. Lemma.** *Let T be a graded k–algebra which is concentrated in degrees ≥ 0 and isomorphic in degree 0 to k. Let $T \to k$ be the homomorphism which sends all elements of positive degree to 0, and M and N two T–modules which themselves are concentrated in degrees ≥ 0 (more generally, are bounded below). Then*

(1) *$M \cong 0$ if and only if $M \otimes_T k \cong 0$,*
(2) *a homomorphism $M \to N$ is surjective if and only if $M \otimes_T k \to N \otimes_T k$ is surjective, and*
(3) *if N is free, a homomorphism $M \to N$ is an isomorphism if and only if $M \otimes_T k \to N \otimes_T k$ is an isomorphism.*

Proof. Part (1) is clear. Part (2) then follows from an application of (1) to the cokernel of $M \to N$. For (3), assume that $M \otimes_T k \to N \otimes_T k$ is surjective. By (2), $M \to N$ is surjective; since N is free, the surjection can be split and M can be written as $N \oplus K$. Clearly $K \otimes_T k \cong 0$, and hence by (1), $K \cong 0$. □

Back to the proof of 1.1. It is a theorem of Serre [7] that R is a polynomial algebra if and only if S is free as a module over R. If υ is an irreducible $k[W]$-module, let $S_i[\upsilon]$ be the υ-isotypic summand of S_i, i.e., the unique summand of S_i which is isomorphic as a W-module to a direct sum of copies of υ. Denote by $S[\upsilon] = \oplus_i S_i[\upsilon]$ the υ-isotypic summand of S. Each $S[\upsilon]$ is an R-submodule of S. Recall that $\rho = P_n$.

3.2. **Lemma.** *$S[\rho]$ is a free rank one R-module.*

Proof. Clearly $S[\rho] \otimes_R k$ is the ρ-isotypic part of P_*, that is, the one–dimensional space P_n (2.2). It follows easily from 3.1(2) that $S[\rho]$ requires only one generator as an R-module; pick such a generator, say, α. Since S is an integral domain, $r\alpha \neq 0 \in S[\rho]$ for any $r \neq 0 \in R$.. It follows that $S[\rho]$ is freely generated by α as an R-module. □

3.3. **Definition.** For $0 \leq k \leq n$, $S(k)$ is the sub-R-module of S generated over R by elements of S of degree k or less.

Note that $S(0) = R$ and $S(n) = S$. Given Serre's result, 1.1 follows from the next proposition.

3.4. **Proposition.** *For $0 \leq k \leq n$, $S(k)$ is a free R-module.*

Proof. This is clear for $k = 0$, since $S(0) = R$. Assume by induction that $S(i)$ is a free R module for all $i < k$. It suffices to show then that $S(k)/S(k-1)$ is a free R-module. But

$$(S(k)/S(k-1)) \otimes_R k \cong P_k.$$

For each i, choose a splitting of the surjection

$$S(i)/S(i-1) \to P_i.$$

Let V_i be the image of this splitting. Taking products with V_{n-i} yields an R-module map

$$(S(i)/S(i-1)) \otimes_k V_{n-i} \to S(n)/S(n-1).$$

Here the target is just $S[\rho]$, which is free over R (3.2). Choose a k–basis $\{v_1, \ldots, v_m\}$ of V_{n-i}. Each basis element v_j produces, via the above multiplication pairing, a map

$$f_j : S(k)/S(k-1) \to S(n)/S(n-1).$$

Taking the sum over all j gives

$$S(k)/S(k-1) \to \oplus_m (S(n)/S(n-1)).$$

This is a map of R-modules. By Poincaré duality in P_*, it is an isomorphism after $- \otimes_R k$. Since the target is a free R-module, it follows from 3.1(3) that the map before tensoring is an isomorphism. Thus $S(k)/S(k-1)$ is a free R-module and the induction is established. □

4. Examples

We give two types of examples; in both, P_* satisfies Poincaré duality, but $|W|$ is not a unit in k. In the first case, the ring of invariants is a polynomial algebra, but in the second it is not.

4.1. *Example.* Let $k = \mathbb{F}_p$, and let $W \subset GL_2(\mathbb{F}_p)$ be generated by the

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

A is a pseudo–reflection of order p. For $S = k[x, y]$, we can choose generators so that $A^T x = x$, $A^T y = y + x$. It is not hard to see that $R = S^W$ is generated by x and $v = y(y + x) \dots (y + (p - 1)x) = y(y^{p-1} - x^{p-1})$. The algebra P_* is isomorphic to

$$k[y]/y^p$$

and thus satisfies Poincaré duality.

In the above example, the group W acts trivially on P_*, and so the semisimplicity needed for our proof of 1.1 dramatically fails. Nevertheless, the ring R of invariants is polynomial.

4.2. *Example.* Let $k = \mathbb{F}_p$ and let $W' \subset GL_4(\mathbb{F}_p)$ be generated by the

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let $S = k[x_1, y_1, x_2, y_2]$; we can choose generators so that $B^T x_i = x_i$ and $B^T y_i = y_i + x_i$. Then $R = S^{W'}$ has generators x_1, v_1, x_2, v_2 with $v_i = y_i(y_i^{p-1} - x_i^{p-1})$ and an additional generator $z = x_1 y_2 - x_2 y_1$. (This is more difficult to see; the argument is below.) There is an isomorphism

$$P_* = k[y_1]/(y_1^p) \otimes k[y_2]/(y_2^p)$$

and so P_* satisfies Poincaré duality. However, $R = S^W$ is not a polynomial algebra.

The case $p = 2$ of this example has been treated by Smith [8].

4.3. *Remark.* Let $H \subset GL_4(\mathbb{F}_p)$ be generated by

$$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then H is isomorphic to the product $W \times W$, where W is the group of 4.1, embedded along the diagonal in $GL_4(\mathbb{F}_p)$. The algebra of invariants S^H is $k[x_1, v_1, x_2, v_2]$, which is isomorphic to the tensor product of two copies of the algebra of invariants from 4.1 . The ideal in S generated by x_1, v_1, x_2, v_2, z is that same as that generated by x_1, v_1, x_2, v_2; this is one way to calculate $P_* = S/SR_+$.

It remains to calculate $R = S^{W'}$. Let T be the subalgebra of $S = k[x_1, y_1, x_2, y_2]$ generated by

$$\{x_1, v_1, x_2, v_2, z\}.$$

It is clear that T is invariant under W', i.e., $T \subset R$, and we must show that $T = R$. Our goal is to use as little explicit calculation as possible to accomplish this. The ring R is integrally closed for general reasons, and so it's enough to show $T \to S$ has the proper degree on the level of fraction fields and that T is integrally closed.

We write $\mathrm{Fr}(T)$, for instance, for the fraction field of T. Note to begin with that $\mathrm{Fr}(T)$ properly contains $\mathrm{Fr}(S^H)$ (where H is the group from 4.3), since z is in $\mathrm{Fr}(T)$ but not in $\mathrm{Fr}(S^H)$. The extension degree $[\mathrm{Fr}(S^H), \mathrm{Fr}(S)]$ is p^2, and hence $[\mathrm{Fr}(T), \mathrm{Fr}(S)]$ properly divides p^2. Since this latter degree is not one, it must be p. Since $[\mathrm{Fr}(R), \mathrm{Fr}(S)]$ is not one and divides $[\mathrm{Fr}(T), \mathrm{Fr}(S)]$, it follows that $[\mathrm{Fr}(R), \mathrm{Fr}(S)]$ also equals p and hence that $\mathrm{Fr}(R) = \mathrm{Fr}(T)$.

It remains to show that T is integrally closed in $\mathrm{Fr}(T)$. For this, we use Serre's R_1 and S_2 criterion; see for example Matsumura [6, p. 183]. Write T as the quotient of the polynomial ring $T' = k[a, b, c, d, e]$ with the surjection ϕ defined by

$$\begin{array}{rclcrcl} \phi(a) & = & x_1 & \quad & \phi(b) & = & v_1 \\ \phi(c) & = & x_2 & & \phi(d) & = & v_2. \\ \phi(e) & = & z & & & & \end{array}$$

Since T is a domain, $\ker(\phi)$ is a prime ideal. The dimension of T' is 5 and that of its image T is 4, so the height of $\ker(\phi)$ is less than or equal to one. But T' is a domain and $\ker(\phi) \neq 0$, so the height of the kernel is one. Since T' is a UFD, any irreducible element generates a prime ideal. Since $\ker(\phi)$ has height one, any irreducible element of it must be a generator. Note that

$$f = e^p - a^{p-1}c^{p-1}e + bc^p - a^p d$$

is in $\ker(\phi)$, since

$$\begin{aligned} & x_1^p y_2^p - x_2^p y_1^p - x_1^{p-1} x_2^{p-1}(x_1 y_2 - x_2 y_1) \\ & \qquad + y_1(y_1^{p-1} - x_1^{p-1})x_2^p - y_2(y_2^{p-1} - x_2^{p-1})x_1^p = 0. \end{aligned}$$

The element f is irreducible, since its image is irreducible in $T'/(c)$. Hence f generates the prime ideal $\ker(\phi)$ and $T = T'/(f)$. $\{f\}$ is a regular sequence of length one, so $T = T'/(f)$ is Cohen-Macaulay. But T being Cohen-Macaulay is equivalent to T satisfying Serre's condition S_n for all n [6, p. 183]. Condition R_1 is that for each height one ideal $\mathcal{P} \in \mathrm{Spec}(T)$, the localization $T_{\mathcal{P}}$ must be regular. This remains to be checked. We first calculate the singular locus of T, using the Jacobian criterion [2, p. 306, 5.7.5]. The 1×1 minors of the Jacobian of f generate the ideal

$$j = (a^{p-2}c^{p-1}e, c^p, a^p, a^{p-1}c^{p-2}e, a^{p-1}c^{p-1}) \ .$$

Its radical is generated by a and c, and the radical $\mathcal{Q}'$ of the ideal in T' generated by f and j is (a, c, e), which has height 3. Now if $\mathcal{P} \subset T$ is a height one prime ideal, then its preimage $\mathcal{P}' \subset T'$ is only height two. Hence $\mathcal{P}'$ cannot contain $\mathcal{Q}'$. That is $T_{\mathcal{P}}$ is regular, since $\mathcal{P}$ is not in the singular locus of T, defined by $\mathcal{Q} = \mathrm{Image}(\mathcal{Q}')$. Here we've used that T is a domain and that $k = \mathbb{F}_p$ is perfect.

4.4. *Example.* Our last example arises as the ring of invariants for the action of the Weyl group $W(F_4)$ acting on the mod 3 cohomology of the classifying space of the maximal torus of the simple compact exceptional Lie group F_4 of rank four. The group $W(F_4)$ has cardinality 1152, and the vector space V on which it acts can be identified with the first homology group of the maximal torus; this has rank 4 over $\mathbb{F}_3$. The algebra $S = S(V^{\#})$ is then the cohomology algebra of the classifying space of the torus, and in order to conform to the topological origin of this algebra, we grade it so that the elements of $V^{\#}$ have degree 2. Toda [10] computed that $R = S^{W(F_4)}$ is isomorphic to $\mathbb{F}_3[y_1, y_2, y_5, y_9, y_{12}]/(f_{15})$, where the dimension of y_i is $4i$, and

$$f_{15} = y_5^3 + y_2^2 y_1 y_5^2 - y_{12} y_1^3 - y_9 y_2^2.$$

Define $R'' = \mathbb{F}_3[y_1, y_2, y_9, y_{12}]$. The fraction field degree of S over R is 1152, and that of S over R'' is three times this. The quotient $S \otimes_{R''} k$ is a Poincaré algebra. Under the inclusion $R \to S$, y_5 is contained in the S-ideal generated by y_1 and y_2. Hence $S \otimes_R k = S \otimes_{R''} k$ is a Poincaré duality algebra, even though R is not a polynomial algebra. Note that in this case W is generated by reflections of order 2 and the algebra of coinvariants satisfies Poincaré duality, but S^W still fails to be a polynomial algebra. We thank Larry Smith for telling us of this example.

4.5. *Remark.* In each of the counterexamples above, the ring of coinvariants is a complete intersection, not just a Gorenstein ring (for algebras like P_*, being Gorenstein is equivalent to satisfying Poincaré duality). It would be interesting to find an algebra which satisfies Poincaré duality but is *not* a complete intersection occurring as a ring of coinvariants.

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